Real Analysis

This is a mix of notes from MIT 18.100A Real Analysis (https://ocw.mit.edu/courses/18-100a-real-analysis-fall-2020/), MIT 18.S190 Introduction To Metric Spaces (https://ocw.mit.edu/courses/18-s190-introduction-to-metric-spaces-january-iap-2023/) and MIT 18.101 Analysis II (https://ocw.mit.edu/courses/18-101-analysis-ii-fall-2005/pages/lecture-notes/), covering the basic topics in real analysis. I previously took MATH320 Real Variables I from UBC, which covers first five chapters in Principles of Mathematical Analysis by Walter Rudin, but didn't do well in it. This set of notes is mostly a review for that course with generalization on $\mathbb{R}^n$.

1 Basic Set Theory

1.1 Definitions

***Definition:* 1.1: Set**

A set is a collection of objects called elements or numbers.

***Definition:* 1.2: Empty Set**

The empty set is the set with no elements, denoted by $\emptyset$.

Notation:

- $a \in S$ (a is an element in S)
- $a \notin S$ (a is not an element in S)
- $\Rightarrow$ (implies)
- $\Leftrightarrow$ (if and only if)

***Definition:* 1.3: Subset**

1. A set A is a subset of B, $A \subset B$ if $a \in A \Rightarrow a \in B$
2. Two sets are equal, $A = B$, if $A \subset B$ and $B \subset A$
3. A is a proper subset of B, $A \subsetneq B$ if $A \subset B$ and $A \neq B$

Set building notation: $\{x \in A : P(x)\}$ or $\{x : P(x)\}$, where $P(x)$ means x satisfies property P.

Example:

1. $\mathbb{N} = \{1, 2, 3, ...\}$: natural numbers
2. $\mathbb{Z} = \{0, 1, -1, 2, -2, ...\}$: integers
3. $\mathbb{Q} = \left\{\frac{m}{n} : m, n \in \mathbb{Z}, n \neq 0\right\}$: rational numbers

4. $\mathbb{R}$: real numbers
5. $\{2m-1 : m \in \mathbb{N}\}$: odd numbers
6. $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$

***Definition:* 1.4: Set Operations**

1. The union of A, B is the set $A \cup B = \{x : x \in A \text{ or } x \in B\}$
2. The intersection of A, B is the set $A \cap B = \{x : x \in A \text{ and } x \in B\}$
3. The set difference of A w.r.t. B is the set $A \setminus B = \{x : x \in A \text{ and } x \notin B\}$
4. The compliment of A is the set $\bar{A} = A^C = \{x : x \notin A\}$
5. A and B are disjoint if $A \cap B = \emptyset$

***Theorem:* 1.1: De Morgan's Law**

If A, B, C are sets, then
1. $(B \cup C)^C = B^C \cap C^C$
2. $(B \cap C)^C = B^C \cup C^C$
3. $A \setminus (B \cup C) = (A \setminus B) \cap (A \setminus C)$
4. $A \setminus (B \cap C) = (A \setminus B) \cup (A \setminus C)$

Proof. For the first rule only. Let B, C be sets. To show that $(B \cup C)^C = B^C \cap C^C$, we need to prove that $(B \cup C)^C \subset B^C \cap C^C$ and $B^C \cap C^C \subset (B \cup C)^C$

1. Let $x \in (B \cup C)^C$. Then $x \notin B \cup C \Rightarrow x \notin B$ and $x \notin C \Rightarrow x \in B^C \cap C^C$. Thus $(B \cup C)^C \subset B^C \cap C^C$.
2. Let $x \in B^C \cap C^C$. Then $x \in B^C$ and $x \in C^C \Rightarrow x \notin B$ and $x \notin C \Rightarrow x \notin B \cup C \Rightarrow x \in (B \cup C)^C$. Thus $B^C \cap C^C \subset (B \cup C)^C$.

By 1. and 2. $(B \cup C)^C = B^C \cap C^C$. □

1.2 Induction

Consider the natural number set $\mathbb{N} = \{1, 2, 3, ...\}$. It has ordering $1 < 2 < 3 < \cdots$

***Axiom:* 1.1: Well Ordering Property of $\mathbb{N}$**

If $S \subset \mathbb{N}$ and $S \neq \emptyset$. then S has a least element. *i.e.* $\exists x \in S$, s.t. $x \leq y$ for all $y \in S$.

***Theorem:* 1.2: Induction**

Let $P(n)$ be a statement depending on $n \in \mathbb{N}$. Assume:
1. (Base case) $P(1)$ is true.
2. (Inductive step) If $P(m)$ is true, then $P(m+1)$ is true.

Then $P(n)$ is true for all $n \in \mathbb{N}$.

Proof. Let $S = \{n \in \mathbb{N} : P(n) \text{ is not true}\}$. We want to show that $S = \emptyset$. We will show this by contradiction.

Suppose $S \neq \emptyset$, by well ordering principle, S has a least element $x \in S$.

Since $P(1)$ is true as base case, $1 \notin S$, and $x \neq 1$. In particular $x > 1$.

Since x is the least element of S and $x - 1 < x$, then $x - 1 \notin S$. Thus, $P(x-1)$ is true by definition of S.

By the inductive step, $P(x)$ is true from $P(x-1)$ is true $\Rightarrow x \notin S$.

We show that $x \in S$ and $x \notin S$, which is a contradiction. Thus $S = \emptyset$. $\square$

Using induction, we want to prove $\forall n \in \mathbb{N}$, $P(n)$ is true. We do 2 things:

1. Prove $P(1)$
2. Prove if $P(m)$ is true, then $P(m+1)$ is true

Example: For all $c \neq 1$, $\forall n \in \mathbb{N}$, $1 + c + \cdots + c^n = \frac{1-c^{n+1}}{1-c}$.

Proof. 1) Base case: $1 + c^1 = \frac{1-c^{1+1}}{1-c}$, since $\frac{1-c^2}{1-c} = \frac{(1-c)(1+c)}{1-c} = 1 + c$
2) Inductive step: Assume $1 + c + \cdots + c^m = \frac{1-c^{m+1}}{1-c}$, we want to prove it holds for $n = m + 1$.
$1 + c + \cdots + c^{m+1} = \frac{1-c^{m+1}}{1-c} + c^{m+1} = \frac{1-c^{m+1}+c^{m+1}-c^{m+2}}{1-c} = \frac{1-c^{m+2}}{1-c}$
Thus, it holds for $n = m + 1$. By induction, it holds for all $n \in \mathbb{N}$. $\square$

Example: If $c \geq -1$, then $\forall n \in \mathbb{N}$, $(1 + c)^n \geq 1 + nc$.

Proof. 1) Base case: $(1 + c)^1 = 1 + 1 \cdot c$
2) Inductive step: Assume $(1 + c)^m \geq 1 + mc$
Then $(1+c)^{m+1} = (1+c)(1+c)^m \geq (1+c)(1+mc) = 1+(m+1)c+mc^2 \geq 1+(m+1)c$ since $mc^2 \geq 0$ $\square$

Note: There are multiple ways to write the induction proof. This is just the way shown by the prof of 18.100A. I personally like to separate the induction hypothesis (the assumption for m) from inductive step.

1.3 Cantor's Theory of Cardinality

Q: When do two sets A and B have the same size?
A: When the elements of the two sets can be paired off.

***Definition:* 1.5: Functions**

If A, B are sets, a function $f : A \to B$ is a mapping that assigns to each $x \in A$ a unique element $f(x) \in B$.

***Definition:* 1.6: Image/Preimage**

Let $f : A \to B$
1. Image: If $C \subset A$, define $f(C) = \{y \in B : \exists x \in C \text{ s.t. } y = f(x)\} = \{f(x) : x \in C\}$
2. Preimage: If $D \subset B$, define $f^{-1}(D) = \{x \in A : f(x) \in D\}$

***Definition:* 1.7: Injection/Surjection/Bijection**

Let $f : A \to B$
1. f is injective or 1-1 if $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ or equivalently $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
2. f is surjectvie or onto if $f(A) = B$
3. f is bijective if f is injective and surjective

Definition: 1.8: Composition and Inverse

Let $f : A \to B$, $g : B \to C$

1. $g \circ f : A \to C$ is defined by $(g \circ f)(x) = g(f(x))$
2. If f is bijective, then define the inverse function $f^{-1} : B \to A$ by: if $y \in B$, then $f^{-1}(y) \in A$ is the unique element in A s.t. $f(f^{-1}(y)) = y$

Definition: 1.9: Cardinality

Two sets A and B have the same cardinality if there exists a bijective function $f : A \to B$

Notation:

1. If A, B have the same cardinality, we write $|A| = |B|$
2. If $|A| = \{1, 2, ..., n\}$, we write $|A| = n$ (A is finite)
3. If $\exists$ injective function $f : A \to B$, we write $|A| \leq B$
4. If $|A| \leq |B|$, and $|A| \neq |B|$, we write $|A| < |B|$

Theorem: 1.3: Cantor-Schroder-Bernstein

If $|A| \leq |B|$ and $|B| \leq |A|$, then $|A| = |B|$.

Definition: 1.10: Countable

If $|A| = |\mathbb{N}|$, then A is countably infinite.
If A is finite or countably infinite, A is countable. Otherwise, A is uncountable.

Theorem: 1.4: Symmetry of Cardinality

If $|A| = |B|$, then $|B| = |A|$

Proof. Suppose $|A| = |B|$, then there exists a bijective function $f : A \to B$.
Then $f^{-1} : B \to A$ is a bijection, so $|B| = |A|$ □

Theorem: 1.5: Transitivity of Cardinality

If $|A| = |B|$ and $|B| = |C|$, then $|A| = |C|$

Proof. Suppose $|A| = |B|$ and $|B| = |C|$, then there exists bijective functions $f : A \to B$ and $g : B \to C$.
Let $h : A \to C$ be the function $h(x) = (g \circ f)(x)$. We want to show that h is bijective.
Injective: If $h(x_1) = h(x_2)$, then $g(f(x_1)) = g(f(x_2))$. Since g is injective, then $f(x_1) = f(x_2)$. Similarly, sice f is injective, $x_1 = x_2$
Surjective: Let $z \in C$. Since g is surjective, $\exists b \in B$ s.t. $g(y) = z$. Since f is surjective, $\exists x \in A$ s.t. $f(x) = y$. Thus, $h(x) = g(f(x)) = g(y) = z$. □

Example: $|\{2n : n \in \mathbb{N}\}| = |\mathbb{N}|$ and $|\{2n - 1 : n \in \mathbb{N}\}| = |\mathbb{N}|$

Proof. let $f : \mathbb{N} \to \{2n : n \in \mathbb{N}\}$ be the function $f(n) = 2n, n \in \mathbb{N}$.
Injective: Suppose $f(n_1) = f(n_2)$, then $2n_1 = 2n_2 \Rightarrow n_1 = n_2$
Surjective: Let $m \in \{2k : k \in \mathbb{N}\}$. Then $\exists n \in \mathbb{N}$ s.t. $m = 2n$. Then $f(n) = 2n = m$. □

Example: $|\mathbb{Z}| = |\mathbb{N}|$

Proof. Define $f : \mathbb{Z} \to \mathbb{N}$ by $f(n) = \begin{cases} 1, n = 0 \\ 2n, n > 0 \\ -2n+1, n < 0 \end{cases}$, $f(n)$ is bijective. □

Example: $|\{q \in \mathbb{Q} : q > 0\}| = |\mathbb{N}|$

Remark 1. Every $q \in \mathbb{Q}$, $q > 0$ can be written as $q = \frac{p_1^{r_1} \cdots p_N^{r_N}}{q_1^{s_1} \cdots q_M^{s_M}}$, where $r_j, s_k \in \mathbb{N}$, $\forall j, k$, $q_j \neq p_k$.

Proof. The function $f : \{q \in \mathbb{Q} : q > 0\} \to \mathbb{N}$ $f(q) = p_1^{2r_1} \cdots p_N^{2r_N} q_1^{2s_1 - 1} \cdots q_M^{2s_M - 1}$ is bijective. □

Example: $|\mathbb{Q}| = |\mathbb{N}|$

Proof. $|\{q \in \mathbb{Q} : q > 0\}| = |\{r \in \mathbb{Q} : r < 0\}|$, since $f(q) = -q$ is a bijection between the two sets.
$\Rightarrow |\{r \in \mathbb{Q} : r < 0\}| = |\mathbb{N}|$.
Then there exist bijections $f : \{q \in \mathbb{Q} : q > 0\} \to \mathbb{N}$ and $g : \{r \in \mathbb{Q} : r < 0\} \to \mathbb{N}$.
Define $h : \mathbb{Q} \to \mathbb{Z}$ by $h(x) = \begin{cases} 0, x = 0 \\ f(x), x > 0 \\ g(x), x < 0 \end{cases}$. $h(x)$ is a bijection, so $|\mathbb{Q}| = |\mathbb{Z}| = |\mathbb{N}|$. □

Definition: 1.11: Power Set

If A is a set, we define the power set of A by $\mathcal{P}(A) = \{B : B \subset A\}$

Examples:

1. $A = \emptyset$, $\mathcal{P}(A) = \{\emptyset\}$
2. $A = \{1\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}\}$
3. $A = \{1, 2\}$, $\mathcal{P}(A) = \{\emptyset, \{1\}, \{2\}, \{1, 2\}\}$

Theorem: 1.6: Size of Power Sets

$|\mathbb{N}| < |\mathcal{P}(\mathbb{N})| < |\mathcal{P}(\mathcal{P}(\mathbb{N}))| < \cdots$

Remark 2. Informally, there are an infinity of infinite sets

Proof. Let A be a set. **Injective:** Define $f : A \to \mathcal{P}(A)$ by $f(x) = \{x\}$. Then $f(x) = f(y) \Rightarrow \{x\} = \{y\} \Rightarrow x = y$. So f is 1-1. *i.e.* $|A| \leq \mathcal{P}(A)$.

We now show that $|A| \neq |\mathcal{P}(A)|$ by contradiction.
Assume $|A| = |\mathcal{P}(A)|$. Then there exists a bijective function $g : A \to \mathcal{P}(A)$.
Define a subset $B \subset A$ by $B = \{x \in A : x \notin g(x)\}$, $B \in \mathcal{P}(A)$.
Since g is surjective (as a bijective function), $\exists b \in A$ s.t. $g(b) = B$

Case 1: $b \in g(b) = B$, then $b \in B \Rightarrow b \notin g(b)$ by definition of B. Contradiction.
Case 2: $b \notin g(B)$, then $b \notin B \Rightarrow b \in g(b)$. Contradiction.
This gives that $b \in g(b) \Leftrightarrow b \notin g(b)$. Contradiction. Thus $|A| \neq |\mathcal{P}(A)|$ $\square$

Side Note: Continuum hypothesis: Does there exist A such that $|\mathbb{N}| < |A| < |\mathcal{P}(\mathbb{N})|$?

There is no set whose cardinality is strictly between that of the integers and the real numbers. Any subset of the real numbers is finite, is countably infinite, or has the same cardinality as the real numbers.

2 Real Numbers

Theorem: 2.1: Existence of Real Numbers

There exists a unique ordered field containing $\mathbb{Q}$ with the least upperbound property, which we denote by $\mathbb{R}$.

2.1 Ordered Sets and Rational Numbers

Definition: 2.1: Ordered Set

An ordered set is a set S with a relation $<$ s.t.

1. $\forall x, y \in S$, either $x = y$, $x < y$ or $y < x$
2. if $x < y$ and $y < z$, then $x < z$

Example:

1. $\mathbb{Z}$ is ordered, $m < n$ if $n - m \in \mathbb{N}$
2. $\mathbb{Q}$ is ordered, $q < r$ if $\exists m, n \in \mathbb{N}$ s.t. $r - q = \frac{m}{n}$
3. Dictionary ordering of $\mathbb{Q} \times \mathbb{Q}$, where $A \times B = \{(a, b) : a \in A, b \in B\}$. We say $(a, b) < (q, r)$ if $a < q$ or ($a = q$ and $b < r$).

Non Example: $S = \mathcal{P}(\mathbb{N})$ is not ordered with the relation $A < B$ if $A \subset B$.
The second property can be satisfied: if $A \subset B$ and $B \subset C$, then $A \subset C$. **i.e.** if $A < B$ and $B < C$, then $A < C$.
However, $\{0\} \neq \{1\}$, but neither $\{0\} < \{1\}$ or $\{1\} < \{0\}$ holds. The first property is not satisfied.

Definition: 2.2: Supremum and Infimum

Let S be an ordered set, and $E \subset S$

1. If $\exists b \in S$ s.t. $\forall x \in E$, $x \leq b$, then we say that E is bounded above and b is an *upper bound* of E
2. If $\exists b \in S$ s.t. $\forall x \in E$, $b \leq x$, then we say that E is bounded below and b is an *lower bound* of E
3. We call $b_0 \in S$ the *least upper bound* of E if
   - b_0 is an upper bound of E
   - if b is any upper bound for E, then $b_0 \leq b$

   We call b_0 the *supremum* of E, and write $b_0 = \sup E$
4. We call $b_0 \in S$ the *greatest lower bound* of E if
   - b_0 is an lower bound of E
   - if b is any lower bound for E, then $b \leq b_0$

   We call b_0 the *infimum* of E, and write $b_0 = \inf E$

Example:

1. $S = \mathbb{Z}$, $E = \{-1, 0, 2\}$, UBs$= \{2, 3, 4, 5, ...\}$, $\sup E = 2$, LBs$= \{..., -3, -2, -1\}$, $\inf E = -1$
2. $S = \mathbb{Q}$, $E = \{q \in \mathbb{Q} : 0 \leq q \leq 1\}$, $\sup E = 1$, $\inf E = 0$
3. $S = \mathbb{Q}$, $E = \{q \in \mathbb{Q} : 0 < q < 1\}$, $\sup E = 1 \notin E$, $\inf E = 0 \notin E$

Definition: 2.3: Least Upperbound Property

An ordered set S has the least upperbound property if every $E \subset S$, which is nonempty and bounded above, has a supremum in S.

Example:

1. $S = \{0\}$ has the least upperbound property
2. $S = \{0, 1\}$ has the least upperbound property. When $E = \{0\}$, $\sup E = 0 \in S$. When $E = \{1\}$, $\sup E = 1 \in S$. When $E = \{0, 1\}$, $\sup E = 1 \in S$.
3. $S = \{-1, -2, -3, -4, ...\}$ has the least upperbound property. If $E \subset S$, E nonempty, then $-E = \{-x : x \in E\} \subset \mathbb{N}$. By well ordering property of $\mathbb{N}$, $\exists m \in -E$ s.t. $m \leq -x$ for all $x \in E$. $\Rightarrow -m \in E$ and $\forall x \in E$, $x \leq -m$. Thus, $\sup E = -m$
4. $\mathbb{Z}$ has the least upperbound property

Claim 1. *$\mathbb{Q}$ does not have the least upper bound property.*

If $E = \{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$, then $\sup E$ DNE in $\mathbb{Q}$.

Theorem: 2.2:

If $x \in \mathbb{Q}$ and $x = \sup\{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$, then $x \geq 1$ and $x^2 = 2$

Proof. Let $E = \{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$ and suppose $x \in \mathbb{Q}$ s.t. $x = \sup E$.
Since $1 \in E$ and $x = \sup E$, then $x \geq 1$

We then prove $x^2 \geq 2$ by contradiction.
Assume x^2. Define $h = \min\left\{\frac{1}{2}, \frac{2-x^2}{2(2x+1)}\right\} < 1$
Then $h > 0$, we show that $x + h \in E$.
$(x+h)^2 = x^2 + 2xh + h^2 < x^2 + 2xh + h$, since $h < 1 = x^2 + (2x+1)h \leq x^2 + (2x+1)\frac{2-x^2}{2(2x+1)} = x^2 + \frac{2-x^2}{2} < x^2 + 2 - x^2 = 2$
Thus $(x + h)^2 < 2$, $x + h \in E$.
However, $x + h > x$. This contradicts to the condition that $x = \sup E$. Thus $x^2 \geq 2$.

We now show $x^2 = 2$. Since $x^2 \geq 2$, we have either $x^2 = 2$ or $x^2 > 2$. We show that $x^2 > 2$ cannot hold by contradiction.
Assume $x^2 > 2$. Define $h = \frac{x^2-2}{2x}$. Note that since $x^2 > 2$, then $0 < h$, and $x - h < x$.
$(x - h)^2 = x^2 - 2xh + h^2 = x^2 - (x^2 - 2) + h^2 = 2 + h^2 > 2$, *i.e.* $2 < (x - h)^2$
Let $q \in E$, *i.e.* $q > 0$ and $q^2 < 2$. Then $q^2 < 2 < (x-h)^2$. Then $0 < (x-h)^2 - q^2 \Rightarrow 0 < (x-h-q)(x-h+q)$ $\Rightarrow 0 < (x - h - q)\left(\frac{x^2+2}{2x} + q\right)$
Since $q > 0$ and $\frac{x^2+2}{2x} > 0$, $\frac{x^2+2}{2x} + q > 0$, and thus $0 < x - h - q$, which implies $q < x - h$.
Thus, $\forall q \in E$, $q < x - h$, $x - h$ is an upper bound for E

Since $x = \sup E$, $x \leq x - h$ by defintion. This means that $h \leq 0$, which contradicts to the fact that $h > 0$. Thus $x^2 = 2$. □

Theorem: 2.3:

The set $E = \{q \in \mathbb{Q} : q > 0 \text{ and } q^2 < 2\}$ is bounded above and has no supremum in $\mathbb{Q}$

Proof. Let $q \in E$. Then $q^2 < 2 < 4 \Rightarrow 0 < 4 - q^2 \Rightarrow 0 < (2-q)(2+q)$. Since $q > 0$, $2 + q > 0$, we must have $2 - q > 0$, *i.e.* $q < 2$.
Thus $\forall q \in E$, $q < 2$. E is bounded above and 2 is an upper bound for E.

We now show $\sup E$ DNE by contradiction.
Assume $\sup E$ exists in $\mathbb{Q}$. Call it $x = \sup E$.
By Theorem 2.2, $x > 1$ and $x^2 = 2$.
Thus $\exists m, n \in \mathbb{N}$ s.t. $m > n$ and $x = \frac{m}{n} > 1$. This also means that $\exists n \in \mathbb{N}$ s.t. $nx \in \mathbb{N}$.
Let $S = \{k \in \mathbb{N} : kx \in \mathbb{N}\}$. S is nonempty since $n \in S$.

By the well ordering principle of $\mathbb{N}$. S has a least element $k_0 \in S$. Define $k_1 = k_0 x - k_0 \in \mathbb{Z}$.
$x^2 = 2$ implies that $x < 2$. Then $k_1 = k_0(x-1) < k_0(2-1) = k_0$. Thus $k_1 \in \mathbb{N}$ ($x \geq 1$, so $k_1 = k_0(x-1) \geq 0$) and $k_1 < k_0$.

Then $xk_1 = x(k_0(x-1)) = x^2 k_0 - xk_0 = 2k_0 - xk_0 = k_0 + (k_0 - xk_0) = k_0 - k_1 \in \mathbb{N}$
Thus $k_1 \in S$ and $k_1 < k_0$. k_0 is not the least element in S. Contradiction. Thus $\sup E$ DNE. □

This also concludes the proof that $\mathbb{Q}$ doesn't have the least upper bound property.

2.2 Fields and Ordered Fields

***Definition:* 2.4: Field**

A set F is a field if it has two operations $+$ and $\cdot$ s.t.
A1) If $x, y \in F$, then $x + y \in F$
A2) (Commutativity) $\forall x, y \in F$, $x + y = y + x$
A3) (Associativity) $\forall x, y, z \in F$, $(x + y) + z = x + (y + z)$
A4) There exists an element $0 \in F$ s.t. $\forall x \in F$, $0 + x = x$
A5) $\forall x \in F$, $\exists -x \in F$ s.t. $x + (-x) = 0$
M1) If $x, y \in F$, then $x \cdot y \in F$
M2) (Commutativity) $\forall x, y \in F$, $x \cdot y = y \cdot x$
M3) (Associativity) $\forall x, y, z \in F$, $(xy)z = x(yz)$
M4) There exists an element $1 \in F$ s.t. $\forall x \in F$, $1 \cdot x = x$
M5) $\forall x \in F \setminus \{0\}$, $\exists x^{-1} \in F$ s.t. $x \cdot x^{-1} = 1$

Example: $\mathbb{Z}_2 = \{0, 1\}$ with $1 + 1 = 0$, $\mathbb{Z}_3 = \{0, 1, 2\}$ and $\mathbb{Q}$ are fields.
$\mathbb{Z}$ is a commutative ring, but not a field.

***Theorem:* 2.4:**

If F is a field, then $\forall x \in F$, $0x = 0$.

Proof. If $x \in F$, $0 = 0 \cdot x + (-0 \cdot x) = (0 + 0)x + (-0x) = 0x + 0x + (-0x) = 0x$ □

Similarly, we can show that $-x = (-1)x$.

Definition: 2.5: Ordered Field

An ordered field is a field F, which is also an ordered set s.t.

1. $\forall x, y, z \in F$, if $x < y$, then $x + z < y + x$
2. If $x > 0$ and $y > 0$, then $xy > 0$.

If $x > 0$ ($x \geq 0$), we say x is positive (non-negative).

Example: $\mathbb{Q}$ is an ordered field. $\mathbb{Z}_2 = \{0, 1\}$ is not an ordered field.

Proof. If $<$ is an order on $\mathbb{Z}_2$, then either $0 < 1$ or $1 < 0$
If $0 < 1$, then $1 + 0 = 1$, $1 + 1 = 0 \Rightarrow 1 + 1 < 1 + 0$, so it doesn't satisfy the first property.
If $1 < 0$, then $1 + 0 = 1$, $1 + 1 = 0 \Rightarrow 1 + 0 < 1 + 1$, so it doesn't satisfy the first property. □

Generally, there is no finite ordered field.

Theorem: 2.5: Properties of Ordered Fields

If F is an ordered field, then

1. $x > 0 \Rightarrow -x < 0$ and $x < 0 \Rightarrow -x > 0$
2. If $x > 0$ and $y < z$, then $xy < xz$
3. If $x < 0$ and $y < z$, then $xy > xz$
4. If $x \neq 0$, then $x^{>}0$
5. If $0 < x < y$, then $0 < \frac{1}{y} < \frac{1}{x}$
6. If $0 < x < y$, then $x^2 < y^2$
7. If $x \leq y$ and $z \leq w$, then $x + z \leq y + w$

Proof. 1) If $0 < x$, then $-x + 0 < -x + x \Rightarrow -x < 0$ (LHS by A1 in Definition 2.4, RHS by A5)
□

Theorem: 2.6: Supremum and Infimum in Ordered Fields

Let F be an ordered field with the least upperbound property. If $A \subset F$, $A \neq \emptyset$ and bounded below, then $\inf F$ exists in F.

Proof. Suppose F is an ordered field with the least upperbound property. Let $A \subset F$, $A \neq \emptyset$ be bounded below.
Then $\exists b \in F$ s.t. $b \leq a$, $\forall a \in A$. Thus $-a \leq -b$, $-b$ is an upper bound for $-A$.
Since F has the least upperbound property, there exists $x \in F$ s.t. $x = \sup(-A)$.
Then $\forall a \in A$, $-a \leq x$, which implies $\forall a \in A$, $-x \leq a$. $-x$ is a lower bound for A.

We now show that if y is a lower bound for A, then $y \leq -x$
Let y be a lower bound for A. Then $-y$ is an upper bound for $-A$.
Since $x = \sup(-A)$, $x \leq y \Rightarrow y \leq -x$.
Thus $\inf A$ exists and $\inf A = -\sup(-A)$. □

2.3 Real Numbers

Theorem: 2.7: Existence of Real Numbers

There exists a unique (up to isomorphism) ordered field containing $\mathbb{Q}$ with the least upperbound property, which we denote by $\mathbb{R}$.

Theorem: 2.8:

There exists a unique $r \in \mathbb{R}$ s.t. $r > 0$ and $r^2 = 2$

Proof. Similar to the proof in rationals. Let $E = \{x \in \mathbb{R} : x > 0 \text{ and } x^2 < 2\}$. Then E is bounded above. By the least upper bound property $\sup E$ exists in $\mathbb{R}$.
Label $r = \sup E$. The same proof in $\mathbb{Q}$ shows $r \geq 1$, $r^2 = 2$.

We now prove r is unique. Suppose $\tilde{r} \in \mathbb{R}$, with $\tilde{r} > 0$ and $\tilde{r}^2 = 2$.
Then $0 = \tilde{r}^2 - r^2 = (\tilde{r} + r)(\tilde{r} - r)$.
Since both $\tilde{r} > 0$ and $r > 0$, then $\tilde{r} + r > 0$ and we must have $\tilde{r} - r = 0$, *i.e.* $\tilde{r} = r$. □

In general, if $x \in \mathbb{R}$, $x > 0$, then $x^{\frac{1}{n}}$ exists in $\mathbb{R}$ for all $n \in \mathbb{N}$.

Fact: If $x, y \in \mathbb{R}$ and $x < y$, then $\exists r \in \mathbb{R}$ s.t. $x < r < y$ (*e.g.* $r = \frac{x+y}{2}$). Same holds for $\mathbb{Q}$.

Theorem: 2.9: Archimedian Property and Density of $\mathbb{Q}$

1. (Archimedian Property) If $x, y \in \mathbb{R}$ and $x > 0$, then $\exists n \in \mathbb{N}$ s.t. $nx > y$
2. (Density of $\mathbb{Q}$) If $x, y \in \mathbb{R}$ $x < y$, then $\exists r \in \mathbb{Q}$, s.t. $x < r < y$.

Proof.
1. Suppose $x, y \in \mathbb{R}$, $x > 0$. We want to show that $n > \frac{y}{x}$ for some n.
   Assume $\forall n \in \mathbb{N}$, $n \leq \frac{y}{x}$. Then $\mathbb{N} \subset \mathbb{R}$ is bounded above.
   By the least upper bound property of $\mathbb{R}$, $\mathbb{N}$ has a supremum $a \in \mathbb{R}$. Since $a = \sup \mathbb{N}$, then $a - 1$ is not an upper bound for $\mathbb{N}$.
   This implies that there exists $m \in \mathbb{N}$ s.t. $a - 1 < m$, then $a < m + 1$. a is not an upper bound for $\mathbb{N}$, thus a cannot be the supremum. Contradiction. Thus $n > \frac{y}{x}$.
2. Suppose $x, y \in \mathbb{R}$, $x < y$. Then there are three cases
   (a) $x < 0 < y$. This is simple, just choose $r = 0 \in \mathbb{Q}$
   (b) $0 \leq x < y$. Then $y - x > 0$. By 1., there exists $n \in \mathbb{N}$ s.t. $n(y - x) > 1$. Then $nx + 1 < ny$.
   Again, by 1., $\exists l \in \mathbb{N}$ s.t. $l > nx$ (choose $y = nx$, $x = 1$, $n = l$). Thus, $S = \{k \in \mathbb{N} : k > nx\} \neq \emptyset$.
   By well ordering property of natural numbers, S has a least element m.
   Since $m \in S$ is the least element of S, $m - 1 \notin S$. Since $nx < m$ by definition of S, and $m - 1 \leq nx$, $m \leq nx + 1$.
   Then $nx < m \leq nx + 1 < ny$. $x < \frac{m}{n} < y$. So $r = \frac{m}{n} \in \mathbb{Q}$ is the choice.
   (c) $x < y \leq 0$. Then $0 \leq -y < -x$. By case (b), $\exists \tilde{r} = \frac{m}{n} \in \mathbb{Q}$ s.t. $-y < \tilde{r} < -x$. Then $x < -\tilde{r} < y$.

□

Theorem: 2.10: Supremum in $\mathbb{R}$

Assume $S \subset \mathbb{R}$ is nonempty and bounded above. Then $x = \sup S$ if and only if

1. x is an upper bound for S
2. $\forall \epsilon > 0$, $\exists y \in S$ s.t. $x - \epsilon < y \leq x$

Remark 3. Similarly, $x = \inf S$ if and only if

1. x is a lower bound for S
2. $\forall \epsilon > 0$, $\exists y \in S$ s.t. $x \leq y < x + \epsilon$

Example: $\sup\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 1$

Proof. Since $1 - \frac{1}{n} < 1$ for all $n \in \mathbb{N}$, 1 is an upper bound for $\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\}$.
Let $\epsilon > 0$. Then by Archimedian Property, there exists $n \in \mathbb{N}$ s.t. $n\epsilon > 1$, *i.e.* $\frac{1}{n} < \epsilon$ and $-\epsilon < -\frac{1}{n}$
Then $1 - \epsilon < 1 - \frac{1}{n} < 1$. Thus $\sup\left\{1 - \frac{1}{n} : n \in \mathbb{N}\right\} = 1$. □

Definition: 2.6:

For $x \in \mathbb{R}$, $A \subset \mathbb{R}$. Define $x + A = \{x + a : a \in A\}$ and $xA = \{xa : a \in A\}$.

Theorem: 2.11: Supremum with Constant Addition and Multiplication

1. If $x \in \mathbb{R}$ and A is bounded above, then $x + A$ is bounded above and $\sup(x + A) = x + \sup A$
2. If $x \in \mathbb{R}$, $x > 0$ and A is bounded above, then xA is bounded above and $\sup(xA) = x \sup A$

Proof.

1. Suppose $x \in \mathbb{R}$ and A is bounded above. Then $\sup A$ exists in $\mathbb{R}$.
Then $\forall a \in A$, $a \leq \sup A$, thus $\forall a \in A$, $x + a \leq x + \sup A$. $x + \sup A$ is an upper bound for $x + A$.
Let $\epsilon > 0$. Then $\exists y \in A$ s.t. $\sup A - \epsilon < y \leq \sup A$.
Then $\exists y \in A$ s.t. $x + \sup A - \epsilon < x + y \leq x + \sup A$. *i.e.* $\exists z \in x + A$, s.t. $(x + \sup A) - \epsilon < z \leq x + \sup A$.
Thus $\sup(x + A) = x + \sup A$
2. $x \in \mathbb{R}$ and A is bounded above. Then $\sup A$ exists in $\mathbb{R}$.
Then $\forall a \in A$, $a \leq \sup A$, thus $\forall a \in A$, $xa \leq x \sup A$. $x \sup A$ is an upper bound for xA.
Let $\epsilon > 0$. Then $\exists y \in A$ s.t. $\sup A - \frac{\epsilon}{x} < y \leq \sup A$.
Then $\exists y \in A$ s.t. $x \sup A - x\frac{\epsilon}{x} < xy \leq x \sup A$. *i.e.* $\exists z \in xA$, s.t. $x \sup A - \epsilon < z \leq x \sup A$.
Thus $\sup(xA) = x \sup A$

□

Theorem: 2.12: Bounded Sets

If $A, B \subset \mathbb{R}$ with A bounded above, B bounded below and $\forall x \in A$, $\forall y \in B$, $x \leq y$. Then $\sup A = \inf B$.

Proof. Let $y \in B$. Then $\forall x \in A$, $x \leq y$. *i.e.* y is an upper bound for A. Thus, $\sup A \leq y$.
This is true for all $y \in B$. Thus $\sup A$ is a lower bound for B. $\sup A \leq \inf B$. □

2.4 Absolute Values

Definition: 2.7: Absolute Value

If $x \in \mathbb{R}$, define the absolute value to be $|x| = \begin{cases} x, \text{ if } x \geq 0 \\ -x, \text{ if } x \leq 0 \end{cases}$

Theorem: 2.13: Properties of Absolute Values

1. $\forall x \in \mathbb{R}$, $|x| \geq 0$ and $|x| = 0 \Leftrightarrow x = 0$
2. $\forall x \in \mathbb{R}$, $|x| = |-x|$
3. $\forall x, y \in \mathbb{R}$, $|xy| = |x||y|$
4. $\forall x \in \mathbb{R}$, $|x^2| = |x|^2$
5. If $x, y \in \mathbb{R}$, then $|x| \leq y \Leftrightarrow -y \leq x \leq y$
6. $\forall x \in \mathbb{R}, x \leq |x|$

Proof.

1. If $x \geq 0$, then $|x| = x \geq 0$. If $x \leq 0$, then $|x| = -x \geq 0$.
   ($\Rightarrow$) If $x = 0$, then $|x| = 0$
   ($\Leftarrow$) Suppose $|x| = 0$, then if $x \geq 0$, $x = |x| = 0$. If $x \leq 0$, then $-x = |x| = 0$, $x = 0$. Thus $x = 0$
2. If $x \geq 0$, then $-x \leq 0$. Thus $|-x| = -(-x) = x = |x|$. If $x \leq 0$, then $-x \geq 0$, $|-x| = -x = |x|$. Thus $|-x| = |x|$.
3. If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$, $|xy| = xy = |x||y|$. WLOG, if $x \geq 0$ and $y \leq 0$, then $xy \leq 0$, $|xy| = -xy = x(-y) = |x||y|$. If $x \leq 0$ and $y \leq 0$, then $-x \geq 0$ and $-y \geq 0$. $|xy| = |(-x)(-y)| = |-x||-y| = |x||y|$.
4. Take $y = x$ in 3.
5. ($\Rightarrow$) suppose $|x| \leq y$. If $x \geq 0$, then $-y \leq 0 \leq x = |x| \leq y$. If $x \leq 0$, then $-x \geq 0$ and $|x| = -x \leq y$, $-y \leq -x \leq y$. Thus $-y \leq x \leq y$
   ($\Leftarrow$) Suppose $-y \leq x \leq y$. If $x \geq 0$, then $|x| =\leq y$. If $x \leq 0$, then $-y \leq x$ and $-x \leq y$. Thus, $|x| \leq y$.
6. Let $y = |x|$ in 5. We get $-|x| \leq x \leq |x|$

□

Theorem: 2.14: Triangle Inequality

$\forall x, y \in \mathbb{R}$, $|x + y| \leq |x| + |y|$

Proof. If $x, y \in \mathbb{R}$, $x + y \leq |x| + |y|$ and $(-x) + (-y) \leq |-x| + |-y| = |x| + |y|$.
Multiply -1 on both sides, $-(|x| + |y|) \leq -((-x) + (-y))$.
Thus $-(|x| + |y|) \leq x + y \leq |x| + |y|$
Then $|x + y| \leq |x| + |y|$. □

Remark 4. (Reversed Triangle Inequality) $\forall x, y \in \mathbb{R}$, $||x| - |y|| \leq |x - y|$

2.5 Decimal Representation and Uncountability of Reals

Typically, we think of $\mathbb{Q}$ in decimal representation. If $x \in \mathbb{Q}$, $x = d_K 10^K + d_{K-1} 10^{K-1} + \cdots + d_0 + d_{-1} 10^{-1} + \cdots + d_{-M} 10^{-M}$, where $\{d_j : -M \leq j \leq K\} \subset 0, 1, ..., 9$. And we write $x = d_K d_{K-1} \cdots .d_{-1} d_{-2} \cdots$

Example: $1 \cdot 10 + 1 \cdot 10^0 + 1 \cdot 10^{-1} = 11.1 = \frac{111}{10}$.

Definition: **2.8: Decimal Representation of Real Numbers**

Let $x \in (0, 1]$ and let $d_{-j} \in \{0, 1, ..., 9\}$ for $j \in \mathbb{N}$. We say x is represented by digits $\{d_{-j} : j \in \mathbb{N}\}$ and write $x = 0.d_{-1}d_{-2}\cdots$ if $x = \sup\{d_{-1}10^{-1} + \cdots + d_{-n}10^{-n} : n \in \mathbb{N}\}$

Example: $0.250 = \sup\left\{\frac{2}{10}, \frac{2}{10} + \frac{5}{100}, \frac{2}{10} + \frac{5}{100} + \frac{0}{1000}\right\} = \frac{1}{4}$.

Theorem: **2.15: Uniqueness of Decimal Representation**

1. For every set of digits $\{d_{-j} : j \in \mathbb{N}\}$ with $d_{-j} \in \{0, 1, ..., 9\}$, there exists a unique $x \in (0, 1]$ s.t. $x = 0.d_{-1}d_{-2}\cdots$ (Note: $\frac{1}{2} = 0.5 = 0.499\cdots$)
2. For every $x \in (0, 1]$, there exist unique digits $\{d_{-j} : j \in \mathbb{N}\}$ s.t. $x = 0.d_{-1}d_{-2}\cdots$ and $0.d_{-1}d_{-2}\cdots < x \le 0.d_{-1}d_{-2}\cdots + 10^{-n}$

Theorem: **2.16: Cantor's Theorem**

$(0, 1]$ is uncountable

Proof. We prove by contradiction.
Assume $|(0, 1]| = |\mathbb{N}|$, *i.e.* $(0, 1]$ is countable.
Then $\exists x : \mathbb{N} \to (0, 1]$ a bijection.

For each n, write $x(n) = 0.d_{-1}^{(n)}d_{-2}^{(n)}\cdots$

The idea is that we can write the $x(n)$ in a matrix form and look at the diagonal elements

- $x(1) = 0.\boxed{d_{-1}^1}d_{-2}^1d_{-3}^1d_{-4}^1\cdots$
- $x(2) = 0.d_{-1}^2\boxed{d_{-2}^2}d_{-3}^2d_{-4}^2\cdots$
- $x(3) = 0.d_{-1}^3d_{-2}^3\boxed{d_{-3}^3}d_{-4}^3\cdots$
- $x(4) = 0.d_{-1}^4d_{-2}^4d_{-3}^4\boxed{d_{-4}^4}\cdots$

  $\vdots$

Let $e_{-j} = \begin{cases} 1, \text{ if } d_{-j}^{(j)} \neq 1 \\ 2, \text{ if } d_{-j}^{(j)} = 1 \end{cases}$

By 1 of Theorem 2.15, there exists a unique $y \in (0, 1]$ s.t. $y = 0.e_{-1}e_{-2}\cdots$

Since all e_{-j}s are either 1 or 2, they are certainly non zero. Then $\forall n \in \mathbb{N}$, $0.e_{-1}e_{-2}\cdots e_{-n} < y \le 0.e_{-1}e_{-2}\cdots e_{-n} + 10^{-n}$

Thus $y = 0.e_{-1}e_{-2}\cdots e_{-n}$ is the unique decimal representation from 2 of Theorem 2.15.

Since x is surjective (as a bijection), $\exists m \in \mathbb{N}$ s.t. $y = x(m)$. Then $d_{-m}^{(m)} = e_{-m} = \begin{cases} 1, \text{ if } d_{-m}^{(m)} \neq 1 \\ 2, \text{ if } d_{-m}^{(m)} = 1 \end{cases} \neq d_{-m}^{(m)}$

This is a contradition. Thus $|\mathbb{N}| < |(0, 1]|$, $(0, 1]$ is uncountable. □

Corollary 1. $\mathbb{R}$ *is uncountable*

Proof. Find a bijection from $(0,1]$ to $\mathbb{R}$, e.g. tangent function and $|\mathbb{N}| < |(0,1]| \leq |\mathbb{R}|$ □

2.6 Complex Field

This is from UBC MATH320 and Rudin Principles of Mathematical Anlysis. The most important part of this subsection is the Cauchy-Schwarz Inequality.

***Definition:* 2.9: Complex Field**

The underlying set $\mathbb{C} = \{(a,b) : a, b \in \mathbb{R}\} = \mathbb{R}^2$. Let $x = (a,b)$, $y = (c,d)$. Define $0 = (0,0)$, $1 = (1,0)$. Define addition $(x+y) = (a+c, b+d)$ and multiplication $xy = (ac-bd, ad+bc)$. We write $x = a+bi$, $\text{Re}x = a$, $\text{Im}x = b$.

***Definition:* 2.10: Conjugate**

In $\mathbb{C}$, the conjugate of $x = a+bi$ is $\bar{x} = a - bi$, with the following properties:

1. $\overline{x+y} = \bar{x} + \bar{y}$
2. $\overline{xy} = \bar{x}\bar{y}$
3. $x + \bar{x} = 2\text{Re}x$
4. $x - \bar{x} = 2i\text{Im}x$
5. $x\bar{x} = a^2 + b^2$

***Definition:* 2.11: Norm of Complex Numbers**

Define $|x| = \sqrt{x\bar{x}}$ to be the norm of $x \in \mathbb{C}$

Remark 5. $\mathbb{R}$ is a subfield of $\mathbb{C}$

***Theorem:* 2.17: Properties of Norms in $\mathbb{C}$**

Let $x, y \in \mathbb{C}$, then

1. $|x| \geq 0$ with equality if and only if $x = 0$
2. $|\bar{x}| = |x|$
3. $|xy| = |x||y|$
4. $|\text{Re}x| \leq |x|$ and $|\text{Im}x| \leq |x|$
5. $|x+y| \leq |x| + |y|$

***Theorem:* 2.18: Cauchy-Schwarz Inequality**

Let $a_1, ..., a_n, b_1, ..., b_n \in \mathbb{C}$, then $|\sum_{j=1}^{n} a_j b_j|^2 \leq \sum_{j=1}^{n} |a_j|^2 \sum_{j=1}^{n} |b_j|^2$. Equality is true if and only if one of the following holds:

1. $\exists \alpha \in \mathbb{C}$ s.t. $a_j = \alpha b_j$, for all j.
2. $\exists \beta \in \mathbb{C}$ s.t. $b_j = \beta a_j$, for all j.

In vector form, $|\langle a, b\rangle|^2 \leq \langle a, a\rangle\langle b, b\rangle$ or $|\langle a, b\rangle| = \|a\|\|b\|$.

3 Sequences and Series

3.1 Sequences and Limits

Definition: 3.1: Sequence

A sequence of real numbers is a function $x : \mathbb{N} \to \mathbb{R}$. We denote $x(n)$ by x_n and the sequence by $\{x_n\}_{n=1}^{\infty}$ or $\{x_n\}$ or $x_1, x_2, ...$

Example:

- $1, 1, 1, ...$ is the sequence $x(n) = 1$ for all n.
- $\left\{\frac{1}{n}\right\}$ is the sequence $1, \frac{1}{2}, \frac{1}{3}, ...$
- $\{(-1)^n\}$ is the sequence $-1, 1, -1, 1, ...$

Definition: 3.2: Bounded Sequence

A sequence $\{x_n\}$ is bounded if $\exists B \geq 0$ s.t. $\forall n \in \mathbb{N}$, $|x_n| \leq B$. Otherwise, $\{x_n\}$ is unbounded.

Remark 6. A sequence $\{x_n\}$ is unbounded if $\forall B \geq 0$, $\exists n \in \mathbb{N}$ s.t. $|x_n| > B$

- $\{1\}$ and $\left\{\frac{1}{n}\right\}$ are bounded
- $\{(-1)^n\}$ is bounded by $B = 1$
- $\{n\}$ is unbounded.

Proof. Let $B \geq 0$, by Archimedian Property, $\exists n \in \mathbb{N}$ s.t. $n \geq B$. Thus $\{n\}$ is unbounded. □

Definition: 3.3: Convergent Sequence

A sequence $\{x_n\}$ converges to $x \in \mathbb{R}$ if $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $|x_n - x| < \epsilon$. If a sequence converges, we say it is convergent. Otherwise, it is divergent.

Definition: 3.4: Divergence

A sequence $\{x_n\}$ does not converge to $x \in \mathbb{R}$ if $\exists \epsilon_0 > 0$, s.t. $\forall M \in \mathbb{N}$, $\exists n \geq M$ s.t. $|x_n - x| \geq \epsilon_0$.

Theorem: 3.1:

Let $x, y \in \mathbb{R}$. If $\forall \epsilon > 0$, $|x - y| < \epsilon$, then $x = y$.

Proof. Assume $x \neq y$. Then $|x - y| > 0$.
Take $\epsilon = \frac{|x-y|}{2} > 0$, we get $|x - y| < \frac{|x-y|}{2}$
$\Rightarrow \frac{|x-y|}{2} < 0$. Contradiction. □

Theorem: 3.2: Uniqueness of Limit

If $\{x_n\}$ converges to x and y, then $x = y$.

Proof. Suppose $\{x_n\}$ converges to x and y.
Let $\epsilon > 0$. Since $\{x_n\}$ converges to x, $\exists M_1 \in \mathbb{N}$, s.t. $\forall n \geq M_1$, $|x_n - x| < \frac{\epsilon}{2}$
Similarly, since $\{x_n\}$ converges to y, $\exists M_2 \in \mathbb{N}$, s.t. $\forall n \geq M_2$, $|x_n - y| < \frac{\epsilon}{2}$
Take $M = \max M_1, M_2$. Then $\forall n \geq M$, $|x_n - x| < \frac{\epsilon}{2}$ and $|x_n - y| < \frac{\epsilon}{2}$
By triangle inequality, $|x - y| = |(x - x_n) + (x_n - y)| \leq |x_n - x| + |x_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. Thus $x = y$ by Theorem 3.1 □

Definition: 3.5: Limit

If $\{x_n\}$ converges to x, we call x the limit of $\{x_n\}$ and write $x = \lim_{n\to\infty} x_n$ or $x_n \to x$.

Example:

- $x_n = 1, \forall n, \lim_{n\to\infty} x_n = 1$.

  Proof. Let $\epsilon > 0$. Choose $M = 1$. Then if $n \geq M$, $|x_n - 1| = |1 - 1| = 0 < \epsilon$ □

- $\lim_{n\to\infty} \dfrac{1}{n} = 0$.

  Proof. Let $\epsilon > 0$. Choose $M \in \mathbb{N}$, s.t. $\frac{1}{M} < \epsilon$ (exists by Archimedian Property). Then if $n \geq M$, $|x_n - 1| = |1 - 1| = 0 < \epsilon$.
  Then if $n \geq M$, $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{M} < \epsilon$ □

General technique to prove $\lim_{n\to\infty} x_n = x$

Proof. Let $\epsilon > 0$. Choose $M \in \mathbb{N}$, s.t. M has some property relevant to ϵ.
Then if $n \geq M$, after some calculation, $|x_n - x| < \epsilon$
How to find M? We start with $|x_n - x| \leq \cdots \leq$ something involving $M < \epsilon$ □

Example: $\lim_{n\to\infty} \dfrac{1}{n^2 + 30n + 1} = 0$

(We want to find M s.t. $\frac{1}{n^2+30n+1} < \epsilon$. Note that $\frac{1}{n^2+30n+1} \leq \dfrac{1}{\underbrace{30}_{n^2+1>0}} \leq \frac{1}{n}$. If $\frac{1}{n} < \epsilon$, $\frac{1}{n^2+30n+1} < \epsilon$)

Proof. Let $\epsilon > 0$. Choose $M \in \mathbb{N}$ s.t. $\frac{1}{M} < \epsilon$. Then for all $n \geq M$, $\left|\frac{1}{n^2+30n+1} - 0\right| = \frac{1}{n^2+30n+1} \leq \frac{1}{30} \leq \frac{1}{n} \leq \frac{1}{M} < \epsilon$ □

Example: $\{(-1)^n\}$ does not converge.

Proof. Let $x \in \mathbb{R}$. We want to show $\{(-1)^n\}$ does not converge x.
Let $\epsilon_0 = 1$, $M \in \mathbb{N}$. Then $2 = |(-1)^M - (-1)^{M+1}| \leq |(-1)^M - x| + |(-1)^{M+1} - x|$.

Thus, either $|(-1)^M - x| \geq 1$ or $|(-1)^{M+1} - x| \geq 1$ □

Theorem: 3.3: Bounded Sequence

If $\{x_n\}$ is convergent, then $\{x_n\}$ is bounded.

Proof. Suppose $x_n \to x$. Then $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $|x_n - x| < 1$.
Then $\forall n \geq M$, $|x_n| = |x_n - x + x| \leq |x_n - x| + |x| \leq 1 + |x|$.
Define $B = |x_1| + |x_2| + \cdots + |x_{M-1}| + (1 + |x|)$ (or max of these values).
Then $\forall n \in \mathbb{N}$, $|x_n| \leq B$. □

Definition: 3.6: Monotone Sequnece

A sequence $\{x_n\}$ is monotone increasing if $\forall n \in \mathbb{N}$, $x_n \leq x_{n+1}$. A sequence is monotone decreasing if $\forall n \in \mathbb{N}$, $x_n \geq x_{n+1}$. If $\{x_n\}$ is monotone increasing or monotone decreasing, then $\{x_n\}$ is monotone or monotonic.

Theorem: 3.4: Convergent Monotone Sequence

A monotonic sequence is convergent if and only if it is bounded.

Proof. Convergent $\Rightarrow$ bounded is proved. We consider the other direction only.

Suppose $\{x_n\}$ is a monotone increasing sequence and bounded. Then $\{x_n : n \in \mathbb{R}\} \subset \mathbb{R}$ is bounded above and below.

Let $x = \sup\{x_n : n \in \mathbb{N}\}$. Claim $\lim_{n\to\infty} x_n = x$.
Let $\epsilon > 0$. Then since $x - \epsilon$ is not an upper bound for $\{x_n : n \in \mathbb{N}\}$, $\exists M_0 \in \mathbb{N}$ s.t. $x - \epsilon < x_{M_0} \leq x$.
Choose $M = M_0$. Then $\forall n \geq M$, $x - \epsilon < x_{M_0} \leq x_n \leq x \Rightarrow x - \epsilon < x_n < x + \epsilon$.

The proof for monotone decreasing sequence is similar. □

Remark 7. If $\{x_n\}$ is bounded and monotone increasing, then $\lim_{n\to\infty} x_n = \sup\{x_n : n \in \mathbb{N}\}$. If $\{x_n\}$ is bounded and monotone decreasing, then $\lim_{n\to\infty} x_n = \inf\{x_n : n \in \mathbb{N}\}$.

Theorem: 3.5: Geometric Sequence

1. If $c \in (0, 1)$, then $\lim_{n\to\infty} c^n = 0$
2. If $c > 1$, then $\{c^n\}$ is unbounded

Proof.

1. Claim $\forall n \in \mathbb{N}$, $0 < c^{n+1} < c^n$. We can prove this by induction
   Base case: since $0 < c < 1$, and $0 < c^2 < c$ (multiply by c).
   Inductive step: suppose $0 < c^{n+1} < c$. Then $0 < c^{n+2} < c^{n+1}$. Thus $\{c^n\}$ is monotone decreasing and bounded. $0 < |c^n| = c^n < c$. By Theorem 3.4, $\{c^n\}$ has a limit L.

   We now want to show $L = 0$.
   Let $\epsilon > 0$. Then $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $|c^n - L| < \frac{(1-c)\epsilon}{2}$, L is the limit of $\{c^n\}$ and $\frac{(1-c)\epsilon}{2} > 0$.
   Thus $(1 - c)|L| = |L - cL| = |L - c^{M+1} + c^{M+1} - cL| \leq |L - c^{M+1}| + c|c^M - L| < \frac{(1-c)\epsilon}{2} + \frac{c(1-c)\epsilon}{2} = \frac{(1+c)}{2}(1 - c)\epsilon$.
   Since $\frac{(1+c)}{2} < 1$ for $c \in (0, 1)$, $|L| < \epsilon$. Thus $|L| = 0 \Rightarrow L = 0$.

2. We want to show $\forall B \geq 0$, $\exists n \in \mathbb{N}$, $c^n > B$.
   Recall that $(1 + x)^n \geq 1 + nx$. Thus $c^n = (1 + (c - 1))^n \geq 1 + n(c - 1) \geq n(c - 1)$
   Let $n \in \mathbb{N}$ s.t. $n > \frac{B}{c-1}$. Then $c^n = (1 + (c - 1))^n \geq n(c - 1) > \frac{B}{c-1}(c - 1) = B$

□

Definition: 3.7: Subsequence

Let $\{x_n\}$ be a sequence and let $\{n_k\}$ be a sequence of natural numbers s.t. $n_1 < n_2 < n_3 < \cdots$ (strictly increasing). Then the sequence $\{x_{n_k}\}_{k=1}^{\infty}$ is called a subsequence of $\{x_n\}$.

- $x_n = 1, 2, 3, 4, 5, ...$
  - $x_{n_k} = 1, 3, 5, 7, ...$ is a subsequence with $n_k = 2k - 1$
  - $x_{n_k} = 2, 4, 6, 8, ...$ is a subsequence with $n_k = 2k$
  - $x_{n_k} = 2, 3, 5, 7, ...$ is a subsequence with $n_k = k^{th} prime$
  - $x_{n_k} = 1, 1, 1, 1, ...$ is not a subsequence, because $n_k = 1$ for all k
  - $x_{n_k} = 1, 1, 3, 3, 5, ...$ is not a subsequence
- $x_n = (-1)^n$
  - $-1, -1, -1, ...$ is a subsequence with $n_k = 2k - 1$
  - $1, 1, 1, ...$ is a subsequence with $n_k = 2k$

Theorem: 3.6: Convergent Subsequence

If $\{x_n\}$ converges to x, and $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$, then $\lim_{k\to\infty} x_{n_k} = x$

Proof. Since $1 \leq n_1 < n_2 < n_3 < \cdots$, then $\forall k \in \mathbb{N}$, $n_k \geq k$
Let $\epsilon > 0$. Since $x_n \to x$, $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $|x_n - x| < \epsilon$.
Choose $M = M_0$. If $k \geq M$, $n_k \geq k \geq M = M_0$. Then $|x_{n_k} - x| < \epsilon$ □

3.2 Facts about Limits

Theorem: 3.7:

$$\lim_{n\to\infty} x_n = x \Leftrightarrow \lim_{n\to\infty} |x_n - x| = 0$$

Proof. From definition and $|x_n - x| = ||x_n - x| - 0|$ □

Theorem: 3.8: Squeeze Theorem

Let $\{a_n\}$, $\{b_n\}$, and $\{x_n\}$ be sequences s.t. $\forall n \in \mathbb{N}$, $a_n \leq x_n \leq b_n$ and $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = x$. Then $\{x_n\}$ converges and $\lim_{n\to\infty} x_n = x$

Proof. Let $\epsilon > 0$. Since $b_n \to x$, $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $|b_n - x| < \epsilon \Rightarrow b_n < x + \epsilon$
Similarly, since $a_n \to x$, $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $|a_n - x| < \epsilon \Rightarrow x - \epsilon < a_n$
Take $M = \max(M_0, M_1)$, then $\forall n \geq M$, $x - \epsilon < a_n \leq x_n \leq b_n < x + \epsilon$. Thus $|x_n - x| < \epsilon$ □

Example: $\lim_{n\to\infty} \dfrac{n^2}{n^2 + n + 1} = 1$

Proof. $\left|\frac{n^2}{n^2+n+1}-1\right| = \left|\frac{n+1}{n^2+n+1}\right| \leq \frac{n+1}{n^2+n} = \frac{1}{n}$, so $0 \leq \left|\frac{n^2}{n^2+n+1}-1\right| \leq \frac{1}{n}$.

By Squeeze Theorem $\lim\limits_{n\to\infty}\left|\dfrac{n^2}{n^2+n+1}-1\right| = 0$ and by Theorem 3.7, $\lim\limits_{n\to\infty}\dfrac{n^2}{n^2+n+1} = 1$. □

How do limits interact with order in $\mathbb{R}$?

Theorem: 3.9: Order Property of Limits

1. If $\{x_n\}$ and $\{y_n\}$ are convergent sequences and $\forall n \in \mathbb{N}$, $x_n \leq y_n$, then $\lim\limits_{n\to\infty} x_n \leq \lim\limits_{n\to\infty} y_n$
2. If $\{x_n\}$ is a convergent sequence and $\forall n \in \mathbb{N}$, $a \leq x_n \leq b$, then $a \leq \lim\limits_{n\to\infty} x_n \leq b$

Remark 8. $\forall n$, $x_n < y_n \not\Rightarrow \lim\limits_{n\to\infty} x_n < \lim\limits_{n\to\infty} y_n$. We can have $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n$

Example: $x_n = 0$, $y_n = \frac{1}{n}$, $x_n < y_n$ for all n, and $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty} y_n = 0$

Proof.
1. Suppose $x_n \to x$ and $y_n \to y$, we want to show $x \leq y$.
   Assume $y < x$.
   Since $y_n \to y$, $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $|y_n - y| < \frac{x-y}{2}$, so $y_n < \frac{x+y}{2} \Rightarrow y_n < x - \frac{x-y}{2}$
   Similarly, since $x_n \to x$, $\exists M_1 \in \mathbb{N}$ s.t. $\forall n \geq M_1$, $|x_n - x| < \frac{x-y}{2}$, so $x - x_n < \frac{x-y}{2} \Rightarrow x - \frac{x-y}{2} < x_n$
   Let $n = M_0 + M_1$, $n \geq M_0$ and $n \geq M_1$, then $y_n < x - \frac{x-y}{2} < x_n$. Contradictions.
2. Follows 1

□

How does limits interact with algebraic operations?

Theorem: 3.10: Algebraic Operations of Limits

Suppose $\lim\limits_{n\to\infty} x_n = x$ and $\lim\limits_{n\to\infty} y_n = y$. Then

1. $\lim\limits_{n\to\infty}(x_n + y_n) = x + y$
2. $\forall c \in \mathbb{R}$, $\lim\limits_{n\to\infty} cx_n = cx$
3. $\lim\limits_{n\to\infty}(x_n y_n) = xy$
4. If $\forall n$, $y_n \neq 0$ and $y \neq 0$, then $\lim\limits_{n\to\infty}\dfrac{x_n}{y_n} = \dfrac{x}{y}$

Proof.
1. Let $\epsilon > 0$. Since $x_n \to x$, $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $|x_n - x| < \frac{\epsilon}{2}$.
   Similarly, since $y_n \to y$, $\exists M_1 \in \mathbb{N}$ s.t. $\forall n \geq M_1$, $|y_n - y| < \frac{\epsilon}{2}$.
   Choose $M = M_0 + M_1$, if $n \geq M$, $|(x_n + y_n) - (x + y)| \leq |x_n - x| + |y_n - y| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$
2. Let $\epsilon > 0$. Since $x_n \to x$, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $|x_n - x| < \frac{\epsilon}{|c|+1}$.
   Then $|cx_n - cx| = |c||x_n - x| < \frac{|c|}{|c|+1}\epsilon < \epsilon$
3. Since $y_n \to y$, $\{y_n\}$ is bounded. *i.e.* $\exists B \geq 0$ s.t. $\forall n \in \mathbb{N}$, $|y_n| \leq B$.
   Then $|x_n y_n - xy| = |x_n y_n - x y_n + x y_n - xy| = |(x_n - x)y_n + (y_n - y)x| \leq |x_n - x||y_n| + |y_n - y||x| \leq |x_n - x|B + |y_n - y||x|$, which converges to 0. By sequeeze theorem, $|x_n y_n - xy| \to 0$ and $x_n y_n \to xy$.

4. We prove that $\frac{1}{y_n} \to \frac{1}{y}$, and the result directly follows 3.
   Claim $\exists b > 0$ s.t. $\forall n \in \mathbb{N}$, $|y_n| \geq b$.
   Since $y_n \to y$ and $y \neq 0$, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $|y_n - y| < \frac{y}{2}$.
   Then $\forall n \geq M$, $|y| = |y - y_n + y_n| \leq |y - y_n| + |y_n| < \frac{|y|}{2} + |y_n|$. So $\frac{y}{2} < |y_n|$.
   Let $b = \inf\{|y_1|, ..., |y_{M-1}|, \frac{|y|}{2}\} > 0$ (infimum of a finite set always exists). Then $\forall n \in \mathbb{N}$, $|y_n| \geq b$.
   Then $0 \leq \left|\frac{1}{y_n} - \frac{1}{y}\right| = \frac{|y_n - y|}{|y_n||y|} \leq \frac{1}{b|y|}|y_n - y|$. By squeeze theorem, $\left|\frac{1}{y_n} - \frac{1}{y}\right| \to 0$, and thus, $\frac{1}{y_n} \to \frac{1}{y}$.

□

Theorem: 3.11: Limits of Square Roots

If $\forall n$, $x_n \geq 0$, and $\lim_{n\to\infty} x_n = x$, then $\lim_{n\to\infty} \sqrt{x_n} = \sqrt{x}$.

Proof. **Case 1:** $x = 0$. Let $\epsilon > 0$. Since $x_n \to 0$. $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $|x_n - 0| = |x_n| = x_n < \epsilon^2$. Choose $M = M_0$. Then $\forall n \geq M$, $|\sqrt{x_n} - 0| = |\sqrt{x_n}| = \sqrt{x_n} < \sqrt{\epsilon^2} = \epsilon$
Case 2: $x > 0$. Then $|\sqrt{x_n} - \sqrt{x}| = \left|(\sqrt{x_n} - \sqrt{x})\frac{\sqrt{x_n}+\sqrt{x}}{\sqrt{x_n}+\sqrt{x}}\right| = \frac{|x_n - x|}{\sqrt{x_n}+\sqrt{x}} < \frac{1}{\sqrt{x}}|x_n - x|$. Since $|x_n - x| \to 0$, by squeeze theorem $|\sqrt{x_n} - \sqrt{x}| \to 0$. □

Theorem: 3.12: Limits of Absolute Values

If $\{x_n\}$ is a convergent sequence and $\lim_{n\to\infty} x_n = x$. Then $\{|x_n|\}$ is convergent and $\lim_{n\to\infty} |x_n| = |x|$

Remark 9. The converse is not true. Consider $x_n = (-1)^n$. $|x_n| = 1$, and thus $|x_n| \to 1$. But $\{x_n\}$ does not converge.

Proof. This directly follows the reverse triangle inequality, $0 \leq ||x_n| - |x|| \leq |x_n - x|$. By squeeze theorem. $|x_n| \to |x|$. □

Theorem: 3.13: Binomial Theorem

$\forall n \in \mathbb{N}, x, y \in \mathbb{R}$, $(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$, where $\binom{n}{k} = \frac{n!}{k!(n-k)!}$.

Theorem: 3.14: Some Special Sequences

1. If $p > 0$, then $\lim_{n\to\infty} \frac{1}{n^p} = 0$
2. If $p > 0$, then $\lim_{n\to\infty} p^{\frac{1}{n}} = 1$
3. $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$

Proof. 1. Let $\epsilon > 0$. Choose $M \in \mathbb{N}$ s.t. $M > \frac{1}{\epsilon^{1/p}}$. Then if $n \geq M$, $\left|\frac{1}{n^p} - 0\right| = \frac{1}{n^p} \leq \frac{1}{M^p} < \epsilon$

2. Three cases:

   (a) $p = 1$: clear

   (b) $p > 1$: $|p^{1/n} - 1| = p^{1/n} - 1$, $p = [1 + (p^{1/n} - 1)]^n \geq 1 + n(p^{1/n} - 1)$ (since $(1+x)^n \geq 1 + nx$ for $x \geq -1$) $\Rightarrow 0 \leq p^{1/n} - 1 \leq \frac{p-1}{n}$. By squeeze theorem, $p^{1/n} - 1 \to 0$ and $p^{1/n} \to 1$

(c) $p < 1$: Take the reciprocal. $\lim_{n\to\infty} p^{1/n} = \lim_{n\to\infty} \frac{1}{\left(\frac{1}{p}\right)^{1/n}}$. $\frac{1}{p} > 1 \to \left(\frac{1}{p}\right)^{1/n} \to 1$

then $\lim_{n\to\infty} p^{1/n} = 1$

3. Let $x_n = n^{1/n} - 1 \geq 0$. We want to show $x_n \to 0$. Then $n = (1 + x_n)^n = \sum_{k=0}^{n} \binom{n}{k} x_n^k \geq \binom{n}{2} x_n^2 = \frac{n(n-1)}{2} x_n^2$

Then, $\forall n > 1$, $0 \leq x_n \leq \sqrt{\frac{2}{n-1}}$. By squeeze theorem, $x_n \to 0$, and $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$.

□

3.3 Limsup, Liminf, and Bolzano-Weierstrass

Does every bounded sequence have a convergent subsequence?

***Definition:* 3.8: limsup and liminf**

Let $\{x_n\}$ be a bounded sequence.

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} (\sup\{x_k : k \geq n\})$$
$$\liminf_{n\to\infty} x_n = \lim_{n\to\infty} (\inf\{x_k : k \geq n\})$$

***Theorem:* 3.15: Infimum and Supremum of Subsets in $\mathbb{R}$**

If $A, B \subset \mathbb{R}$, $A, B \neq \emptyset$, s.t. A, B are bounded and $A \subset B$, then $\inf B \leq \inf A \leq \sup A \leq \sup B$.

Proof. The middle inequality is obvious, we consider $\inf B \leq \inf A$ and $\sup A \leq \sup B$ only.
Since $\sup B$ is an upper bound for B, and $A \subset B$, then $\sup B$ is an upper bound for A. Thus $\sup A \leq \sup B$.
Similar argument goes for $\inf B \leq \inf A$ □

***Theorem:* 3.16: Existence of limsup and liminf**

Let $\{x_n\}$ be a bounded sequence, and let $a_n = \sup\{x_k : k \geq n\}$, $b_n = \inf\{x_k : k \geq n\}$. Then

1. $\{a_n\}$ is monotone decreasing and bounded. $\{b_n\}$ is monotone increasing and bounded. Thus $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.
2. $\liminf x_n \leq \limsup x_n$

Proof.

1. Since $\{x_k : k \geq n+1\} \subset \{x_k : k \geq n\}$, $a_{n+1} = \sup\{x_k : k \geq n+1\} \leq \sup\{x_k : k \geq n\} = a_n$ by Theorem 3.15, and similarly $b_{n+1} \geq b_n$
   Since $\{x_n\}$ is bounded, $\exists B \geq 0$, s.t. $\forall n \in \mathbb{N}$, $-B \leq x_n \leq B$. Then $-B \leq \inf\{x_k : k \geq n\} \leq \sup\{x_k : k \geq n\} \leq B$. Thus $-B \leq b_n \leq a_n \leq B$. And $|b_n| \leq B$, $|a_n| \leq B$. $\{a_n\}$ and $\{b_n\}$ are monotone and bounded. Thus $\lim_{n\to\infty} a_n$ and $\lim_{n\to\infty} b_n$ exist.
2. Since $-B \leq b_n \leq a_n \leq B$, we have $\liminf x_n = \lim_{n\to\infty} b_n \leq \lim_{n\to\infty} a_n = \limsup x_n$ by Theorem 3.9

□

Example: $x_n = (-1)^n$, $\{(-1)^k : k \geq n\} = \{-1, 1\}$.
$\sup\{(-1)^k : k \geq n\} = 1$, $\limsup x_n = 1$. $\inf\{(-1)^k : k \geq n\} = -1$, $\liminf x_n = -1$.

Example: $x_n = \frac{1}{n}$. $\left\{\frac{1}{k} : k \geq n\right\} = \left\{\frac{1}{n}, \frac{1}{n+1}, \frac{1}{n+2}, \cdots\right\}$.
$\sup\left\{\frac{1}{k} : k \geq n\right\} = \frac{1}{n}$, $\limsup x_n = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$. $\inf\left\{\frac{1}{k} : k \geq n\right\} = 0$, $\liminf x_n = \lim\limits_{n\to\infty} 0 = 0$

***Theorem:* 3.17: Convergent Subsequence to liminf and limsup**

Let $\{x_n\}$ be a bounded sequence. There exist subsequences $\{x_{n_k}\}$ and $\{x_{m_k}\}$ s.t. $\lim\limits_{k\to\infty} x_{n_k} = \limsup x_n$ and $\lim\limits_{k\to\infty} x_{m_k} = \liminf x_n$.

Proof. Let $a_n = \sup\{x_k : k \geq n\}$, $\exists n_1 \geq 1$ s.t. $a_1 - 1 < x_{n_1} \leq a_1$ by Theorem 2.10, since $a_1 = \sup\{x_k : k \geq 1\}$.
Since $a_{n_1+1} = \sup\{x_k : k \geq n_1 + 1\}$, $\exists n_2 \geq n_1 + 1 > n_1$ s.t. $a_{n_1+1} - \frac{1}{2} < x_{n_2} \leq a_{n_1+1}$.
Continuing in this manner, we obtain a sequence of natural numbers $n_1 < n_2 < n_3 < \cdots$ s.t. $\forall k \in \mathbb{N}$, $a_{n_{k-1}+1} - \frac{1}{k} < x_{n_k} \leq a_{n_{k-1}+1}$.
Since $n_1 < n_2 < n_3 < \cdots$, $n_1 + 1 < n_2 + 1 < n_3 + 1 < \cdots$. And $\{a_{n_{k-1}+1}\}$ is a subsequence of $\{a_n\}$.
$\{a_n\}$ converges and $\{a_{n_{k-1}+1}\}$ is a subsequence of $\{a_n\}$. Thus by Theorem 3.6, $\lim\limits_{k\to\infty} a_{n_{k-1}+1} = \lim\limits_{n\to\infty} a_n = \limsup x_n$. By squeeze theorem, $\lim\limits_{k\to\infty} x_{n_k} = \limsup x_n$.
Similarly for liminf, we can have $a_{n_{k-1}+1} \leq x_{n_k} < a_{n_{k-1}+1} + \frac{1}{k}$. □

***Theorem:* 3.18: Bolzano-Weierstrass**

Every bounded sequence has a convergent subsequence.

Proof. Directly follows Theorem 3.17, by choosing the subsequence to be $a_n = \sup\{x_k : k \geq n\}$. □

***Theorem:* 3.19:**

$\{x_n\}$ converges $\Leftrightarrow \limsup x_n = \liminf x_n$.
If $\{x_n\}$ converges, then $\limsup x_n = \lim\limits_{n\to\infty} x_n = \liminf x_n$

Proof. ($\Leftarrow$) Suppose $L = \limsup x_n = \liminf x_n$. Since $\forall n$, $\inf\{x_k : k \geq n\} \leq x_n \leq \sup\{x_k : k \geq n\}$. By squeeze theorem, $x_n \to L$.
($\Rightarrow$) Let $L = \lim\limits_{n\to\infty} x_n$. By Theorem 3.17, there exists subsequence $\{x_{n_k}\}$ s.t. $\lim\limits_{k\to\infty} x_{n_k} = \limsup x_n \Rightarrow L = \limsup x_n$.
Similarly, there exists subsequence $\{x_{m_k}\}$ s.t. $\lim\limits_{k\to\infty} x_{m_k} = \liminf x_n \Rightarrow L = \liminf x_n$. □

3.4 Cauchy Sequences

***Definition:* 3.9: Cauchy Sequence**

A sequence $\{x_n\}$ is Cauchy if $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ s.t. for all $n \geq M$ and $k \geq M$, $|x_n - x_k| < \epsilon$.

Remark 10. The negation: $\{x_n\}$ is not Cauchy if $\exists \epsilon_0 > 0$ s.t. $\forall M \in \mathbb{N}$, $\exists n \geq M$ and $k \geq M$ s.t. $|x_n - x_k| \geq \epsilon_0$.

Example: $x_n = \frac{1}{n}$ is Cauchy.

Proof. Let $\epsilon > 0$. Choose $M \in \mathbb{N}$ s.t. $\frac{1}{M} < \frac{\epsilon}{2}$.
Then if $n \geq M$, $k \geq M$, $\left|\frac{1}{n} - \frac{1}{k}\right| \leq \frac{1}{n} + \frac{1}{k} \leq \frac{2}{M} < \epsilon$. □

Example: $x_n = (-1)^n$ is not Cauchy.

Proof. Let $\epsilon_0 = 2$. Let $M \in \mathbb{N}$. Choose $n = M$ and $k = M+1$. Then $|(-1)^n - (-1)^k| = |1-(-1)| = 2$. □

***Theorem:* 3.20:**

If $\{x_n\}$ is Cauchy, then $\{x_n\}$ is bounded.

Proof. Since $\{x_n\}$ is Cauchy, $\exists M \in \mathbb{N}$ s.t. for all $n \geq M$ and $k \geq M$, $|x_n - x_k| < 1$.
Then $\forall n \geq M$, $|x_n - x_M| < 1$. $|x_n| = |x_n - x_M + x_M| \leq |x_n - x_M| + |x_M| < 1 + |x_M|$.
i.e. $\forall n \geq M$, $|x_n| \leq |x_M| + 1$.
Let $B = \max\{|x_1|, ..., |x_{M-1}|, |x_M| + 1\}$.
Then $\forall n \geq M$, $|x_n| \leq |x_M| + 1 \leq B$ and for $1 \leq n \leq M$, $|x_n| \leq B$. □

***Theorem:* 3.21:**

If $\{x_n\}$ is Cauchy and there exists subsequence $\{x_{n_k}\}$ converging to x, then $\{x_n\}$ converges to x.

Proof. Let $\epsilon > 0$. Since $\{x_n\}$ is Cauchy, $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$ and $m \geq M_0$, $|x_n - x_m| < \frac{\epsilon}{2}$.

Since there exists a subsequence $\{x_{n_k}\}$ convergint to x, $\exists M_1 \in \mathbb{N}$ s.t. $\forall k \geq M_1$, $|x_{n_k} - x| < \frac{\epsilon}{2}$.

Choose $M = M_0 + M_1$. Since $n_k \geq k$ for all $k \in \mathbb{N}$, then $n_M \geq M \geq M_0$ and $n_M \geq M_1$.

If $n \geq M$, $|x_n - x| = |x_n - x_{n_M} + x_{n_M} - x| \leq |x_n - x_{n_M}| + |x_{n_M} - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

***Theorem:* 3.22:**

$\{x_n\}$ is Cauchy $\Leftrightarrow$ $\{x_n\}$ is convergent.

Proof. ($\Rightarrow$) If $\{x_n\}$ is Cauchy, then by Theorem 3.20, $\{x_n\}$ is bounded. Then $\{x_n\}$ has a convergent subsequence by Theorem 3.18. Thus $\{x_n\}$ is convergent by Theorem 3.21.

($\Leftarrow$) If $\{x_n\}$ is convergent.
Let $\epsilon > 0$. Since $x_n \to x$, $\exists M_0 \in \mathbb{N}$, s.t. $\forall n \geq M_0$, $|x_n - x| < \frac{\epsilon}{2}$.
Choose $M = M_0$. Then if $n \geq M$ and $k \geq M$, $|x_n - x_m| \leq |x_n - x| + |x_m - x| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$ □

Remark 11. If we only work in rationals, then convergence $\Rightarrow$ Cauchy, but Cauchy $\not\Rightarrow$ convergence. The equivalence is satisified in a complete metric space and $\mathbb{R}$ is complete.

Example: Take $\{x_n\}$ s.t. $x_n \in \mathbb{Q}$ and $x_n \to \sqrt{2}$ in $\mathbb{R}$. $\{x_n\}$ is Cauchy, but does not converge in $\mathbb{Q}$.

3.5 Series

Definition: 3.10: Series

Given $\{x_n\}$, the symbol $\sum_{n=1}^{\infty} x_n$ or $\sum x_n$ is the series associated to $\{x_n\}$. We say $\sum x_n$ converges if the sequence $\left\{s_m = \sum_{n=1}^{m} x_n = x_1 + \cdots + x_m\right\}_{m=1}^{\infty}$ (Partial sums) converges.
If $s = \lim_{n\to\infty} s_m$, then we write $s = \sum x_n$ and treat $\sum x_n$ as a real number.

Remark 12. The series doesn't necessarily have to start at $n = 1$.

Example: $\sum_{n=0}^{\infty} \frac{1}{n(n+1)}$ converges.

Proof. $s_m = \sum_{n=0}^{\infty} \frac{1}{n(n+1)} = \sum_{n=0}^{\infty} \frac{1}{n} - \sum_{n=0}^{\infty} \frac{1}{n+1} = \left(1 + \frac{1}{2} + \cdots + \frac{1}{m}\right) - \left(\frac{1}{2} + \cdots + \frac{1}{m} + \frac{1}{m+1}\right) = 1 - \frac{1}{m+1}$.
Then $\lim_{n\to\infty} s_m = 1$. □

Example: $\sum_{n=0}^{\infty} (-1)^n$ does not converge.

Proof. $s_m = (-1) + 1 + \cdots + (-1)^m = \begin{cases} -1, \text{ if } m \text{ is odd} \\ 0, \text{ if } m \text{ is even} \end{cases}$ Thus s_m does not converge. □

Theorem: 3.23: Geometric Series

If $|r| < 1$, then $\sum_{n=0}^{\infty} r^n$ converges and $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$

Proof. $s_m = \sum_{n=0}^{m} r^n = \frac{1 - r^{m+1}}{1-r}$. If $|r| < 1$, then $\lim_{n\to\infty} r^m = 0$ and $\lim_{n\to\infty} s_m = \frac{1}{1-r}$ □

Theorem: 3.24:

Let $\{x_n\}$ be a sequence and let $m \in \mathbb{N}$. Then $\sum_{n=1}^{\infty} x_n$ converges $\Leftrightarrow$ $\sum_{n=M}^{\infty} x_n$ converges.

Proof. The partial sums satisfy that $\forall m \in \mathbb{N}$, $\sum_{n=1}^{m} x_n = \sum_{n=M}^{m} x_n + \sum_{n=1}^{M-1} x_n$. Since $\sum_{n=1}^{M-1} x_n$ is a finite number. $\sum_{n=1}^{m} x_n$ converges if and only if $\sum_{n=M}^{m} x_n$ converges □

Definition: 3.11: Cauchy Series

$\sum x_n$ is Cauchy if $\{s_m = \sum_{n=1}^{m} x_n\}_{m=1}^{\infty}$ is Cauchy.

Theorem: 3.25:

$\sum x_n$ is Cauchy $\Leftrightarrow \sum x_n$ is convergent.

Proof. Directly follows Theorem 3.22. □

Theorem: 3.26: Partial Sum of Cauchy Series

$\sum x_n$ is Cauchy $\Leftrightarrow \forall \epsilon > 0, \exists M \in \mathbb{N}$ s.t. $\forall l > m \geq M, \left| \sum_{n=m+1}^{l} x_n \right| < \epsilon.$

Proof. ($\Rightarrow$) Suppose $\sum x_n$ is Cauchy. Then $\{s_m\}$ is Cauchy. Let $\epsilon > 0$, $\exists M_0 \in \mathbb{N}$, s.t. $\forall m \geq M_0$ and $l \geq M_0$, $|s_m - s_l| < \epsilon$.

Choose $M = M_0$. If $l > m \geq M$. $\left| \sum_{n=m+1}^{l} x_n \right| = |s_l - s_m| < \epsilon$.

($\Leftarrow$) Follows directly from $\left| \sum_{n=m+1}^{l} x_n \right| = |s_l - s_m| < \epsilon$. □

Theorem: 3.27:

If $\sum x_n$ converges, then $\lim_{n\to\infty} x_n = 0$.
If $\lim_{n\to\infty} x_n \neq 0$, then $\sum x_n$ does not converge.

Proof. Since $\sum x_n$ converges, then $\sum x_n$ is Cauchy by Theorem 3.25.

Then by Theorem 3.26. Let $\epsilon > 0$, $\exists M_0 \in \mathbb{N}$, s.t. $\forall l > m \geq M_0$, $\left| \sum_{n=m+1}^{l} x_n \right| < \epsilon$.

Choose $M = M_0 + 1$. Then if $m \geq M$, $|x_m| = \left| \sum_{n=m}^{m} x_n \right| < \epsilon$ □

Theorem: 3.28:

If $|r| > 1$, then $\sum_{n=0}^{\infty} r^n$ does not converge.

Proof. If $|r| > 1$, then $\lim_{n\to\infty} r^n$ does not exist, $\sum r^n$ does not converge. □

Remark 13. $\lim_{n\to\infty} x_n = 0 \not\Rightarrow \sum x_n$ converges.

Theorem: 3.29: Harmonic Series

The series $\displaystyle\sum_{n=0}^{\infty} \frac{1}{n}$ does not converge.

Proof. We show that there exists a subsequence of partial sums $s_{m_k} = \displaystyle\sum_{n=1}^{m_k} \frac{1}{n}$ that is unbounded.

Let $l \in \mathbb{N}$. Consider $s_{2^l} = \displaystyle\sum_{n=1}^{2^l} \frac{1}{n}$.

$$
\begin{aligned}
s_{2^l} &= 1 + \frac{1}{2} + \left(\frac{1}{3} + \frac{1}{4}\right) + \left(\frac{1}{5} + \cdots + \frac{1}{8}\right) + \cdots + \left(\frac{1}{2^{l-1}+1} + \cdots + \frac{1}{2^l}\right) \\
&= 1 + \sum_{\lambda=1}^{l} \sum_{n=2^{\lambda-1}+1}^{2^\lambda} \frac{1}{n} \\
&\geq 1 + \sum_{\lambda=1}^{l} \sum_{n=2^{\lambda-1}+1}^{2^\lambda} \frac{1}{2^\lambda} \\
&= 1 + \sum_{lambda=1}^{l} \frac{1}{2} = 1 + \frac{l}{2}
\end{aligned}
$$

As $l \to \infty$, $\{s_{2^l}\}$ is unbounded. Thus $\sum \frac{1}{n}$ does not converge. ☐

Theorem: 3.30: Algebraic Operations on Series

Let $\alpha \in \mathbb{R}$. $\sum x_n$ and $\sum y_n$ are convergent series. Then $\sum(\alpha x_n + y_n)$ converges and $\sum(\alpha x_n + y_n) = \alpha \sum x_n + \sum y_n$.

Proof. The partial sums satisfy $\displaystyle\sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \sum_{n=1}^{m} x_n + \sum_{n=1}^{m} y_n$ by linearity.

By Theorem 3.10, $\displaystyle\lim_{m\to\infty} \sum_{n=1}^{m}(\alpha x_n + y_n) = \alpha \lim_{m\to\infty} \sum_{n=1}^{m} x_n + \lim_{m\to\infty} \sum_{n=1}^{m} y_n = \alpha \sum x_n + \sum y_n$. ☐

Theorem: 3.31:

if $\forall n \in \mathbb{N}$, $x_n \geq 0$, then $\displaystyle\sum_{n=1}^{\infty} x_n$ converges $\Leftrightarrow$ $\{s_m = \sum_{n=1}^{m} x_n\}_{m=1}^{\infty}$ is bounded.

Proof. We have $\forall m \in \mathbb{N}$, $s_{m+1} = \displaystyle\sum_{n=1}^{m+1} x_n = \sum_{n=1}^{m} x_n + x_{m+1} = s_m + x_{m+1} \geq s_m$, since $x_{m+1} \geq 0$.
Thus $\{s_m\}$ is monotone increasing. By Theorem 3.4, $\{s_m\}$ converges $\Leftrightarrow$ $\{s_m\}$ bounded. ☐

Definition: 3.12: Absolute Convergence

$\sum x_n$ converges absolutely if $\sum |x_n|$ converges.

Theorem: 3.32: Series Triangle Inequality

If $m \geq 2$, and $x_1, ..., x_m \in \mathbb{R}$, then $\left|\sum_{n=1}^{m} x_m\right| \leq \sum_{n=1}^{m} |x_n|$. (When $m = 2$, this gives the usual triangle inequality $|x_1 + x_2| \leq |x_1| + |x_2|$)

Proof. Base case: $m = 2$. Triangle inequality.
Inductive step: Suppose it holds for $m = l$.

$$\left|\sum_{n=1}^{l+1} x_n\right| = \left|\sum_{n=1}^{l} x_n + x_{l+1}\right| \leq \left|\sum_{n=1}^{l} x_n\right| |x_{l+1}| \leq \sum_{n=1}^{l} |x_n| + |x_{l+1}| = \sum_{n=1}^{l+1} |x_n|.$$ □

Theorem: 3.33:

If $\sum x_n$ converges absolutely, then $\sum x_n$ converges.

Proof. We will show that $\sum x_n$ is Cauchy. *i.e.* $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ s.t. $\forall l > m \geq M$, $\left|\sum_{n=m+1}^{l} x_n\right| < \epsilon$.

Let $\epsilon > 0$. Since $\sum |x_n|$ converges, then $\sum |x_n|$ is Cauchy.

Thus $\exists M_0 \in \mathbb{N}$ s.t. $\forall l > m \geq M$, $\left|\sum_{n=m+1}^{l} |x_n|\right| = \sum_{n=m+1}^{l} |x_n| < \epsilon$.

Choose $M = M_0$, if $l > m \geq M$, then $\left|\sum_{n=m+1}^{l} x_n\right| < \sum_{n=m+1}^{l} |x_n| < \epsilon$ by Theorem 3.32. □

Remark 14. $\sum x_n$ converges $\not\Rightarrow$ $\sum x_n$ converges absolutely.

Example: $\sum \frac{(-1)^n}{n}$ converges, but $\sum \frac{1}{n}$ does not converge.

3.5.1 Convergence Tests

Theorem: 3.34: Comparison Test

Suppose $\forall n \in \mathbb{N}$, $0 \leq x_n \leq y_n$. Then

1. If $\sum y_n$ converges, then $\sum x_n$ converges.
2. If $\sum x_n$ diverges, then $\sum y_n$ diverges.

Proof. 1. If $\sum y_n$ converges, then $\left\{\sum_{n=1}^{m} y_n\right\}_{m=1}^{\infty}$ is bounded.

Therefore, $\exists B \geq 0$ s.t. $\exists m_0 \in \mathbb{N}$, $\sum_{n=1}^{m_0} y_n \leq B$ (We can drop the absolute value here because $y_n \geq 0$).

Then $\forall m \in \mathbb{N}$, $\sum_{n=1}^{m} x_n \leq \sum_{n=1}^{m} y_n \leq B$ by Theorem 3.9. $\left\{\sum_{n=1}^{m} x_n\right\}_{m=1}^{\infty}$ is bounded and thus $\sum x_n$ converges.

2. If $\sum x_n$ diverges, then $\left\{\sum_{n=1}^{m} x_n\right\}_{m=1}^{\infty}$ is unbounded. $\forall B \geq 0$, $\exists m_0 \in \mathbb{N}$ s.t. $\sum_{n=1}^{m_0} x_n > B$.

   Since $y_n \geq x_n \geq 0$, choose $m = m_0$, $\sum_{n=1}^{m_0} y_n \geq \sum_{n=1}^{m_0} x_n > B$. $\left\{\sum_{n=1}^{m} y_n\right\}_{m=1}^{\infty}$ is unbounded and thus $\sum y_n$ diverges.

□

Theorem: 3.35: p-series

For $p \in \mathbb{R}$, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges $\Leftrightarrow p > 1$.

Proof. ($\Rightarrow$) Suppose $\sum \frac{1}{n^p}$ converges.
Assume $p \leq 1$. Then $\frac{1}{n^p} \geq \frac{1}{n}$ for all n. Since $\sum \frac{1}{n}$ diverges. $\sum \frac{1}{n^p}$ diverges by comparison test.

($\Leftarrow$) Suppose $p > 1$.

$$\begin{aligned}
\text{We have } s_{2^k} &= 1 + \frac{1}{2^p} + \left(\frac{1}{3^p} + \frac{1}{4^p}\right) + \cdots + \left(\frac{1}{(2^{k-1}+1)^p} + \frac{1}{2^{kp}}\right) \\
&= 1 + \sum_{l=1}^{k} \sum_{n=2^{l-1}+1}^{2^l} \frac{1}{n^p} \\
&\leq 1 + \sum_{l=1}^{k} \sum_{n=2^{l-1}+1}^{2^l} \frac{1}{(2^{l-1}+1)^p} \\
&\leq 1 + \sum_{l=1}^{k} \sum_{n=2^{l-1}+1}^{2^l} \frac{1}{2^{p(l-1)}} = 1 + \sum_{l=1}^{k} 2^{-(p-1)(l-1)} \\
&= 1 + \frac{1}{1 - 2^{-(p-1)}}
\end{aligned}$$

Let $m \in \mathbb{N}$. Since $2^m > m$, $s_m \leq s_{2^m} \leq 1 + \frac{1}{1-2^{-(p-1)}}$.

Thus $\{s_m\}$ is bounded and $\sum \frac{1}{n^p}$ converges. □

Example: $\sum_{n=1}^{\infty} \frac{1}{n^2 + 2020n}$ converges.

Proof. $\frac{1}{n^2+2020n} \leq \frac{1}{n^2}$. Since $\sum \frac{1}{n^2}$ converges, $\sum \frac{1}{n^2+2020n}$ converges by comparison test. □

Theorem: 3.36: Ratio Test

Suppose $x_n \neq 0 \ \forall n \in \mathbb{N}$ and $L = \lim_{n\to\infty} \frac{|x_{n+1}|}{|x_n|}$ exists. Then

1. If $L > 1$, $\sum x_n$ diverges
2. If $L < 1$, $\sum x_n$ converges absolutely

Remark 15. No info for $L = 1$. $x_n = 1$, $\forall n \in \mathbb{N}$, $L = 1$, but $\sum 1$ diverges;
$x_n = \frac{1}{n^2}$. $L = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} = \lim_{n\to\infty} \frac{1}{(1+1/n)^2} = 1$, but $\sum x_n$ converges.

Proof.

1. Suppose $L > 1$. Since $\frac{|x_{n+1}|}{|x_n|} \to L$, then $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $\frac{|x_{n+1}|}{|x_n|} \geq 1$. Thus $\forall n \geq M_0$, $|x_{n+1}| \geq |x_n|$
   Then we have $|x_{M_0}| \leq |x_{M_0+1}| \leq |x_{M_0+2}| \leq \cdots$. Thus $|x_n| \not\to 0$ as $n \to \infty$. Otherwise $x_n = 0$, $\forall n$.
2. Suppose $L < 1$. Let $L < \alpha < 1$. Then since $\frac{|x_{n+1}|}{|x_n|} \to L < \alpha$. $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $\frac{|x_{n+1}|}{|x_n|} \leq \alpha$. Thus $\forall n \geq M_0$, $|x_{n+1}| \leq \alpha |x_n|$
   Then we have $\forall n \geq M_0 + 1$ $|x_n| \leq \alpha |x_{n-1}| \leq \alpha^2 |x_{n-2}| \leq \cdots \leq \alpha^{n-M_0} |x_{M_0}|$.
   Let $m \in \mathbb{N}$, $m \geq M_0$. Then

$$\begin{aligned}
\sum_{n=1}^{m} |x_n| &= \sum_{n=1}^{M_0} |x_n| + \sum_{n=M_0+1}^{m} |x_n| \\
&\leq \sum_{n=1}^{M_0} |x_n| + |x_{M_0+1}| \sum_{n=M_0+1}^{m} \alpha^{l-(M_0+1)} \\
&= \sum_{n=1}^{M_0} |x_n| + |x_{M_0+1}| \sum_{n=0}^{m-(M_0+1)} \alpha^n \\
&\leq \sum_{n=1}^{M_0} |x_n| + |x_{M_0+1}| \frac{1}{1-\alpha}
\end{aligned}$$

   Thus $\sum |x_n|$ converges.

□

Example: $\forall x \in \mathbb{R}$, $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges absolutely.

Proof. $L = \lim_{n\to\infty} \frac{|x^{n+1}|}{(n+1)!} \frac{n!}{|x|^n} = \lim_{n\to\infty} \frac{|x|}{n+1} = 0 < 1$. Thus converges asbolutely by ratio test. □

Theorem: 3.37: Root Test

Let $\sum x_n$ be a series and suppose $L = \lim_{n\to\infty} |x_n|^{\frac{1}{n}}$ exists. Then

1. If $L > 1$, $\sum x_n$ diverges
2. If $L < 1$, $\sum x_n$ converges absolutely

Proof.

1. Suppose $L > 1$. Since $|x_n|^{\frac{1}{n}} \to L > 1$, $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $|x_n|^{\frac{1}{n}} > 1$. Then $\forall n \geq M_0$, $|x_n| > 1$, $x_n \not\to 0$. $\sum x_n$ diverges.

2. Suppose $L < 1$. Let $L < \alpha < 1$. Since $|x_n|^{\frac{1}{n}} \to L < \alpha$. $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $|x_n|^{\frac{1}{n}} < \alpha$. Thus $\forall n \geq M_0$, $|x_n| < \alpha^n$.
   Then $\forall m \in \mathbb{N}$,

$$\begin{aligned}\sum_{n=1}^{m} |x_n| &= \sum_{n=1}^{M_0} |x_n| + \sum_{n=M_0+1}^{m} \alpha^n \\ &\leq \sum_{n=1}^{M_0} |x_n| + \frac{1}{1-\alpha}\end{aligned}$$

□

Theorem: 3.38: Alternating Series

Let $\{x_n\}$ be a monotone decreasing sequence converging to 0 (therefore $\forall n \in \mathbb{N}$, $x_n \geq 0$). Then $\sum_{n=1}^{\infty}(-1)^n x_n$ converges.

Proof. We firstly show that the subsequence $\{s_{2k}\}_{k=1}^{\infty}$ converges.
For $k \in \mathbb{N}$, $s_{2k} = \sum_{n=1}^{2k}(-1)^n x_n = -x_1 + x_2 - x_3 + \cdots + x_{2k} = (x_2 - x_1) + (x_4 - x_3) + \cdots + (x_{2k} - x_{2k-1}) \geq (x_2 - x_1) + (x_4 - x_3) + \cdots + (x_{2k} - x_{2k-1}) + (x_{2k+2} - x_{2k+1})$ (Note: $x_{2k+2} - x_{2k+1} \leq 0$)
Then $s_{2k} \leq s_{2(k+1)}$. Thus $\{s_{2k}\}$ is monotone decreasing.
Also $\forall k \in \mathbb{N}$, $s_{2k} = -x_1 + (x_2 - x_3) + \cdots + (x_{2k-2} - x_{2k-1}) + x_{2k} \geq -x_1 + x_{2k} \geq -x_1$ because $x_{2k-2} \geq x_{2k-1}$.
Thus $\forall k \in \mathbb{N}$, $-x_1 \leq s_{2k} \leq s_2$, $\{s_{2k}\}$ is bounded and monotone. By Theorem 3.4, $\{s_{2k}\}$ converges.

Let $L = \lim_{k\to\infty} s_{2k}$.

We show that $\lim_{m\to\infty} s_m = L$.
Let $\epsilon > 0$. Since $s_{2k} \to L$, $\exists M_0 \in \mathbb{N}$ s.t. $\forall k \geq M_0$, $|s_{2k} - L| < \frac{\epsilon}{2}$.
Since $x_n \to 0$, $\exists M_1 \in \mathbb{N}$ s.t. $\forall n \geq M_1$, $|x_n| < \frac{\epsilon}{2}$.
Choose $M = \max\{2M_0 + 1, M_1\}$. Let $m \geq M$.
If m is even, then $\frac{m}{2} \geq \frac{M}{2} \geq M_0$, then $|s_m - L| = |s_{2\frac{m}{2}} - L| < \frac{\epsilon}{2} < \epsilon$ by the convergence of $\{s_{2k}\}$.
If m is odd. Let $k = \frac{m-1}{2} \in \mathbb{N}$. $m = 2k+1$. Then $m \geq M$ means that $2k + 1 \geq 2M_0 + 1$. Thus $k \geq M_0$.
Also $m \geq M_1$. Then $|s_m - L| = |s_{m-1} + (-1)^m x_m - L| = |s_{2k} - x_m - L| \leq |s_{2k} - L| + |x_m| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
Then if $m \geq M$, $|s_m - L| < \epsilon$ □

4 Continuous Functions

4.1 Limits of Functions

Definition: 4.1: Cluster Point

Let $S \subset \mathbb{R}$, $x \in \mathbb{R}$. We say x is a cluster point of S if $\forall \delta > 0$, $(x-\delta, x+\delta) \cap S \setminus \{x\} \neq \emptyset$ or $\forall \delta > 0$, $\exists y \in S$ s.t. $0 < |x-y| < \delta$.

Remark 16. The negation: x is not a cluster point of S if $\forall \delta > 0$, $(x-\delta, x+\delta) \cap S \setminus \{x\} \neq \emptyset$.

Examples:

1. $S = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. 0 is a cluster point of S. Let $\delta > 0$. Choose $n \in \mathbb{N}$ s.t. $0 < \frac{1}{n} < \delta$. Then $\frac{1}{n} \in (-\delta, \delta) \cap S \setminus \{0\} \neq \emptyset$.
2. $S = (0, 1)$. The set of cluster points is $[0, 1]$.
3. $S = \mathbb{Q}$. The set of cluster points is $\mathbb{R}$.
4. $S = \{0\}$ has no cluster points. If $x > 0$, then x is not a cluster point of S. Choose $\delta_0 = \frac{x}{2}$. Then $(x-\delta_0, x+\delta_0) = (\frac{x}{2}, \frac{3x}{2})$. $0 \notin (\frac{x}{2}, \frac{3x}{2})$ and $(x-\delta_0, x+\delta_0) \cap S \setminus \{x\} = \emptyset$.
5. In general, any finite set has no cluster point.
6. $S = \mathbb{Z}$ has no cluster point.

Theorem: 4.1: Cluster Points and Sequences

Let $S \subset \mathbb{R}$. Then x is a cluster point of $S \Leftrightarrow \exists$ sequence $\{x_n\}$ of lements of $S \setminus \{x\}$ s.t. $x_n \to x$.

Definition: 4.2: Limit of Functions

Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Let $f : S \to \mathbb{R}$, we say $f(x)$ converges to L as $x \to c$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $x \in S$ and $0 < |x-c| < \delta \Rightarrow |f(x) - L| < \epsilon$.

i.e. If x is near c but not at c, then $f(x)$ is near L.
Notation: $f(x) \to L$ as $x \to c$ or $\lim\limits_{x \to c} f(x) = L$.

Theorem: 4.2: Uniqueness of Limit of Functions

Let c be a cluster point of $S \subset \mathbb{R}$ and let $f : S \to \mathbb{R}$. If $f(x) \to L_1$ and $f(x) \to L_2$ as $x \to c$, then $L_1 = L_2$.

Proof. We want to show that $\forall \epsilon > 0$, $|L_1 - L_2| < \epsilon$.

Let $\epsilon > 0$. Since $f(x) \to L_1$ as $x \to c$, then $\exists \delta_1 > 0$ s.t. $0 < |x-c| < \delta_1 \Rightarrow |f(x) - L_1| < \frac{\epsilon}{2}$.
Similarly, since $f(x) \to L_2$ as $x \to c$, then $\exists \delta_2 > 0$ s.t. $0 < |x-c| < \delta_2 \Rightarrow |f(x) - L_2| < \frac{\epsilon}{2}$.
Take $\delta = \min\{\delta_1, \delta_2\}$. Since c is a cluster point of S, $\exists x \in S$ s.t. $0 < |x-c| < \delta$. Then $|L_1 - L_2| = |L_1 - f(x) + f(x) - L_2| \leq |f(x) - L_1| + |f(x) - L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. □

Example: $\lim\limits_{x \to c}(ax + b) = ac + b$

Proof. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{|a|}$. If $0 < |x-c| < \delta$, then $|f(x) - (ac+b)| = |(ax+b) - (ac+b)| = |a(x-c)| = |a||x-c| < |a|\frac{\epsilon}{|a|} = \epsilon$. □

Example: $\lim_{x \to c} \sqrt{x} = \sqrt{c}$

Proof. Let $\epsilon > 0$. Choose $\delta = \epsilon\sqrt{c}$. If $0 < |x-c| < \delta$, then $|f(x) - \sqrt{c}| = |\sqrt{x} - \sqrt{c}| = \left|(\sqrt{x} - \sqrt{c})\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}}\right| = \left|\frac{x-c}{\sqrt{x}+\sqrt{c}}\right| < \frac{\delta}{\sqrt{x}+\sqrt{c}} \leq \frac{\delta}{\sqrt{c}} = \frac{\epsilon\sqrt{c}}{\sqrt{c}} = \epsilon$ □

Example: $f(x) = \begin{cases} 1, x = 0 \\ 2, x \neq 0 \end{cases}$. $\lim_{x \to 0} f(x) = 2 \neq f(0)$

Proof. Let $\epsilon > 0$. Choose $\delta = 1$. Then $0 < |x| < \delta \Rightarrow x \neq 0$, $|f(x) - 2| = |2 - 2| = 0 < \epsilon$. □

***Theorem:* 4.3:**

Let $S \subset \mathbb{R}$, c a cluster point of S and let $f : S \to \mathbb{R}$. Then $\lim_{x \to c} f(x) = L \Leftrightarrow \forall$ sequnce $\{x_n\}$ in $S \setminus \{c\}$ s.t. $x_n \to c$. We have $f(x_n) \to L$.

Proof. $(\Rightarrow)$ Suppose $\lim_{x \to c} f(x) = L$. Let $\{x_n\}$ be a sequence of elements in $S \setminus \{c\}$ s.t. $x_n \to c$ as $n \to \infty$. We want to show that $f(x_n) \to L$ as $n \to \infty$.
Let $\epsilon > 0$. Since $f(x) \to L$, $\exists \delta > 0$ s.t. if $0 < |x - c| < \delta$, then $|f(x) - L| < \epsilon$. Since $x_n \to c$, then $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $0 < |x_n - c| < \delta \Rightarrow |f(x_n) - L| < \epsilon$.

$(\Leftarrow)$ Suppose $\forall$ sequence $\{x_n\}$ in $S \setminus \{c\}$ s.t. $x_n \to c$, we have $f(x_n) \to L$.
Assume $\lim_{x \to c} f(x) \neq L$.
Then $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x$ s.t. $0 < |x - c| < \delta$ and $|f(x) - L| \geq \epsilon_0$.
Since $x_n \to c$ $\forall n \in \mathbb{N}$, $\exists x_n$ s.t. $0 < |x_n - c| < \frac{1}{n}$ and $|f(x_n) - L| \geq \epsilon_0$.
However, by definition $f(x_n) \to L$, we get $0 = \lim_{n \to \infty} |f(x_n) - L| \geq \epsilon_0 > 0$. Contradiction. □

Remark 17. The negation: $\lim_{x \to c} f(x) \neq L \Leftrightarrow \forall$ sequnce $\{x_n\}$ in $S \setminus \{c\}$ s.t. $x_n \to c$. We have either $\lim_{n \to \infty} f(x_n) \neq L$ or $\lim_{n \to \infty} f(x_n)$ does not exist.

Example: $\forall c \in \mathbb{R}$, $\lim_{x \to c} x^2 = c^2$.

Proof. Let $\{x_n\}$ be a sequence s.t. $x_n \to c$ as $n \to \infty$. By product rule in Theorem 3.10, $x_n^2 \to c^2$. Thus by Theorem 4.3, $\lim_{x \to c} x^2 = c^2$. □

Example:

1. $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$
2. $\lim_{x \to 0} \sin \dfrac{1}{x}$ does not exist

Proof. 1. Suppose $x_n \to 0$, we want to show $x_n \sin \frac{1}{x_n} \to 0$. $0 \leq |x_n \sin \frac{1}{x_n}| \leq |x_n||\sin \frac{1}{x_n}| \leq |x_n|$, since $|\sin \frac{1}{x_n}| \leq 1$. By squeeze theorem, $x_n \sin \frac{1}{x_n} \to 0$. Thus by Theorem 4.3, $\lim_{x \to 0} x \sin \dfrac{1}{x} = 0$.

2. We want to show that $\exists x_n \to 0$ s.t. $\lim_{n\to\infty} \sin \dfrac{1}{x_n}$ DNE.

   Let $x_n = \frac{2}{(2n-1)\pi}$. Note for all n, $|x_n| \leq \frac{2}{[n+(n-1)]\pi} \leq \frac{2}{n\pi}$. Thus $x_n \to 0$, but $\{\sin \frac{1}{x_n}\} = \{\sin \frac{(2n-1)\pi}{2}\} = \{(-1)^{n+1}\}$. The sequence does not converge. Thus $\lim_{n\to\infty} \sin \dfrac{1}{x_n}$ DNE.

$\square$

Theorem: 4.4: Order Property of Limits of Functions

Let $S \subset \mathbb{R}$, c a cluster point of S and suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$. If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist and $\forall x \in S$, $f(x) \leq g(x)$, then $\lim_{x\to c} f(x) \leq \lim_{x\to c} g(x)$.

Proof. Let $L1 = \lim_{x\to c} f(x), L_2 = \lim_{x\to c} g(x)$. Let $\{x_n\}$ be a sequence in $S \setminus \{c\}$ s.t. $x_n \to c$.
Then $\forall n \in \mathbb{N}$, $f(x_n) \leq g(x_n)$. Since $f(x_n) \to L_1$ and $g(x_n) \to L_2$ by Theorem 4.3, then $L_1 = \lim_{x\to c} f(x) \leq \lim_{x\to c} g(x) = L_2$ by Theorem 3.9. $\square$

We also have the following analogs for limits of functions:

Theorem: 4.5: Squeeze Theorem for Limits of Functions

Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$, $g : S \to \mathbb{R}$ and $h : S \to \mathbb{R}$ s.t. $f(x) \leq g(x) \leq h(x)$ for all $x \in S \setminus \{c\}$. If $\lim_{x\to c} f(x) = \lim_{x\to c} h(x)$, then $\lim_{x\to c} g(x) = \lim_{x\to c} f(x) = \lim_{x\to c} h(x)$

Theorem: 4.6: Algebraic Operations of Limits of Functions

Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$ s.t. $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ exist. Then

1. $\lim_{x\to c}(f(x) \pm g(x)) = \lim_{x\to c} f(x) \pm \lim_{x\to c} g(x)$
2. $\lim_{x\to c}(f(x)g(x)) = \lim_{x\to c} f(x) \lim_{x\to c} g(x)$
3. If $\lim_{x\to c} g(x) \neq 0$ and $g(x) \neq 0$ for all $x \in S \setminus \{c\}$, then $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$

Theorem: 4.7: Absolute Values of Limits of Functions

Let $S \subset \mathbb{R}$ and let c be a cluster point of S. Suppose $f : S \to \mathbb{R}$ s.t. $\lim_{x\to c} f(x)$ exists. Then $\lim_{x\to c} |f(x)| = |\lim_{x\to c} f(x)|$.

Definition: 4.3: Left and Right Limits of Functions

Let $S \subset \mathbb{R}$ and suppose c is a cluster point of $(-\infty, c) \cap S$.

1. Left Limit: We say $f(x) \to L$ as $x \to c^-$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $c - \delta < x < c \Rightarrow |f(x) - L| < \epsilon$. We write $\lim_{x\to c^-} f(x) = L$.
2. Right Limit: We say $f(x) \to L$ as $x \to c^+$ if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $c < x < c+\delta \Rightarrow |f(x) - L| < \epsilon$. We write $\lim_{x\to c^+} f(x) = L$.

Example: The Heaviside function $f(x) = \begin{cases} 0, x \leq 0 \\ 1, x > 0 \end{cases}$.

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} 0 = 0$ and $\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} 1 = 1$.

***Theorem:* 4.8:**

Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$. c is a cluster point of $(-\infty, c) \cap S$ and $(c, \infty) \cap S$. Then $\lim_{x \to c} f(x) = L \Leftrightarrow \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$

Proof. ($\Rightarrow$) By definition.

($\Leftarrow$) Assume $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = L$
Let $\epsilon > 0$. Since $\lim_{x \to c^-} f(x) = L$, then $\exists \delta_1 > 0$ s.t. $c - \delta_1 < x < c \Rightarrow |f(x) - L| < \epsilon$.
Similarly, since $\lim_{x \to c^+} f(x) = L$, then $\exists \delta_2 > 0$ s.t. $c < x < c + \delta_2 \Rightarrow |f(x) - L| < \epsilon$.
Take $\delta = \min\{\delta_1, \delta_2\}$. Then for $c - \delta_1 \leq c - \delta < x < c + \delta \leq c + \delta_2$, $x \neq c$, *i.e.* $0 < |x - c| < \delta$, $|f(x) - L| < \epsilon$.
Thus $\lim_{x \to c} f(x) = L$. ☐

4.2 Continuous Functions

The continuity of a function studies how a function behaves near a point compared to the function at the point.

***Definition:* 4.4: Continuity**

Let $S \subset \mathbb{R}$, $c \in S$. We say f is continuous at c if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x \in S$, $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

Remark 18. The negation: f is not continuous at c if $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x$ s.t. $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.

If f is continuous at every $c \in S$, we say f is *continuous*. If f is not continuous, we say f is *discontinuous*.

Example: $f(x) = ax + b$ is continuous.

Proof. Let $c \in \mathbb{R}$. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{1+|a|}$.
If $|x - c| < \delta$, then $|f(x) - f(c)| = |a(x - c)| = |a||x - c| < |a|\delta = \frac{|a|\epsilon}{1+|a|} < \epsilon$. ☐

Example: $f(x) = \begin{cases} 1, x = 0 \\ 2, x \neq 0 \end{cases}$ is discontinuous at $x = 0$.

Proof. Choose $\epsilon_0 = 1$. Let $\delta > 0$. Choose $x = \frac{\delta}{2}$. Then $|x - \delta| = \frac{\delta}{2} < \delta$ and $|f(x) - f(0)| = 2 - 1 = 1 \geq \epsilon_0$.
So f is not continuous. ☐

Theorem: 4.9: Conditions of Continuity

Suppose $S \subset \mathbb{R}$, $c \in S$, $f : S \to \mathbb{R}$

1. If c is not a cluster point of S, then f is continuous at c
2. Suppose c is a cluster point of S. Then f is continuous at $c \Leftrightarrow \lim_{x\to c} f(x) = f(c)$
3. f is continuous at $c \Leftrightarrow \forall$ sequence $\{x_n\}$ of elements of S s.t. $x_n \to c$, we have $f(x_n) \to f(c)$.

Remark 19. The negation of 3: f is discontinuous at $c \Leftrightarrow \exists$ sequence $\{x_n\}$ s.t. $x_n \to c$ and $\{f(x_n)\}$ does not converge to $f(c)$.

Proof.

1. Let $\epsilon > 0$. Since c is not a cluster point of S, then $\exists \delta_0 > 0$ s.t. $(c - \delta_0, c + \delta_0) \cap S \setminus \{c\} = \emptyset$. Thus, $(c-\delta_0, c+\delta_0) \cap S = \{c\}$. Choose $\delta = \delta_0$. If $x \in S$, $|x - c| < \delta$, then $x = c$, and $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$.
2. Suppose c is a cluster point of S.

   ($\Rightarrow$) Suppose f is continuouse at c. Let $\epsilon > 0$, $\exists \delta_0 > 0$ s.t. $0 < |x-c| < \delta_0 \Rightarrow |f(x) - f(c)| < \epsilon$. Then choose $L = f(c)$, $\delta = \delta_0$. $0 < |x - c| < \delta \Rightarrow |f(x) - L| = |f(x) - f(c)| < \epsilon$. Thus, $\lim_{x\to c} f(x) = f(c)$.

   ($\Leftarrow$) Suppose $\lim_{x\to c} f(x) = f(c)$. Let $\epsilon > 0$. $\exists \delta_0 > 0$ s.t. if $x \in S$, $0 < |x - c| < \delta_0 \Rightarrow |f(x) - f(c)| < \epsilon$. Choose $\delta = \delta_0$. Suppose $|x - c| < \delta$. If $x = c$, then $|f(x) - f(c)| = |f(c) - f(c)| = 0 < \epsilon$. If $x \neq c$, then $0 < |x - c| < \delta = \delta_0$. Thus $|f(x) - f(c)| < \epsilon$. f is continuous.
3. ($\Rightarrow$) Suppose f is continuous at c. Let $\{x_n\}$ be a sequence in S s.t. $x_n \to c$. We want to show that $\lim_{x\to c} f(x) = f(c)$.
   Let $\epsilon > 0$. Since f is continuous at c, $\exists \delta > 0$ s.t. $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$
   Since $x_n \to c$, $\exists M_0 \in \mathbb{N}$, $\forall n \geq M_0$, $|x_n - c| < \delta$.
   Choose $M = M_0$. Then if $n \geq M$, $|x_n - c| < \delta \Rightarrow |f(x_n) - f(c)| < \epsilon$.

   ($\Leftarrow$) Suppose $\forall$ sequence $\{x_n\}$ of elements of S s.t. $x_n \to c$, we have $f(x_n) \to f(c)$.
   Assume that f is not continuous at c. *i.e.* $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x \in S$, $|x-c| < \delta$ and $|f(x)-f(c)| \geq \epsilon_0$.
   Then $\exists x_1 \in S$ s.t. $|x_1 - c| < 1$ and $|f(x_1) - f(c)| \geq \epsilon_0$ and $\forall n \in \mathbb{N}$, $\exists x_n \in S$ s.t, $|x_n - c| < \frac{1}{n}$ and $|f(x_n) - f(c)| \geq \epsilon_0$.
   Then $0 \leq |x_n - c| < \frac{1}{n}$. By squeeze theorem, $x_n \to c$. Then $f(x_n) \to f(c)$. $0 = \lim_{n\to\infty} |f(x_n) - f(c)| \geq \epsilon$.
   Contradition. Thus f must be continuous at c.

$\square$

From the definition of sin and cos via unit circle, we have $\forall x \in \mathbb{R}$:

1. $\sin^2 x + \cos^2 x = 1$
2. $|\sin x| \leq 1$, $|\cos x| \leq 1$
3. $|\sin x| \leq |x|$

Angle Formula:

1. $\sin(a + b) = \sin a \cos b + \cos a \sin b$
2. $\sin a - \sin b = 2 \sin \frac{a-b}{2} \cos \frac{a+b}{2}$

Theorem: 4.10: Continuity of Sine and Cosine

$f(x) = \sin x$ and $g(x) = \cos x$ are continuous.

Proof. 1. $\sin x$. Let $c \in \mathbb{R}$, $\epsilon > 0$. Choose $\delta = \epsilon$
Then $|x - c| < \delta \Rightarrow |\sin x - \sin c| = |2 \sin \frac{x-c}{2} \cos \frac{x-c}{2}| \leq 2|\sin \frac{x-c}{2}| \leq 2\frac{|x-c|}{2} < \delta = \epsilon$

2. $\cos x$. Let $c \in \mathbb{R}$, $\{x_n\}$ be a sequence s.t. $x_n \to c$. We want to show that $\cos x_n \to \cos c$
$\forall x \in \mathbb{R}$, we have $\cos x = \sin\left(x + \frac{\pi}{2}\right)$. If $x_n \to c$, then $x_n + \frac{\pi}{2} \to c + \frac{\pi}{2}$ by Theorem 3.10. Since $\sin x$ is continuous, then $\sin\left(x + \frac{\pi}{2}\right) \to \sin\left(c + \frac{\pi}{2}\right)$ by Theorem 4.9. Thus $\cos(x_n) \to \cos c$.

□

Theorem: 4.11: Dirichlet Function

Let $f(x) = \begin{cases} 1, x \in \mathbb{Q} \\ 0, x \notin \mathbb{Q} \end{cases}$. Then f is discontinuous at every $c \in \mathbb{R}$.

Proof. Let $c \in \mathbb{R}$.

Case 1: $c \in \mathbb{Q}$. For every $n \in \mathbb{N}$, $\exists x_n \in \mathbb{Q}^C$ s.t. $c < x_n < c + \frac{1}{n}$ (Density of irrational numbers)
By squeeze theorem, $x_n \to c$. But $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 0 = 0 \neq 1 = f(c)$.

Case 2: $c \in \mathbb{Q}^C$. For every $n \in \mathbb{N}$, $\exists x_n \in \mathbb{Q}$ s.t. $c < x_n < c + \frac{1}{n}$ by Theorem 2.9.
By squeeze theorem, $x_n \to c$. But $\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} 1 = 1 \neq 0 = f(c)$. □

Theorem: 4.12: Algebraic Operations of Continuous Functions

Suppose $S \subset \mathbb{R}$, $c \in S$, $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$. If f and g are continuous at c, then

1. $f + g$ is continuous at c
2. fg is continuous at c
3. If $g(x) \neq 0$ for all $x \in S$, then $\frac{f}{g}$ is continuous at c

Theorem: 4.13: Composition of Continuous Functions

Suppose $A, B \subset \mathbb{R}$ and $c \in A$. Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. If g is continuous at c and f is continuous at $g(c) \in A$, then $f \circ g$ is continuous at c.

Proof. Let $\{x_n\}$ be a sequence in B s.t. $x_n \to c$. Since $x_n \to c$ and g is continuous at c, then by Theorem 4.9, $g(x_n) \to g(c)$.
Since $g(x_n) \to g(c)$ and f is continuous at $g(c)$, then $f(g(x_n)) \to f(g(c))$. *i.e.* $\lim_{n\to\infty} f \circ g(x_n) = f \circ g(c)$ □

Example: $\forall n \in \mathbb{N}$, $f(x) = x^n$ is continuous.

Proof. Base case: $n = 1$, $f(x) = x$ is continuous as shown by $f(x) = ax + b$.
Inductive step: suppose x^m is continuous. Then $x^{m+1} = x \cdot x^{m+1}$ is the product of two continuous functions, then by Theorem 4.6, x^{m+1} is continuous. □

Example: $\forall n \in \mathbb{N}$, $a_0, ..., a_n \in \mathbb{R}$, the function $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ is continuous.

Proof. By previous example + addition property in Theorem 4.6. ☐

Example: $f(x) = \frac{1}{3+(\sin x)^4}$ is continuous.

Proof. By Theorem 4.13, $(\sin x)^4$ is continuous. Then by Theorem 4.6, $3 + (\sin x)^4$ is continuous, and thus $\frac{1}{3+(\sin x)^4}$ is continuous since $3 + (\sin x)^4 \neq 0$, $\forall x$. ☐

4.3 Extreme and Intermediate Value Theorems

If $f : [a, b] \to \mathbb{R}$ is continuous, then it is well-behaved. *i.e.* $f([a, b]) = [c, d]$.

***Definition:* 4.5: Bounded Functions**

$f : S \to \mathbb{R}$ is bounded if $\exists B \geq 0$ s.t. $\forall x \in S$, $|f(x)| \leq B$.

Remark 20. The negation: $f : S \to \mathbb{R}$ is unbounded if $\forall B \geq 0$, $\exists x$ s.t. $|f(x)| > B$.

Example: $f : [0, 1] \to \mathbb{R}$, $f(x) = 3x + 1$. f is bounded.

Proof. $|f(x)| = |3x + 1| \leq 3|x| + 1 \leq 3 + 1 = 4$ ☐

Example: $f : [0, 1] \to \mathbb{R}$, $f(x) = \begin{cases} 0, x = 0 \\ \frac{1}{x}, x \neq 0 \end{cases}$. f is unbounded.

Proof. Let $B \geq 0$, $x \in \mathbb{R}$ s.t. $0 < x < \frac{1}{B}$, then $|f(x)| = \frac{1}{x} > B$. ☐

***Theorem:* 4.14:**

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded.

Proof. Assume $f : [a, b] \to \mathbb{R}$ and f is not bounded.
Then $\forall n \in \mathbb{N}$, $\exists x_n \in [a, b]$ s.t. $|f(x_n)| \geq n$. Then $\{x_n\}$ is bounded. By Theorem 3.18, there exists a subsequence $\{x_{n_k}\}_k$ and $x \in [a, b]$ s.t. $x_{n_k} \to x$.
Since $\forall k \in \mathbb{N}$, $a \leq x_{n_k} \leq b$, then by squeeze theorem, $a \leq \lim_{k\to\infty} x_{n_k} \leq b$, *i.e.* $a \leq x \leq b$.
Since f is continuous and $x_{n_k} \to x$, $f(x) = \lim_{k\to\infty} f(x_{n_k})$. Thus, by Theorem 4.7, $|f(x)| = \lim_{k\to\infty} |f(x_{n_k})|$.
Then $\{|f(x_{n_k})|\}_k$ is bounded and since $n_k \leq |f(x_{n_k})|$, then $\{n_k\}$ is bounded. Contradiction. ☐

***Definition:* 4.6: Min/Max of Functions**

LEt $f : S \to \mathbb{R}$, f achieves an absolute min at $c \in S$ if $\forall x \in S$, $f(c) \leq f(x)$. f achieves an absolute max at $d \in S$ if $\forall x \in S$, $f(x) \leq f(d)$.

***Theorem:* 4.15: Extreme Value Theorem**

Let $f : [a, b] \to \mathbb{R}$. If f is continuous, then f achieves an absolute min and absolute max on $[a, b]$.

Remark 21. $\exists c, d \in [a, b]$ s.t. $f([a, b]) \subset [f(c), f(d)]$

Proof. If $f : [a, b] \to \mathbb{R}$ is continuous, then f is bounded by Theorem4.14. Then the set $E = \{f(x) : x \in [a, b]\}$ is bounded above.
Let $L = \sup E$ (exists by least upper bound property of $\mathbb{R}$). Then $\exists$ sequence $\{f(x_n)\}$ s.t. $f(x_n) \to L$ by Theorem 4.3.
By Theorem 3.18, there exists subsequence $\{x_{n_k}\}_k$ of $\{x_n\}$ and $d \in [a, b]$ s.t. $\lim_{k\to\infty} x_{n_k} = d$.
Then since f is continuous at d, $f(d) = \lim_{k\to\infty} f(x_{n_k}) = L$.
Since $f(x_n) \to L$ and $\{f(x_{n_k})\}_k$ is a subsequence of $\{f(x_n)\}$. Then $\forall x \in [a, b]$, $f(x) \leq f(d)$.
Thus f achieves an absolute max. The proof for absolute min is similar. ☐

Note: if $f : (a, b) \to \mathbb{R}$ continuous, f does not necessarily achieve an absolute min or max. *e.g.* $f(x) = \frac{1}{x} - \frac{1}{1-x}$ is continuous on $(0, 1)$, but has no absolute max or min.

Theorem: 4.16: Bisection Method

Let $f : [a, b] \to \mathbb{R}$ be continuous. If $f(a) < 0$ and $f(b) > 0$, then $\exists c \in (a, b)$ s.t. $f(c) = 0$

Remark 22. If $f : [a, b] \to \mathbb{R}$ is not continuous, this theorem is not necessarily true. *e.g.* $f(x) = \begin{cases} x-1, x \neq 1 \\ \frac{1}{2}, x = 1 \end{cases}$ on $[0, 2]$. $\nexists c \in [0, 2]$ s.t. $f(c) = 0$ even though $f(0) < 0$, $f(0) > 0$.

Proof. We first define two sequences $\{a_n\}$ and $\{b_n\}$. Let $a_1 = a$, $b_1 = b$. For $n \in \mathbb{N}$ and knowing a_n, b_n, we define a_{n+1} and b_{n+1} as follows:

1. If $f\left(\frac{a_n+b_n}{2}\right) \geq 0$, then $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$
2. If $f\left(\frac{a_n+b_n}{2}\right) < 0$, then $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$

Then, the following are true:

1. $\forall n \in \mathbb{N}$, $a \leq a_n \leq a_{n+1} \leq b_{n+1} \leq b_n \leq b$
2. $\forall n \in \mathbb{N}$, $b_{n+1} - a_{n+2} = \frac{b_n - a_n}{2}$
3. $\forall n \in \mathbb{N}$, $f(a_n) \geq 0$ and $f(b_n) < 0$

By 1, $\{a_n\}$ and $\{b_n\}$ are bounded monotone sequences. Thus, $\{a_n\}$ and $\{b_n\}$ converges. *i.e.* $\exists c, d \in [a, b]$ s.t. $a_n \to c$ and $b_n \to d$.
By 2, $b_n - a_n = \frac{1}{2}(b_{n-1} - a_{n-1}) = \cdots = \frac{1}{2^{n-1}}(b_1 - a_1) = \frac{1}{2^{n-1}}(b - a)$
Thus $d - c = \lim_{n\to\infty} b_n - a_n = \lim_{n\to\infty} \dfrac{1}{2^{n-1}}(b - a) = 0$
By 3 and continuity, $f(c) = \lim_{n\to\infty} f(a_n) \leq 0$, and $f(c) = f(d) = \lim_{n\to\infty} f(b_n) \geq 0$.
Thus $f(c) = f(d) = 0$ ☐

Theorem: 4.17: Bolzano's Intermediate Value Theorem (IVT)

Let $f : [a, b] \to \mathbb{R}$ be continuous.

- If $f(a) < y < f(b)$, then $\exists c \in (a, b)$ s.t. $f(c) = y$
- If $f(b) < y < f(a)$, then $\exists c \in (a, b)$ s.t. $f(c) = y$

Proof. Suppose $f(a) < f(b)$ and $y \in (f(a), f(b))$. Let $g : [a, b] \to \mathbb{R}$, $g(x) = f(x) - y$. g is continuous by Theorem 4.12.
Also, $g(a) = f(a) - y < 0$, $g(b) = f(b) - y > 0$. By Theorem 4.16, $\exists c \in (a, b)$ s.t. $g(c) = 0$. *i.e.* $f(c) = y$.
The other case is similar, by choosing $g(x) = y - f(x)$. ☐

Theorem: 4.18:

Suppose $f : [a, b] \to \mathbb{R}$. If f achieves an absolute min at c and absolute max at d, then $f([a, b]) = [f(c), f(d)]$

Proof. Apply IVT to $f : [c, d] \to \mathbb{R}$, $[f(c), f(d)] \subset f([c, d]) \subset f([a, b]) \subset [f(c), f(d)]$
Thus $f([a, b]) = [f(c), f(d)]$. □

Theorem: 4.19:

If $f(x)$ is a polynomial of odd degree, then $f(x)$ has at least one real root.

4.4 Uniform Continuity

Definition: 4.7: Continuous Functions

$f : S \to \mathbb{R}$ is continuous if $\forall c \in S$, $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon, c) > 0$ s.t. $\forall x \in S$, $|x-c| < \delta \Rightarrow |f(x)-f(c)| < \epsilon$.

Example: $f(x) = \frac{1}{x}$ is continuous on $(0, 1)$.

Proof. Let $c \in (0, 1)$, $\epsilon > 0$. Choose $\delta = \min\{\frac{c}{2}, \frac{c^2}{2}\epsilon\}$. Suppose $x \in (0, 1)$, $|x - c| < \delta$.
Then $|c| = |c - x + x| \leq |c - x| + |x| < \delta + |x| \leq \frac{c}{2} + |x|$. So $\frac{c}{2} < x$.
Then $|f(x) - f(c)| = |\frac{1}{x} - \frac{1}{c}| = \frac{|x-c|}{xc} < \frac{\delta}{xc} \leq \frac{2\delta}{c^2} \leq \frac{2\frac{c^2}{2}\epsilon}{c^2} = \epsilon$. □

Definition: 4.8: Uniform Continuity

Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$. We say f is uniformly continuous on S if $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon) > 0$ s.t. $\forall x, c \in S$, $|x - c| < \delta \Rightarrow |f(x) - f(c)| < \epsilon$.

Remark 23. The negation: $f : S \to \mathbb{R}$ is not uniformly continuous if $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x, c \in S$ s.t. $|x - c| < \delta$, but $|f(x) - f(c)| \geq \epsilon_0$.

Example: $f : [0, 1] \to \mathbb{R}$, $f(x) = x^2$ is unifornmly continuous.

Proof. Let $\epsilon > 0$. Choose $\delta = \frac{\epsilon}{2}$. Suppose $x, c \in [0, 1]$ and $|x - c| < \delta$.
Then $|x^2 - c^2| = |x + c||x - c| \leq (|x| + |c|)|x - c| < 2\delta = \epsilon$. □

Example: $f : (0, 1) \to \mathbb{R}$, $f(x) = \frac{1}{x}$ is not unifornmly continuous.

Proof. Choose $\epsilon_0 = 2$. Let $\delta > 0$. Choose $c = \min\{\delta, \frac{1}{2}\}$, $x = \frac{c}{2}$
Then $|x - c| = \frac{c}{2} \leq \frac{\delta}{2} < \delta$ and $\left|\frac{1}{x} - \frac{1}{c}\right| = \left|\frac{2}{c} - \frac{1}{c}\right| = \frac{1}{c} \geq 2 = \epsilon$. □

Example: $f : \mathbb{R} \to \mathbb{R}$, $f(x) = x^2$ is not unifornmly continuous.

Proof. Choose $\epsilon_0 = 1$. Let $\delta > 0$. Choose $c = \frac{1}{\delta}$. $x = c + \frac{\delta}{2} = \frac{1}{\delta} + \frac{\delta}{2}$.
Then $|x - c| = \frac{\delta}{2} < \delta$ and $|x^2 - c^2| = |x + c||x - c| = (\frac{2}{\delta} + \frac{\delta}{2})\frac{\delta}{2} = 1 + \frac{\delta^2}{4} \geq 1 = \epsilon_0$. □

Theorem: 4.20:

Let $f : [a, b] \to \mathbb{R}$. Then f is continuous on $[a, b] \Leftrightarrow f$ is uniformly continuous on $[a, b]$.

Remark 24. In general f uniformly continuous $\Rightarrow f$ continuous, but f continuous $\not\Rightarrow f$ uniformly continuous.

Proof. ($\Rightarrow$) Suppose f is continuous on $[a, b]$.
Assume that f is not uniformly continuous on $[a, b]$.
Then $\exists \epsilon_0 > 0$ s.t. $\forall \delta > 0$, $\exists x, c \in [a, b]$ s.t. $|x - c| < \delta$ and $|f(x) - f(c)| \geq \epsilon_0$.
Then by Theorem 4.9, $\forall n \in \mathbb{N}$, $\exists x_n, c_n \in [a, b]$ s.t. $|x_n - c_n| < \frac{1}{n}$ and $|f(x_n) - f(c_n)| \geq \epsilon_0$.
By Theorem 3.18, there exists sequence $\{x_{n_k}\}_k$ of $\{x_n\}$ and $x \in [a, b]$ s.t. $\lim\limits_{k\to\infty} x_{n_k} = x$.
Since $\{c_{n_k}\}_k$ is bounded between a and b. By Theorem 3.18, there exists subsequence $\{c_{n_{k_j}}\}_j$ of $\{c_{n_k}\}$ and $c \in [a, b]$ s.t. $\lim\limits_{j\to\infty} c_{n_{k_j}} = c$.
In summary, the sequences $\{x_{n_{k_j}}\}_j$ and $\{c_{n_{k_j}}\}_j$ are subsequences of $\{x_n\}$ and $\{c_n\}$. And $\exists x, c \in [a, b]$ s.t. $c_{n_{k_j}} \to c$ and $x_{n_{k_j}} \to x$.
Now $0 \leq |x_{n_{k_j}} - c_{n_{k_j}}| \leq \frac{1}{n_{k_j}} \leq \frac{1}{j}$. Thus by squeeze theorem, $x = c$.
Since f is continuous at c, $0 = |f(c) - f(c)| = \lim\limits_{j\to\infty} |f(x_{n_{k_j}}) - f(c_{n_{k_j}})| \geq \epsilon_0 > 0$.
Contradiction. Thus f is uniformly continuous.

($\Leftarrow$) Suppose f is uniformly continuous on $[a, b]$.
Let $\epsilon > 0$. Since f is uniformly continuous on $[a, b]$, $\exists \delta_0 > 0$ s.t. $\forall x, c \in S$, $|x - c| < \delta_0$ we have $|f(x) - f(c)| < \epsilon$.
Then $\forall c \in S$. Choose $\delta = \delta_0$ s.t. $\forall x \in S$, $|x - c| < \delta$, we have $|f(x) - f(c)| < \epsilon$. Thus f is continuous. $\square$

5 Derivatives

5.1 Derivative

Definition: 5.1: Differetiability and Derivative

Let I be an interval, $f : I \to \mathbb{R}$, $c \in I$. We say f is differentiable at c if $\lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ exists. We write $f'(c) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$.
If f is differentiable at every point of I, we write the derivative f' or $\frac{df}{dx}$.

Remark 25. $L = \lim_{x\to c} \frac{f(x)-f(c)}{x-c} \Leftrightarrow \forall$ sequence $\{x_n\}$ with $x_n \neq c$ and $x_n \to c$, we have
$L = \lim_{n\to\infty} \frac{f(x_n)-f(c)}{x_n - c}$.

Example: Let $\alpha \in \mathbb{R}$, $n \in \mathbb{N} \cup \{0\}$. Then $f(x) = \alpha x^n$ is differentiable and $f'(c) = n\alpha c^{n-1}$, $\forall c \in \mathbb{R}$.

Proof. We compute:

$$\begin{aligned}(x-c)\sum_{j=0}^{n-1} x^{n-1-j}c^j &= \sum_{j=0}^{n-1} x^{n-j}c^j - \sum_{l=0}^{n-1} x^{n-1-l}c^{l+1} \\ &= \sum_{j=0}^{n-1} x^{n-j}c^j - \sum_{j=1}^{n} x^{n-j}c^j \text{ (Let } j = l+1) \\ &= x^{n-0}c^0 - x^{n-n}c^n = x^n c^n\end{aligned}$$

Then $f'(c) = \lim_{x\to c} \frac{\alpha x^n - \alpha c^n}{x-c} = \alpha \lim_{x\to c} \sum_{j=0}^{n-1} x^{n-1-j}c^j = \alpha \sum_{j=0}^{n-1} c^{n-1} = \alpha n c^{n-1}$. □

Theorem: 5.1:

If $f : I \to \mathbb{R}$ is differentiable at $c \in I$, then f is continuous at c, *i.e.* $\lim_{x\to c} f(x) = f(c)$.

Proof. We compute $\lim_{x\to c} f(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}(x-c) + f(c) = f'(c)\cdot 0 + f(c) = f(c)$. □

Remark 26. The converse is false. f is continuous at $c \not\Rightarrow f$ is differentiable at c.

Example: $f(x) = |x|$. f is continuous but not differentiable at $c = 0$

Proof. We find $\{x_n\}$ s.t. $x_n \neq 0$ $\forall n$, $x_n \to 0$, and $\left\{\frac{f(x_n)-f(0)}{x_n-0}\right\}$ diverges.

Let $x_n = \frac{(-1)^n}{n}$. Then $\forall n$, $x_n \neq 0$ and $x_n \to 0$.

We compute $\frac{f(x_n)-f(0)}{x_n-0} = \frac{\left|\frac{(-1)^n}{n}\right|}{\frac{(-1)^n}{n}} = (-1)^n$. Thus $\left\{\frac{f(x_n)-f(0)}{x_n-0}\right\}$ diverges. □

If $f:\mathbb{R}\to\mathbb{R}$ is continuous, there still may not exist $c\in\mathbb{R}$ s.t. f is differentiable at c. (Weierstrass)

Goal: Construct a continuous function $f:\mathbb{R}\to\mathbb{R}$ which is no where differentiable

Theorem: 5.2:

1. $\forall x,y\in\mathbb{R}$, $|\cos x-\cos y|\le|x-y|$
2. $\forall c\in\mathbb{R}$, $\forall K\in\mathbb{N}$, $\exists y\in(c+\frac{\pi}{K},c+\frac{3\pi}{K})$ s.t. $|\cos Kc-\cos Ky|\ge 1$

Proof. 1. We have shown that $|\sin x-\sin y|\le|x-y|$ for Theorem 4.10. Then $|\cos x-\cos y| = |\sin(x+\frac{\pi}{2})-\sin(y+\frac{\pi}{2})|\le|x-y|$.

2. The function $g(x)=\cos Kx$ is $\frac{2\pi}{K}$ periodic. Thus $g((c+\frac{\pi}{K},c+\frac{3\pi}{K}))\supset[-1,1]\setminus\{-\cos Kc\}$

   If $\cos Kc\ge 0$, we choose $y\in(c+\frac{\pi}{K},c+\frac{3\pi}{K})$ s.t. $\cos Ky=-1$
   If $\cos Kc<0$, we choose $y\in(c+\frac{\pi}{K},c+\frac{3\pi}{K})$ s.t. $\cos Ky=1$

□

Theorem: 5.3:

$\forall a,b,c\in\mathbb{R}$, $|a+b+c|\ge|a|-|b|-|c|$

Proof. Apply Triangle Inequality twice:

$$|a|=|a+b+c-(b+c)|\le|a+b+c|+|b+c|\le|a+b+c|+|b|+|c|$$

Thus $|a+b+c|\ge|a|-|b|-|c|$. □

Theorem: 5.4:

1. $\forall x\in\mathbb{R}$, the series $\displaystyle\sum_{k=0}^{\infty}\frac{\cos 160^k x}{4^k}$ is absolutely convergent.
2. Let $f:\mathbb{R}\to\mathbb{R}$ be $f(x)=\displaystyle\sum_{k=0}^{\infty}\frac{\cos 160^k x}{4^k}$. Then f is bounded and continuous

Proof. 1. $\left|\frac{\cos 160^k x}{4^k}\right|\le\left|\frac{1}{4^k}\right|$. By comparison test, $\displaystyle\sum\left|\frac{\cos 160^k x}{4^k}\right|$ is convergent, thus $\displaystyle\sum\frac{\cos 160^k x}{4^k}$ is absolutely convergent.

2. Let $x\in\mathbb{R}$. Then

$$|f(x)|=\left|\lim_{m\to\infty}\sum_{k=0}^{m}\frac{\cos 160^k x}{4^k}\right|=\lim_{m\to\infty}\left|\sum_{k=0}^{m}\frac{\cos 160^k x}{4^k}\right|\le\lim_{m\to\infty}\sum_{k=0}^{m}\left|\frac{\cos 160^k x}{4^k}\right|\le\lim_{m\to\infty}\sum_{k=0}^{m}4^{-k}=\frac{4}{3}$$

   Let $c\in\mathbb{R}$, $\{x_n\}$ be a sequence s.t. $x_n\to c$. We want to show $|f(x_n)-f(c)|\to 0$.
   Equivalently, we show $\limsup_n|f(x_n)-f(c)|=0$.
   Claim: $\forall\epsilon>0$, $\limsup_n|f(x_n)-f(c)|<\epsilon$

Let $\epsilon > 0$. Let $M_0 \in \mathbb{N}$ s.t. $\displaystyle\sum_{k=M_0+1}^{\infty} 4^{-k} < \frac{\epsilon}{2}$. Then

$$\limsup_n |f(x_n) - f(c)| = \limsup_n \left| \sum_{k=0}^{M_0} \frac{1}{4^k}[\cos 160^k x_n - \cos 160^k c] + \sum_{k=M_0+1}^{\infty} \frac{1}{4^k}[\cos 160^k x_n - \cos 160^k c] \right|$$

$$\leq \limsup_n \left| \sum_{k=0}^{M_0} \frac{1}{4^k}[\cos 160^k x_n - \cos 160^k c] \right| + \limsup_n \left| \sum_{k=M_0+1}^{\infty} \frac{1}{4^k}[\cos 160^k x_n - \cos 160^k c] \right|$$

$$\leq \limsup_n \sum_{k=0}^{M_0} \frac{1}{4^k}|\cos 160^k x_n - \cos 160^k c| + \limsup_n \sum_{k=M_0+1}^{\infty} \frac{1}{4^k}|\cos 160^k x_n| + |\cos 160^k c|$$

$$\leq \limsup_n \sum_{k=0}^{M_0} \frac{1}{4^k}|160^k x_n - 160^k c| + \limsup_n \sum_{k=M_0+1}^{\infty} \frac{2}{4^k} \text{ (By Theorem 5.2)}$$

$$< \limsup_n \left[\sum_{k=0}^{M_0} 40^k \right] |x_n - c| + \epsilon < \epsilon \text{ as } x_n \to c$$

Thus $\limsup_n |f(x_n) - f(c)| = 0$ and f is bounded and continuous.

□

Theorem: 5.5: Weierstrass's Example

The function $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{\cos 160^k x}{4^k}$ is no where differentiable.

Proof. Let $c \in \mathbb{R}$. We will find a sequence x_n s.t. $x_n \neq c$, $x_n \to c$ and $\left\{ \frac{f(x_n)-f(c)}{x_n-c} \right\}$ is unbounded. $\forall n \in \mathbb{N}$, there exists x_n s.t.

(a) $\frac{\pi}{160^n} < x_n - c < \frac{3\pi}{160^n}$

(b) $|\cos 160^n x_n - \cos 160^n c| \geq 1$

By (a), $\forall n$, $x_n \neq c$ and by squeeze theorem $x_n \to c$. Let $f_k(x) = \frac{\cos 160^k x}{4^k}$, then $f(x) = \sum_{k=0}^{\infty} f_k(x)$

Goal: find a lower bound on $\left| \frac{f(x_n)-f(c)}{x_n-c} \right|$. If the lower bound is unbounded, then the value is unbounded.

$$|f(x_n) - f(c)| = \underbrace{f_n(x_n) - f_n(c)}_{a_n} + \underbrace{\sum_{k=0}^{n-1}(f_k(x_n) - f_k(c))}_{b_n} + \underbrace{\sum_{k=n+1}^{\infty}(f_k(x_n) - f_k(c))}_{c_n}$$

Then $|f(x_n) - f(c)| = |a_n + b_n + c_n| \geq |a_n| - |b_n| - |c_n|$ by Theorem 5.3.

By (b), $|a_n| = 4^{-n}|\cos 160^n x_n - \cos 160^n c| \geq 4^{-n}$

$$|b_n| = \left| \sum_{k=0}^{n-1}(f_k(x_n) - f_k(c)) \right| \leq \sum_{k=0}^{n-1} |f_k(x_n) - f_k(c)| = \sum_{k=0}^{n-1} 4^{-k}|\cos 160^k x_n - \cos 160^k c| \leq \sum_{k=0}^{n-1} 40^k |x_n - c| <$$

$$\frac{3\pi}{160^n}\sum_{k=0}^{n-1}40^k = \frac{3\pi}{160^n}\frac{40^n-1}{39} < \frac{4^{-n}\pi}{13} < \frac{1}{13}4^{-n+1}$$

$$|c_n| = \left|\sum_{k=n+1}^{\infty}(f_k(x_n)-f_k(c))\right| \le \sum_{k=n+1}^{\infty}|f_k(x_n)-f_k(c)| \le \sum_{k=n+1}^{\infty}|f_k(x_n)|+|f_k(c)| \le 2\sum_{k=n+1}^{\infty}4^{-k} = \frac{2}{3}4^{-n}$$

Then $|f(x_n)-f(c)| \ge |a_n|-|b_n|-|c_n| \ge 4^{-n}-\frac{1}{13}4^{-n+1}-\frac{2}{3}4^{-n} = \frac{1}{39}4^{-n}$

Since $\frac{1}{|x_n-c|} = \frac{160^n}{3\pi}$, then $\left|\frac{f(x_n)-f(c)}{x_n-c}\right| \ge \frac{1}{|x_n-c|}4^{-n}\frac{1}{39} \ge \frac{40^n}{117\pi}$

Thus $\left\{\left|\frac{f(x_n)-f(c)}{x_n-c}\right|\right\}$ is unbounded. $f(x)$ is no where differentiable. □

Theorem: 5.6: Algebraic Operations of Derivatives

Let $f: I \to \mathbb{R}$, $g: I \to \mathbb{R}$, $c \in i$. If f and g are differentiable at c, then:

1. (Linearity) $\forall \alpha \in \mathbb{R}$, $\alpha f + g: I \to \mathbb{R}$ is differentiable at c and $(\alpha f+g)'(c) = \alpha f'(c)+g'(c)$
2. (Product Rule) $fg: I \to \mathbb{R}$ is differentiable at c and $(fg)'(c) = f'(c)g(c)+f(c)g'(c)$
3. (Quotient Rule) if $g(x) \ne 0$, $\forall x \in I$, then $\frac{f}{g}: I \to \mathbb{R}$ is differentiable at c and $\left(\frac{f}{g}\right)'(c) = \frac{f'(c)g(c)-f(c)g'(c)}{(g(c))^2}$

Proof.

1. $\lim_{x\to c}\frac{(\alpha f+g)(x)-(\alpha f+g)(c)}{x-c} = \lim_{x\to c}\left[\alpha\frac{f(x)-f(c)}{x-c}+\frac{g(x)-g(c)}{x-c}\right] = \alpha f'(c)+g'(c)$
2. Since g is differentiable at c, g is continuous at c. *i.e.* $\lim_{x\to c} g(x) = g(c)$.
   Then $\lim_{x\to c}\frac{f(x)g(x)-f(c)g(c)}{x-c} = \lim_{x\to c}\frac{f(x)-f(c)}{x-c}g(x)+f(c)\frac{g(x)-g(c)}{x-c}) = f'(c)g(c)+f(c)g'(c)$
3. Consider $\frac{1}{g(x)}$ first. Since $g(x) \ne 0$, $\frac{1}{g(x)}$ is well-defined and continuous.
   $\lim_{x\to c}\frac{\frac{1}{g(x)}-\frac{1}{g(c)}}{x-c} = \lim_{x\to c}\frac{g(c)-g(x)}{g(x)g(c)(x-c)} = -\frac{g'(c)}{g(c)^2}$. Then apply Product Rule.

□

Theorem: 5.7: Chain Rule

Let I_1, I_2 be intervals, $g: I_1 \to I_2$, $f: I_2 \to \mathbb{R}$ and suppose g is differentiable at c, f is differentiable at $g(c)$. Then $f \circ g: I_1 \to \mathbb{R}$ is differentiable at c and $(f\circ g)'(c) = f'(g(c))g'(c)$.

Proof. Let $h(x) = f(g(x))$, $d = g(c)$. We want to show $h'(c) = f'(d)g'(c)$.

Define $u(y) = \begin{cases} \frac{f(y)-f(d)}{y-d}, y \ne d \\ f'(d), y = d \end{cases}$ and $v(x) = \begin{cases} \frac{g(x)-g(c)}{x-c}, x \ne c \\ g'(c), x = c \end{cases}$

Then $f(y)-f(d) = u(y)(y-d)$, $g(x)-g(c) = v(x)(x-c)$

Note: $u(y)$ is continuous at d, $v(x)$ is continuous at c.

$$\lim_{y\to d} u(y) = \lim_{y\to d}\frac{f(y)-f(d)}{y-d} = f'(d) = u(d), \text{ and } \lim_{x\to c} v(x) = \lim_{x\to c}\frac{g(x)-g(c)}{x-c} = g'(c) = v(c)$$

Then $h(x)-h(c) = f(g(x))-f(g(c)) = u(g(x))(g(x)-g(c)) = u(g(x))v(x)(x-c)$.

$$\lim_{x\to c}\frac{h(x)-h(c)}{x-c} = \lim_{x\to c}\frac{u(g(x))v(x)(x-c)}{x-c}u(g(c))v(c) = u(d)v(c) = f'(g(c))g'(c)$$

□

5.2 Mean Value Theorem

***Definition:* 5.2: Relative Min/Max**

Let $S \subset \mathbb{R}$, $f : S \to \mathbb{R}$. f has a relative max at $c \in S$ if $\exists \delta > 0$ s.t. $\forall x \in S$, $|x-c| < \delta \Rightarrow f(x) \leq f(c)$. f has a relative min at $c \in S$ if $\exists \delta > 0$ s.t. $\forall x \in S$, $|x-c| < \delta \Rightarrow f(x) \geq f(c)$.

***Theorem:* 5.8:**

If $f : [a,b] \to \mathbb{R}$ has a relative min or max at $c \in (a,b)$ and f is differentiable at c, then $f'(c) = 0$.

Proof. Suppose f has a relative max at $c \in (a,b)$. Then $\exists \delta > 0$ s.t.

1. $(c-\delta, c+\delta) \subset (a,b)$ (definition of open sets)
2. $\forall x \in (c-\delta, c+\delta)$, $f(x) \leq f(c)$ (relative max)

Let $x_n = c - \frac{\delta}{2n} \in (c-\delta, c)\ \forall n$. Then $x_n \to c$. so $f'(c) = \lim_{n\to\infty} \dfrac{f(x_n) - f(c)}{x_n - c} \geq 0$.
Let $x_n = c + \frac{\delta}{2n} \in (c, c+\delta)\ \forall n$. Then $x_n \to c$. so $f'(c) = \lim_{n\to\infty} \dfrac{f(x_n) - f(c)}{x_n - c} \leq 0$.
Thus $f'(c) = 0$. The same proof applies to relative min □

***Theorem:* 5.9: Rolle's Theorem**

Let $f : [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b). If $f(a) = f(b) = 0$, then $\exists c \in (a,b)$ s.t. $f'(c) = 0$.

Remark 27. Absolute max is a relative max. Absolute min is a relative min.

Proof. Since f is continuous on $[a,b]$, f achieves a relative max at $c_1 \in [a,b]$ and a relative min at $c_2 \in [a,b]$.
If $f(c_1) > 0$, then $c_1 \in (a,b)$, $f'(c_1) = 0$. Set $c = c_1$.
If $f(c_2) < 0$, then $c_2 \in (a,b)$, $f'(c_2) = 0$. Set $c = c_2$.
If $f(c_1) \leq 0 \leq f(c_2)$. By definition, $f(c_1) \geq f(c_2)$. Then $f(c_1) = f(c_2) = 0$, and $\forall x \in [a,b]$, $f(c_2) \leq f(x) \leq f(c_1) = f(c_2)$. Thus $\forall x \in [a,b]$, $f(x) = f(c_2)$. *i.e.* f is constant. Set $c = \frac{a+b}{2}$. □

***Theorem:* 5.10: Mean Value Theorem**

Let $f : [a,b] \to \mathbb{R}$ be continuous, differentiable on (a,b). Then $\exists c \in (a,b)$ s.t. $f(b) - f(a) = f'(c)(b-a)$.

Proof. Define $g : [a,b] \to \mathbb{R}$ by $g(x) = f(x) - f(b) + \frac{f(b)-f(a)}{b-a}(b-x)$
Then g is continuous on $[a,b]$ and differentiable on (a,b).
$g(a) = f(a) - f(b) + \frac{f(b)-f(a)}{b-a}(b-a) = 0$, $g(b) = f(b) - f(b) + \frac{f(b)-f(a)}{b-a}(b-b) = 0$
By Theorem 5.9, $\exists c \in (a,b)$ s.t. $0 = g'(c) = f'(c) - \frac{f(b)-f(a)}{b-a}$.
Thus $f(b) - f(a) = f'(c)(b-a)$. □

Theorem: 5.11:

Let $f : I \to \mathbb{R}$ be differentiable. Then

1. f is increasing ($x < y \Rightarrow f(x) \leq f(y)$) $\Leftrightarrow \forall x \in I$, $f'(x) \geq 0$
2. f is decreasing ($x < y \Rightarrow f(x) \geq f(y)$) $\Leftrightarrow \forall x \in I$, $f'(x) \leq 0$

Proof. We only prove the increasing case.
($\Leftarrow$) Suppose $f'(x) \geq 0, \forall x \in I$. Let $a, b \in I$ with $a < b$. Then f is continuous on $[a, b] \subset I$ and differentiable on (a, b).
Then by Theorem 5.10, $\exists c \in (a, b)$ s.t. $f(b) - f(a) = f'(c)(b - a) \geq 0$, $f(a) \leq f(b)$.
($\Rightarrow$) Suppose f is increasing and $c \in I$. Let $\{x_n\}$ be a sequence in I s.t. $x_n \to c$ and either (a) $\forall n, x_n < c$ or (b) $\forall n, x_n > c$. Such a sequence always exists since I is an interval.
In case (a), $\forall n$, $f(x_n) - f(c) \leq 0$ since f is increasing $\Rightarrow \forall n$, $\frac{f(x_n)-f(c)}{x_n-c} \geq 0 \Rightarrow f'(c) \geq 0$
In case (b), $\forall n$, $f(x_n) - f(c) \geq 0 \Rightarrow \forall n$, $\frac{f(x_n)-f(c)}{x_n-c} \geq 0 \Rightarrow f'(c) \geq 0$.
Thus $f'(c) \geq 0$.

For the second part of theorem, f is decreasing $\Leftrightarrow -f$ is increasing $\Leftrightarrow -f'(x) \geq 0$, $\forall x \in I$. □

Theorem: 5.12:

Let $f : I \to \mathbb{R}$ be differentiable. Then f is constant $\Leftrightarrow f'(x) = 0$, $\forall x \in i$.

Proof. f is constant $\Leftrightarrow$ f is increasing and decreasing $\Leftrightarrow \forall x \in I$, $f'(x) \geq 0$ and $f'(x) \leq 0 \Leftrightarrow \forall x \in I$, $f'(x) = 0$. □

5.3 Taylor's Theorem

Remark 28. Taylor's theorem is essentially the Mean Value Theorem for higher order derivatives.

Definition: 5.3: n-times Differentiable

We say $f : I \to \mathbb{R}$ is n-times differentiable on $J \subset I$ if $f', f'', ..., f^{(n)}$ exist at every point in J, where the n-th derivative is denoted as $f^{(n)}$.

Theorem: 5.13: Taylor's Theorem

Suppose $f : [a, b] \to \mathbb{R}$ is continuous and has n continuous derivatives on $[a, b]$ s.t. $f^{(n+1)}$ exists on (a, b). Given $x_0, x \in [a, b]$, there exists $c \in (x_0, x)$ s.t.

$$f(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k + \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1},$$

where $P_n(x) = \sum_{k=0}^{n} \frac{1}{k!} f^{(k)}(x_0)(x - x_0)^k$ is the n-th order Taylor polynomial for f at x_0, $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x - x_0)^{n+1}$ is the n-th order remainder.

Note: $R_n(x)$ doesn't need to be small.

Example: $f(x) = \begin{cases} e^{-\frac{1}{x}}, x > 0 \\ 0, x = 0 \end{cases}$. f is differentiable as many times as we want, but $f^{(n)}(0) = 0$, $\forall n$. Then $P_n(x) = 0$, $f(x) = R_n(x)$.

Proof. Let $x_0 \neq x$. Let $M_{x,x_0} = \frac{f(x)-P_n(x)}{(x-x_0)^{n+1}}$, so $f(x) = P_n(x) + M_{x,x_0}(x-x_0)^{n+1}$.
Goal: show $\exists c \in (a,b)$ s.t. $M_{x,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}$.
For $0 \le k \le n$, $f^{(k)}(x_0) = P_n^{(k)}(x_0)$. Let $g(s) = f(s) - P_n(s) - M_{x,x_0}(s-x_0)^{n+1}$. $g(s)$ is $n+1$ differentiable in s.
$g(x_0) = f(x_0) - P_n(x_0) = 0$ and $g(x) = f(x) - P_n(x) - M_{x,x_0}(x-x_0)^{n+1} = 0$.
By Theorem 5.10, $\exists x_1$ between x_0 and x s.t. $g'(x_1) = 0$.
Also $g'(x_0) = f'(x_0) - P_n'(x_0) = 0$, so $\exists x_2$ between x_0 and x_1 (thus between x_0 and x) s.t. $g''(x_2) = 0$
Continuing in this way, for $0 \le k \le n$, $\exists x_k$ between x_0 and x s.t. $g^{(k)}(x_k) = 0$.
Since $g^{(n)}(x_0) = f^{(n)}(x_0) - P_n^{(n)}(x_0) - M_{x,x_0}(n+1)!(x_0 - x_0) = f^{(n)}(x_0) - P_n^{(n)}(x_0) = 0$ and $g^{(n)}(x_n) = 0$
By Theorem 5.10 applied to $g^{(n)}$, there exists c between x and x_0 s.t. $g^{(n+1)}(c) = 0$.
Then $f^{(n+1)}(c) - 0 - M_{x,x_0}(n+1)! = 0$. Thus $M_{x,x_0} = \frac{f^{(n+1)}(c)}{(n+1)!}$.
Thus $f(x) = P_n(x) + \frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}$ □

***Theorem:* 5.14: Second Derivative Test**

Suppose $f : (a,b) \to \mathbb{R}$ has two continuous derivatives on (a,b). If $x_0 \in (a,b)$ s.t. $f'(x_0) = 0$ and $f''(x_0) > 0$, then f has a strict relative min at x_0. *i.e.* $\exists \delta > 0$ s.t. $\forall x$, $0 < |x - x_0| < \delta \Rightarrow f(x) > f(x_0)$.

Proof. Since f'' is continuous, $\lim\limits_{c \to x_0} f''(c) = f''(x_0) > 0$. Then $\exists \delta_0 > 0$ s.t. $\forall 0 < |c - x_0| < \delta_0$, $f''(c) > 0$.
Choose $\delta = \delta_0$. Let $0 < |x - x_0| < \delta = \delta_0$. Then by Theorem 5.13, $\exists c$ between x and x_0 s.t. $f(x) = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(c)}{2}(x-x_0)^2$.
Since $f'(x_0) = 0$, $f(x) = f(x_0) + \frac{f''(c)}{2}(x-x_0)^2 > f(x_0)$. Since $x \neq x_0$, $f''(c) > 0$. □

6 Riemann Integration

6.1 The Riemann Integrals

Notation: $C([a,b]) = \{f : [a,b] \to \mathbb{R} : f \text{ is continuous}\}$.

Definition: 6.1: Tagged Partition

A partition of $[a,b]$ is a finite set $\underline{x} = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. The norm of a partition $\|\underline{x}\| = \max\{x_1 - x_0, ..., x_n - x_{n-1}\}$. A tag for a partition $\underline{x}$ is a finite set $\underline{\xi} = \{\xi_1, ..., \xi_n\}$ s.t. $x_0 \le \xi_1 \le x_1 \le \xi_2 \le x_2 \le \cdots \le \xi_n \le x_n$. The pair $(\underline{x}, \underline{\xi})$ is a tagged partition.

Example: $\underline{x} = \{1, \frac{3}{2}, 2, 3\}$, $\underline{\xi} = \{\frac{5}{4}, \frac{7}{4}, \frac{5}{2}\}$. Then $\|\underline{x}\| = \max\{\frac{3}{2} - 1, 2 - \frac{3}{2}, 3 - 2\} = 1$.

Definition: 6.2: Riemann Sum

Let $f \in C([a,b])$, $(\underline{x}, \underline{\xi})$ is a tagged partition. The Riemann Sum of f corresponding to $(\underline{x}, \underline{\xi})$ is the number $S_f(\underline{x}, \underline{\xi}) = \displaystyle\sum_{k=1}^{n} f(\xi_k)(x_k - x_{k-1})$.

Definition: 6.3: Modulus of Continuity

For $f \in C([a,b])$, $\eta > 0$, we define the modulus of continuity $\omega_f(\eta) = \sup\{|f(x) - f(y)| : |x - y| \le \eta\}$. $\forall x, y$, $|f(x) - f(y)| \le \omega_f(|x - y|)$. If $\eta_1 \le \eta_2$, then $\omega_f(\eta_1) \le \omega_f(\eta_2)$.

Example: $f(x) = ax + b$. Then $|f(x) - f(y)| = |a||x - y|$. So if $|x - y| \le \eta$, then $|f(x) - f(y)| \le |a|\eta$. Thus $\omega_f(\eta) = |a|\eta$.

Theorem: 6.1:

$\forall f \in C([a,b])$, $\lim_{\eta \to 0} \omega_f(\eta) = 0$. *i.e.* $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall \eta$, $0 < \eta < \delta$, $\omega_f(\eta) < \epsilon$.

Proof. Let $\epsilon > 0$. Since $f \in C([a,b])$, f is uniformly continuous by Theorem 4.20. *i.e.* $\exists \delta_0 > 0$ s.t. $\forall x, y$, $|x - y| < \delta_0 \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$.
Chooose $\delta = \delta_0$. Suppose $\eta < \delta = \delta_0$. If $|x - y| \le \eta < \delta_0$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$. Then $\frac{\epsilon}{2}$ is an upper bound for the set $\{|f(x) - f(y)| : |x - y| \le \eta\}$.
Thus $\omega_f(\eta) = \sup\{|f(x) - f(y)| : |x - y| \le \eta\} \le \frac{\epsilon}{2} < \epsilon$. Thus $\lim_{\eta \to 0} \omega_f(\eta) = 0$. □

Theorem: 6.2:

If $(\underline{x}, \underline{\xi})$ and $(\underline{x}', \underline{\xi}')$ are tagged partitions of $[a,b]$ s.t. $\underline{x} \subset \underline{x}'$. *i.e.* $\underline{x}'$ is a *refinement* of $\underline{x}$ and $f \in C([a,b])$, then $|S_f(\underline{x}, \underline{\xi}) - S_f(\underline{x}', \underline{\xi}')| \le \omega_f(\|\underline{x}\|)(b - a)$.

Proof. For $k = 1, ..., n$, let $\underline{y^{(k)}} = \{x_{k-1} = x'_l < x'_{l+1} < \cdots < x'_m = x_k\}$, $\underline{\eta^{(k)}} = \{\xi'_{l+1}, ..., \xi'_m\}$ be a partition of $[x_{k-1}, x_k]$.

Then $(\underline{x}', \underline{\xi}') = \cup_{k=1}^{n}(\underline{y^{(k)}}, \underline{\eta^{(k)}})$. Thus $S_f(\underline{x}', \underline{\xi}') = \displaystyle\sum_{k=1}^{n} S_f(\underline{y^{(k)}}, \underline{\eta^{(k)}})$.

Then

$$
\begin{aligned}
|f(\xi_k)(x_k - x_{k-1}) - S_f(\underline{y^{(k)}}, \underline{\eta^{(k)}})| &= \left| f(\xi_k) \sum_{j=l+1}^{m} (x'_j - x'_{j-1}) - \sum_{j=l+1}^{m} f(\xi'_j)(x'_j - x'_{j-1}) \right| \\
&= \left| \sum_{j=l+1}^{m} (f(\xi_k) - f(\xi'_j))(x'_j - x'_{j-1}) \right| \\
&\le \sum_{j=l+1}^{m} |f(\xi_k) - f(\xi'_j)|(x'_j - x'_{j-1}) \\
&\le \sum_{j=l+1}^{m} \omega_f(|x_k - x_{k-1}|)(x'_j - x'_{j-1}) \text{ (by Definition 6.3)} \\
&= \omega_f(|x_k - x_{k-1}|)(x_k - x_{k-1})
\end{aligned}
$$

Then

$$
\begin{aligned}
|S_f(\underline{x}, \underline{\xi}) - S_f(\underline{x'}, \underline{\xi'})| &\le \sum_{k=1}^{n} |f(\xi_k)(x_k - x_{k-1}) - S_f(\underline{y^{(k)}}, \underline{\eta^{(k)}})| \\
&\le \sum_{k=1}^{n} \omega_f(|x_k - x_{k-1}|)(x_k - x_{k-1}) \text{ (By previous calculation)} \\
&\le \sum_{k=1}^{n} \omega_f(\|\underline{x}\|)(x_k - x_{k-1}) \\
&= \omega_f(\|\underline{x}\|)(b-a)
\end{aligned}
$$

□

Theorem: 6.3:

If $(\underline{x}, \underline{\xi})$ and $(\underline{x'}, \underline{\xi'})$ are any tagged partitions and $f \in C([a,b])$, then

$$|S_f(\underline{x}, \underline{\xi}) - S_f(\underline{x'}, \underline{\xi'})| \le [\omega_f(\|\underline{x}\|) + \omega_f(\|\underline{x'}\|)](b-a)$$

Proof. Define $\underline{x''} = \underline{x} \cup \underline{x'}$ amd $\underline{\xi''} = \underline{\xi} \cup \underline{\xi'}$. Then $\underline{x} \subset \underline{x'}$ and $\underline{x} \subset \underline{x''}$. So $\underline{x''}$ is a refinement of both $\underline{x}$ and $\underline{x'}$ Then by Theorem 6.2,

$$
\begin{aligned}
|S_f(\underline{x}, \underline{\xi}) - S_f(\underline{x'}, \underline{\xi'})| &\le |S_f(\underline{x}, \underline{\xi}) - S_f(\underline{x''}, \underline{\xi''})| + |S_f(\underline{x''}, \underline{\xi''}) - S_f(\underline{x'}, \underline{\xi'})| \\
&\le \omega_f(\|\underline{x}\|)(b-a) + \omega_f(\|\underline{x'}\|)(b-a)
\end{aligned}
$$

□

Theorem: 6.4: Riemann Integral

Let $f \in C([a,b])$. Then there exists unique number $\int_a^b f(x)dx$ with the following property: $\forall$ sequences of partitions $\{(\underline{x^{(r)}}, \underline{\xi^{(r)}})\}_r$ s.t. $\lim_{r\to\infty} \|\underline{x^{(r)}}\| = 0$. We have $\lim_{r\to\infty} S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = \displaystyle\int_a^b f(x)dx$.
We denote $\displaystyle\int_a^b f(x)dx = \int_a^b f$.

Proof. Let $\{(\underline{y^{(r)}}, \underline{\zeta^{(r)}})\}_r$ be a sequence of tagged partitions of $[a, b]$ s.t. $\|\underline{y^{(r)}}\| \to 0$.

We claim that $\{S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}})\}_r$ converges. We prove that $\{S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}})\}_r$ is Cauchy.

Let $\epsilon > 0$. By Theorem 6.1, $\exists \delta > 0$ s.t. $\forall 0 < \eta < \delta$, $\omega_f(\eta) < \frac{\epsilon}{2(b-a)}$.
Since $\|\underline{y^{(r)}}\| \to 0$, $\exists M_0 \in \mathbb{R}$ s.t. $\forall r \geq M_0$, $\|\underline{y^{(r)}}\| < \delta$ and thus $\forall r \geq M_0$, $\omega_f(\|\underline{y^{(r)}}\|) < \frac{\epsilon}{2(b-a)}$.
Choose $M = M_0$.
Then $\forall r, r' \geq M = M_0$, $|S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}})| \leq [\omega_f(\|\underline{y^{(r)}}\|) + \omega_f(\|\underline{y^{(r')}}\|)](b-a) < \left(\frac{\epsilon}{2(b-a)} + \frac{\epsilon}{2(b-a)}\right)(b-a) = \epsilon$.
Thus $\{S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}})\}_r$ is Cauchy and thus converges.
Let $I = \lim\limits_{r\to\infty} S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}})$.

Let $\{(\underline{x^{(r)}}, \underline{\xi^{(r)}})\}_r$ be any sequence of tagged partitions with $\lim\limits_{r\to\infty} \|\underline{x^{(r)}}\| = 0$.
Claim: $\lim\limits_{r\to\infty} S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = I$.

We have by triangle inequality that

$$\begin{aligned} |S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) - I| &\leq |S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) - S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}})| + |S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}}) - I| \\ &\leq [\omega_f(\|\underline{x^{(r)}}\|) + \omega_f(\|\underline{y^{(r)}}\|)](b-a) - |S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}}) - I| \text{ (By Theorem 6.3)} \end{aligned}$$

Since ω_fs converges to 0 and $I = \lim\limits_{r\to\infty} S_f(\underline{y^{(r)}}, \underline{\zeta^{(r)}})$, by squeeze theorem, $\lim\limits_{r\to\infty} |S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) - I| = 0$.
Thus $\lim\limits_{r\to\infty} S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = I$. □

Theorem: 6.5: Linearity of Riemann Integral

If $f, g \in C([a, b])$ and $\alpha \in \mathbb{R}$, then $\int_a^b (\alpha f + g) = \alpha \int_a^b f + \int_a^b g$.

Proof. Let $\{(\underline{x^{(r)}}, \underline{\xi^{(r)}})\}_r$ be a sequence of tagged positions with $\|\underline{x^{(r)}}\| \to 0$.
Then $S_{\alpha f+g}(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = \alpha S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) + S_g(\underline{x^{(r)}}, \underline{\xi^{(r)}})$
As $r \to \infty$, $\int_a^b (\alpha f + g) = \alpha \int_a^b f + \int_a^b g$. □

Theorem: 6.6: Additivity of Riemann Integrals

If $f \in C([a, b])$ and $a < c < b$, then $\int_a^b f = \int_a^c f + \int_c^b f$.

Proof. Let $\{(\underline{y^{(r)}}, \underline{\eta^{(r)}})\}_r$ be a sequence of tagged partitions of $[a, c]$ s.t. $\|\underline{y^{(r)}}\| \to 0$,
and $\{(\underline{z^{(r)}}, \underline{\zeta^{(r)}})\}_r$ be a sequence of tagged partitions of $[c, b]$ s.t. $\|\underline{z^{(r)}}\| \to 0$.

Define $\underline{x^{(r)}} = \underline{y^{(r)}} \cup \underline{z^{(r)}}$, $\underline{\xi^{(r)}} = \underline{\eta^{(r)}} \cup \underline{\zeta^{(r)}}$. Then $\{(\underline{x^{(r)}}, \underline{\xi^{(r)}})\}_r$ is a sequence of tagged partitions of $[a, b]$.
Note $\|\underline{x^{(r)}}\| = \max(\|\underline{y^{(r)}}\|, \|\underline{z^{(r)}}\|) \to 0$ as $r \to \infty$.
Then by Theorem 6.4, $S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) \to \int_a^b f$, $S_f(\underline{y^{(r)}}, \underline{\eta^{(r)}}) \to \int_a^c f$, $S_f(\underline{z^{(r)}}, \underline{\zeta^{(r)}}) \to \int_c^b f$.
Since $S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = S_f(\underline{y^{(r)}}, \underline{\eta^{(r)}}) + S_f(\underline{z^{(r)}}, \underline{\zeta^{(r)}})$, we have $\int_a^b f = \int_a^c f + \int_c^b f$. □

Theorem: 6.7: Order Property of Riemann Integrals

Suppose $f, g \in C([a,b])$
1. If $\forall x \in [a,b]$, $f(x) \le g(x)$, then $\int_a^b f \le \int_a^b g$
2. Triangle Inequality: $|\int_a^b f| \le \int_a^b |f|$

Proof. 1. Let $\{(\underline{x^{(r)}}, \underline{\xi^{(r)}})\}_r$ be a sequence of tagged partitions of $[a,b]$ s.t. $\|\underline{x^{(r)}}\| \to 0$. Then $\forall r$,

$$\begin{aligned} S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) &= \sum_{j=1}^{n(r)} f(\xi_j^{(r)})(x_j^{(n)} - x_{j-1}^{(n)}) \\ &\le \sum_{j=1}^{n(r)} g(\xi_j^{(r)})(x_j^{(n)} - x_{j-1}^{(n)}) = S_g(\underline{x^{(r)}}, \underline{\xi^{(r)}}) \end{aligned}$$

As $r \to \infty$, we have $\int_a^b f \le \int_a^b g$

2. $\pm f \le |f| \Rightarrow \int_a^b \pm f \le \int_a^b |f| \Rightarrow$ (By Theorem 6.5) $\pm \int_a^b f \le \int_a^b |f| \Rightarrow -\int_a^b |f| \le \int_a^b f \le \int_a^b |f| \Rightarrow |\int_a^b f| \le \int_a^b |f|$

□

Remark 29. If $f \in C([a,b])$ and $f > 0$, then $\int_a^b f > 0$.

Theorem: 6.8:

$$\int_a^b 1 = b - a$$

Proof. Let $\{(\underline{x^{(r)}}, \underline{\xi^{(r)}})\}_r$ s.t. $\|\underline{x^{(r)}}\| \to 0$.

Then $S_1(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = \sum_{j=1}^{n(r)} (x_j^{(r)} - x_{j-1}^{(r)}) = x_n^{(r)} - x_0^{(r)} = b - a$

Thus $\displaystyle\int_a^b 1 = \lim_{r\to\infty} S_1(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = b - a.$ □

Theorem: 6.9: Bounds of Riemann Integrals

If $f \in C([a,b])$, $m_f = \inf\{f(x) : x \in [a,b]\}$. $M_f = \sup\{f(x) : x \in [a,b]\}$, then $m_f(b-a) \le \int_a^b \le M_f(b-a)$.

Proof. Since $m_f \le f(x) \le M_f$, $\forall x \in [a,b]$, then $\int_a^b m_f \le \int_a^b f \le \int_a^b M_f$ by Theorem 6.7.
Thus $m_f(b-a) \le \int_a^b f \le M_f(b-a)$ □

Remark 30. 1. $\int_a^a f = 0$

2. If $b < a$, $\int_a^b f = -\int_b^a f$

Proof. 1 is consistent with $\lim_{b \to a} \int_a^b f = 0$.
2 is consistent with Theorem 6.6 and $0 = \int_a^a f = \int_a^b f + \int_b^a f$. ☐

6.2 Fundamental Theorem of Calculus

Theorem: 6.10: Fundamental Theorem of Calculus

Let $f \in C([a,b])$. Then

1. If $F : [a,b] \to \mathbb{R}$ is differentiable and $F' = f$, then $\int_a^b f = F(b) - F(a)$. *i.e.* $\int_a^b F' = F(b) - F(a)$
2. The function $G(x) = \int_a^x f : [a,b] \to \mathbb{R}$ is differentiable on $[a,b]$ and $\begin{cases} G' = f \\ G(a) = 0 \end{cases}$

Proof.

1. Let $\{\underline{x^{(r)}}\}_r$ be a sequence of points with $\|\underline{x^{(r)}}\| \to 0$
   By Theorem 5.10, $\forall j$, $\exists \xi_j^{(r)} \in (x_{j-1}^{(r)}, x_j^{(r)})$ s.t. $F(x_j^{(r)}) - F(x_{j-1}^{(r)}) = f(\xi_j^{(r)})(x_j^{(r)} - x_{j-1}^{(r)})$.
   $$S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = \sum_{j=1}^{n(r)} f(\xi_j^{(r)})(x_j^{(r)} - x_{j-1}^{(r)}) = \sum_{j=1}^{n(r)} F(x_j^r) - F(x_{j-1}^{(r)}) = F(x_{n(r)}^{(r)}) - F(x_0^{(r)}) = F(b) - F(a)$$
   Thus $\int_a^b f = \lim_{r \to \infty} S_f(\underline{x^{(r)}}, \underline{\xi^{(r)}}) = F(b) - F(a)$.

2. let $c \in [a,b]$, we want to show $\lim_{x \to c} \frac{\int_a^x f - \int_a^c f}{x - c} = f(c)$
   Let $\epsilon > 0$. Since f is continuous at c, $\exists \delta_0 > 0$ s.t. $|t - c| < \delta \Rightarrow |f(t) - f(c)| < \frac{\epsilon}{2}$.
   Choose $\delta = \delta_0$.
   Suppose $c < x < c + \delta$. If $t \in [c, x]$, $|t - c| = t - c \le x - c < \delta = \delta_0$, then
   $$\begin{aligned} \left| \frac{1}{x-c} \left(\int_a^x f(t)dt - \int_a^c f(t)dt \right) - f(c) \right| &= \left| \frac{1}{x-c} \int_c^x f(t)dt - f(c) \right| \\ &= \left| \frac{1}{x-c} \int_c^x f(t)dt - \frac{f(c)}{x-c} \int_c^x dt \right| \\ &= \frac{1}{x-c} \left| \int_c^x [f(t) - f(c)]dt \right| \\ &\le \frac{1}{x-c} \int_c^x |f(t) - f(c)|dt \text{ (By Theorem 6.7)} \\ &\le \frac{1}{x-c} \int_c^x \frac{\epsilon}{2} dt = \frac{\epsilon}{2} \frac{1}{x-c}(x-c) = \frac{\epsilon}{2} < \epsilon \end{aligned}$$
   Similar proofs can be applied to the other case.
   Thus $0 < |x - c| < \delta \Rightarrow \left| \frac{\int_a^x f - \int_a^c f}{x-c} - f(c) \right| < \epsilon$

☐

6.3 Integration Techniques

Theorem: 6.11: Integration By Parts

Suppose $f, g \in C([a,b])$ are continuously differentiable, *i.e.* $f', g' \in C([a,b])$. Then

$$\int_a^b f'g = f(b)g(b) - f(a)g(a) - \int_a^b fg'$$

Proof. Since $(fg)' = f'g - fg'$ by Theorem 5.6.
Then by Theorem 6.10,

$$\int_a^b f'g + fg' = \int_a^b (fg)' = f(b)g(b) - f(a)g(a)$$

□

Theorem: 6.12: Change of Variable

Let $\phi : [a, b] \to [c, d]$ be continuously differentiable s.t. $\phi' > 0$ on $[a, b]$, $\phi(a) = c$ and $\phi(b) = d$. Then,

$$\int_c^d f(u)du = \int_a^b f(\phi(x))\phi'(x)dx$$

Proof. Let $F : [a, b] \to \mathbb{R}$ s.t. $F' = f$ (always exists by Theorem 6.10). Then $[F(\phi(x))]' = F'(\phi(x))\phi'(x) = f(\phi(x))\phi'(x)$ by Theorem 5.7.
Thus

$$\int_a^b f(\phi(x))\phi'(x)dx = \int_a^b [F(\phi(x))]'dx = F(\phi(b)) - F(\phi(a)) - F(d) - F(c) = \int_c^d F'(u)du = \int_c^d f(u)du$$

□

7 Sequence of Functions

7.1 Motivation

Fourier Series: Suppose $f : [-\pi, \pi] \to \mathbb{R}$ is 2π periodic. Can $f(x) = \sum_{n=0}^{infty} [a_n \sin nx + b_n \cos nx]$ *i.e.* Can a series of sine and cosine function converge to f?

Analogs: suppose $\vec{x} = (x_1, ..., x_m) \in \mathbb{R}^m$, $\vec{x} = \sum_{n=1}^{m} a_n e_n$, where e_n is the n-th orthonormal basis vector.

To compute a_n, we compute $\vec{x} \cdot \vec{e_l} = \sum_{n=1}^{m} a_n e_n \cdot e_l = \sum_{n=1}^{m} a_n \delta_{nl} = a_l$. *i.e.* $a_l = x \cdot e_l$.

Back to functions. $\int_{-\pi}^{\pi} f(x) \sin lx dx = \int_{-\pi}^{\pi} \sum_{n=0}^{\infty} [a_n \sin nx \sin lx + b_n \cos nx \sin lx] dx$.

If we can switch the limit process $\int_{-\pi}^{\pi}$ and $\sum_{n=0}^{\infty}$, we get

$$\int_{-\pi}^{\pi} f(x) \sin lx dx = \sum_{n=0}^{\infty} a_n \int_{-\pi}^{\pi} \sin nx \sin lx dx + b_n \int_{-\pi}^{\pi} \cos nx \sin lx dx = \sum_{n=0}^{\infty} a_n \pi \delta_{nl} = \pi a_l$$

Similarly, we get $\pi b_l = \int_{-\pi}^{\pi} f(x) \cos lx dx$.

***Definition:* 7.1: Fourier Coefficients**

If $f : [-\pi, \pi] \to \mathbb{R}$ is continuous and 2π-periodic, the numbers $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx$ and $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx$ are the Fourier coeffcients of f.

***Theorem:* 7.1: Riemann Lebesgue Lemma**

If $f : [a, b] \to \mathbb{R}$ is continuously differentiable, then $\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = 0$

Proof. We will show $b_n \to 0$, as a_n is similar.

$$\begin{aligned} b_n &= \int_{-\pi}^{\pi} \cos nx f(x) dx = \int_{-\pi}^{\pi} (-\frac{1}{n} \sin nx)' f(x) dx \\ &= \frac{1}{n} [\sin n\pi f(\pi) - \sin n(-\pi) f(-\pi)] - \frac{1}{n} \int_{-\pi}^{\pi} \sin nx f'(x) dx \text{ (By IBP)} \\ &= -\frac{1}{n} \int_{-\pi}^{\pi} \sin nx f'(x) dx \end{aligned}$$

Then

$$\begin{aligned} 0 \le |b_n| \le \left| \frac{1}{n} \int_{-\pi}^{\pi} \sin nx f'(x) dx \right| & \qquad \le \frac{1}{n} \int_{-\pi}^{\pi} |\sin nx||f'(x)| dx \text{ (By Theorem 6.7)} \\ \le \frac{1}{n} \int_{-\pi}^{\pi} |f'(x)| dx & \end{aligned}$$

$\frac{1}{n}\int_{-\pi}^{\pi} |f'(x)|dx \to 0$ as $n \to \infty$ since $\int_{-\pi}^{\pi} |f'(x)|dx < \infty$.
Thus $b_n \to 0$ by squeeze theorem. □

7.2 Pointwise and Uniform Convergence

Definition: 7.2: Power Series

A power series about x_0 is a series of the form $\sum_{j=0}^{\infty} a_j(x-x_0)^j$

Theorem: 7.2: Convergence of Power Series

Suppose $R = \lim_{j\to\infty} |a_j|^{1/j}$ exists and define $\rho = \begin{cases} \frac{1}{R}, R > 0, \\ \infty, R = 0 \end{cases}$. Then $\sum a_j(x-x_0)^j$ converges absolutely if $|x-x_0| < \rho$ and diverges if $|x-x_0| > \rho$. ρ is called the radius of convergence.

Proof.

$$\lim_{j\to\infty} |a_j(x-x_0)^j|^{1/j} = |x-x_0| \lim_{j\to\infty} |a_j|^{1/j} = |x-x_0|R \begin{cases} < 1, |x-x_0| < \rho \\ > 1, |x-x_0| > \rho \end{cases}$$

The Theorem then follows the Root Test (Theorem 3.37). □

If the power series converges absolutely, we then define $f : (x_0-\rho, x_0+\rho) \to \mathbb{R}$ by $f(x) = \sum_{j=0}^{\infty} a_j(x-x_0)^j$.

Example: $\sum_{j=0}^{\infty} x_j = \frac{1}{1-x}$ for $x \in (-1,1) \setminus \{0\}$

Example: Let $a_j = \frac{1}{j!}$, $x_0 = 0$. Then $\sum_{j=0}^{\infty} \frac{1}{j!}x^j = e^x$ has radius of convergence $\rho = \infty$.

Let $f(x) = \lim_{n\to\infty} f_n(x)$, where $f_n(x) = \sum_{j=0}^{mn} a_j(x-x_0)^j$, $\forall x \in (x_0-\rho, x_0+\rho)$. We have the following questions:

1. Is f continuous?
2. If 1, then is f differentiable? and does $f' = \lim_{n\to\infty} f_n'$?
3. If 1), does $\int_a^b f = \lim_{n\to\infty}\int_a^b f_n$?

Definition: 7.3: Pointwise Convergence

For $n \in \mathbb{N}$, let $f_n : S \to \mathbb{R}$ and let $f : S \to \mathbb{R}$. We say $\{f_n\}$ converges pointwise to f, if $\forall x \in S$, $\lim_{n\to\infty} f_n(x) = f(x)$.
i.e. for each point x, the sequence $f_n(x)$ converges to $f(x)$.

Example: $f(x) = \frac{1}{1-x}$, $f_n(x) = \sum_{j=0}^{n} x_j$, then $\forall x \in (-1,1) \setminus \{0\}$, $\lim_{n\to\infty} f_n(x) = f(x)$

Example: $f_n(x) = x^n$, $x \in [0,1]$
If $x = 1$, $\lim_{n\to\infty} f_n(x) = 1$.
If $x \in (0,1]$, then $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = 0$.
Thus $\forall x \in [0,1]$, $\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} 0, x \in [0,1) \\ 1, x = 1 \end{cases}$

Example: Let $f_n : [0,1] \to \mathbb{R}$, $f_n(x) = \begin{cases} 4n^2x, x \in [0, \frac{1}{2n}] \\ 4n - 4n^2x, x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0, x \in [\frac{1}{n}, 1] \end{cases}$. Then $\forall x \in [0,1]$, $f_n(x) \to 0$

Proof. If $x = 0$, then $\lim_{n\to\infty} f_n(0) = \lim_{n\to\infty} 0 = 0$
Suppose $x \in (0,1]$, we want to show $\lim_{n\to\infty} f_n(x) = 0$
Let $M \in \mathbb{N}$ s.t. $\frac{1}{M} < x$. Then $\{f_n(x) = \{f_1(x), ..., f_{M-1}(x), f_M(x) = 0, 0, ...\}\}$. $f_n(x) = 0$, $\forall n \geq M$. Thus $f_n(x) \to 0$. ☐

Definition: 7.4: Uniform Convergence

For $n \in \mathbb{N}$, let $f_n : S \to \mathbb{R}$ and $f : S \to \mathbb{R}$. Then we say $\{f_n\}$ converges uniformly to $f(x)$ if $\forall \epsilon > 0$, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $\forall x \in S$, $|f_n(x) - f(x)| < \epsilon$.
i.e. For any point $x \in S$, the approximated value $f_n(x)$ is always within ϵ distance from $f(x)$

Remark 31. The negation: f_n does not converge uniformly to f ($f_n \not\to f$ uniformly) on S if $\exists \epsilon_0 > 0$ s.t. $\forall M \in \mathbb{N}$, $\exists n \geq M$, $\exists x \in S$ s.t. $|f_n(x) - f(x)| \geq \epsilon_0$.

Theorem: 7.3:

If $f_n : S \to \mathbb{R}$, $f : S \to \mathbb{R}$ and $f_n \to f$ uniformly on S, then $f_n \to f$ pointwise on S.

Proof. Let $c \in S$. We want to show $\lim_{n\to\infty} f_n(c) = f(c)$.
Let $\epsilon > 0$, since $f_n \to f$ uniformly, $\exists M_0 \in \mathbb{N}$ s.t. $\forall n \geq M_0$, $\forall x \in S$, $|f_n(x) - f(x)| < \epsilon$.
Choose $M = M_0$, $x = c$. Then $\forall n \geq M$, $|f_n(c) - f(c)| < \epsilon$. Thus $\lim_{n\to\infty} f_n(c) = f(c)$. ☐

Theorem: 7.4:

Let $f_n(x) = x^n$, $f(x) = \begin{cases} 0, x \in [0,1) \\ 1, x = 1 \end{cases}$. Then

1. $\forall 0 \leq b < 1$, $f_n \to f$ uniformly on $[0,b]$
2. $f_n \not\to f$ uniformly on $[0,1]$.

Proof. 1. Let $b \in [0,1)$. Then $b^n \to 0$. Let $\epsilon > 0$, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $b^n < \epsilon$.
Then $\forall n \geq M$, $\forall x \in [0,b]$, $|f_n(x) - f(x)| = |x^n - 0| = x^n \leq b^n < \epsilon$

2. Choose $\epsilon_0 = \frac{1}{4}$. Let $M \in \mathbb{N}$.
Choose $n = M$, $x = \left(\frac{1}{4}\right)^{1/M} < 1$. Then $f(x) = 0$, but $f_M(x) = \left(\left(\frac{1}{4}\right)^{1/M}\right)^M = \frac{1}{4}$.
$|f_M(x) - f(x)| = \frac{1}{4} \geq \epsilon_0$.

□

Example: $f_n(x) = \begin{cases} 4n^2 x, x \in [0, \frac{1}{2n}] \\ 4n - 4n^2 x, x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0, x \in [\frac{1}{n}, 1] \end{cases}$. $f_n(x) \to 0$ pointwise, but $f_n \not\to 0$ uniformly on $[0,1]$

Proof. Choose $\epsilon_0 = 1$. let $M \in \mathbb{N}$.
Choose $n = M$, $x = \frac{1}{2M}$. Then $|f_M(x) - 0| = f_M\left(\frac{1}{2M}\right) = 2M \geq 1 = \epsilon_0$. □

7.3 Interchange of Limits

Example:

$$\lim_{k\to\infty}\left[\lim_{n\to\infty}\frac{n/k}{n/k+1}\right] = \lim_{k\to\infty} 1 = 1$$
$$\lim_{n\to\infty}\left[\lim_{k\to\infty}\frac{n/k}{n/k+1}\right] = \lim_{n\to\infty} 0 = 0$$

We cannot interchange the limit in this case.

We have the following questions:

1. Suppose $f_n : S \to \mathbb{R}$, $f : S \to \mathbb{R}$ and $f_n \to f$ (pointwise or uniformly), and f_n is continuous $\forall n$. Then is f continuous? *i.e.* Suppose $x \in S$ and $\exists$ sequence $\{x_k\}$ s.t. $x_k \to x$, then can we do $\lim_{k\to\infty} f(x_k) = \lim_{k\to\infty}\lim_{n\to\infty} f_n(x_k) = \lim_{n\to\infty}\lim_{k\to\infty} f_n(x_k) = \lim_{n\to\infty} f_n(x) = f(x)$?
2. Suppose $f_n : [a,b] \to \mathbb{R}$ is differentiable $\forall n$. $f : [a,b] \to \mathbb{R}$ and $f_n \to f$, $f_n' \to g$. Is f differentiable and is $g = f'$?
3. Suppose $f_n \in C([a,b])$, $f \in C([a,b])$ and $f_n \to f$. Does $\lim_{n\to\infty}\int_a^b f_n = \int_a^b f$?

The answer is yes if we have uniform convergence and no if we only have pointwise convergence.

Example: $f_n(x) = x^n$ on $[0,1]$. $f(x) = \begin{cases} 0, x \in [0,1) \\ 1, x = 1 \end{cases}$. $\forall n$, $f_n \in C([0,1])$, $f_n \to f$ pointwise, but $f \notin C([0,1])$.

Example: $f_n(x) = \frac{x^n}{n}$ on $[0,1]$. Then $f_n \to f = 0$, $f_n' \to g(x) = \begin{cases} 0, x \in [0,1) \\ 1, x = 1 \end{cases}$ pointwise, but $g \neq f'$.

Example: $f_n(x) = \begin{cases} 4n^2x, x \in [0, \frac{1}{2n}] \\ 4n - 4n^2x, x \in [\frac{1}{2n}, \frac{1}{n}] \\ 0, x \in [\frac{1}{n}, 1] \end{cases}$. $f_n(x) \to f = 0$ pointwise. $\forall n$, $\int_0^1 f_n = \frac{1}{2}\frac{1}{n}2n = 1$, but $\int_0^1 f_n \not\to \int_0^1 f = 0$.

***Theorem:* 7.5:**

Suppose $f_n : S \to \mathbb{R}$, $f : S \to \mathbb{R}$, f_n continuous $\forall n$ and $f_n \to f$ uniformly on S. Then f is continous.

Proof. Let $c \in S$, $\epsilon > 0$. Since $f_n \to f$ uniformly, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $\forall y \in S$, $|f_M(y) - f(y)| < \frac{\epsilon}{3}$.
Since $f_M : S \to \mathbb{R}$ is continous, $\exists \delta > 0$ s.t. $\forall |x - c| < \delta \Rightarrow |f_M(x) - f_M(c)| < \frac{\epsilon}{3}$.
Then $\forall |x - c| < \delta$, using triangle inequality:

$$|f(x) - f(c)| \leq |f(x) - f_M(x)| + |f_M(x) - f_M(c)| + |f_M(c) - f(c)| < \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon$$

□

***Theorem:* 7.6:**

Suppose $f_n \in C([a,b])$, $f : [a,b] \to \mathbb{R}$ and $f_n \to f$ uniformly. Then $\lim_{n\to\infty} \int_a^b f_n = \int_a^b f$.

Proof. Let $\epsilon > 0$. Since $f_n \to f$ uniformly, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $\forall x \in [a,b]$, $|f_n(x) - f(x)| < \frac{\epsilon}{b-a}$.
Then $\forall n \geq M$, by Theorem 6.7,

$$\left|\int_a^b f_n - \int_a^b f\right| = \left|\int_a^b (f_n - f)\right| \leq \int_a^b |f_n - f| < \int_a^b \frac{\epsilon}{b-a} = \frac{\epsilon}{b-a}(b-a) = \epsilon$$

□

***Theorem:* 7.7:**

Suppose $f_n : [a,b] \to \mathbb{R}$ is continuously differentiable $\forall n$. $f, g : [a,b] \to \mathbb{R}$. $f_n \to f$ pointwise on $[a,b]$ and $f_n' \to g$ uniformly on $[a,b]$. Then f is continuously differentiable and $g = f'$.

Proof. Let $x \in [a,b]$. Then by Theorem 6.10 and Theorem 7.6,

$$f_n(x) - f_n(a) = \int_a^x f_n' \Rightarrow f(x) - f(a) = \lim_{n\to\infty}[f_n(x) - f_n(a)] = \lim_{n\to\infty}\int_a^x f_n' = \int_a^x g$$

Then $f(x) = f(a) + \int_a^x g$, f is differentiable and $f' = (\int_a^x g)'$ □

Theorem: 7.8: Weierstrass M-test

Let $f_j : S \to \mathbb{R}$ and suppose $\exists\{M_j\}$ s.t.

a) $\forall x \in S$, $|f_j(x)| \le M_j$

b) $\sum_{j=1}^{\infty} M_j < \infty$

Then

1. $\forall x \in S$, $\sum_{j=1}^{\infty} f_j(x)$ converges absolutely
2. Let $f(x) = \sum_{j=1}^{\infty} f_j(x)$. Then $\sum_{j=1}^{n} f_j \to f$ uniformly on S as $n \to \infty$.

Proof. 1 follows directly from a), b) and comparison test (Theorem 3.34).

Let $\epsilon > 0$. Since $\sum M_j$ converges, $\exists N \in \mathbb{N}$ s.t. $\sum_{j=N+1}^{\infty} M_j = \sum_{j=1}^{\infty} M_j - \sum_{j=1}^{N} M_j < \epsilon$.

Then $\forall n \ge N$, $\forall x \in S$, $\left| f(x) - \sum_{j=1}^{n} f_j(x) \right| = \| \sum_{j=n+1}^{\infty} f_j(x) \| \le \sum_{j=n+1}^{\infty} |f_j(x)| \le \sum_{j=n+1}^{\infty} M_j \le \sum_{j=N+1}^{\infty} M_j < \epsilon$ □

Example: $f_j(x) = \frac{\cos 160^j x}{4^j}$, $x \in \mathbb{R}$. Then

1. $|f_j(x)| \le 4^{-j}$
2. $\sum_{j=1}^{\infty} 4^{-j}$ converges

Thus $\sum_{j=1}^{\infty} f_j(x)$ converges uniformly on $\mathbb{R}$.

7.4 Power Series

Theorem: 7.9: Uniform Convergence of Power Series

Let $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ be a power series with radius of convergence $\rho = \left(\lim_{j\to\infty} |a_j|^{1/j} \right)^{-1} \in (0, \infty]$. Then $\forall r \in [0, \rho)$, $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ converges uniformly on $[x_0 - r, x_0 + r]$.

Proof. Let $r \in [0, \rho)$. Then $\forall j \in \mathbb{N} \cup \{0\}$, $\forall x \in [x_0 - r, x_0 + r]$, $|a_j(x-x_0)^j| \le |a_j||x-x_0|^j \le |a_j|r^j$.

We have $\lim_{j\to\infty} [|a_j|r^j]^{1/j} = r \lim_{j\to\infty} |a_j|^{1/j} = \begin{cases} \frac{r}{\rho}, \rho < \infty \\ 0, \rho = \infty \end{cases} < 1$.

Thus $\sum_{j=0}^{\infty} |a_j|r^j$ converges.

By Theorem 7.8, $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ converges uniformly on $[x_0 - r, x_0 + r]$. □

Theorem: 7.10: Differentiation and Integration of Power Series

Let $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ be a power series with radius of convergence $\rho \in (0,\infty]$. Then

1. $\forall c \in (x_0-\rho, x_0+\rho)$, $\sum_{j=0}^{\infty} a_j(x-x_0)^j$ is differentiable at c and $\frac{d}{dx}\sum_{j=0}^{\infty} a_j(x-x_0)^j\Big|_{x=c} = \sum_{j=1}^{\infty} \frac{d}{dx}(a_j(x-x_0)^j)\Big|_{x=c}$
2. $\forall a, b$ with $x_0-\rho < a < b < x+\rho$, $\int_a^b \sum_{j=0}^{\infty} a_j(x-x_0)^j dx = \sum_{j=0}^{\infty}\int_a^b a_j(x-x_0)^j dx$.

Proof. Claim: $\sum_{j=0}^{\infty} \frac{d}{dx}a_j(x-x_0)^j = \sum_{j=0}^{\infty} a_{j+1}(j+1)(x-x_0)^j$ has radius of convergence ρ.

$$\lim_{j\to\infty} |a_{j+1}(j+1)|^{1/j} = \lim_{j\to\infty}\left[|a_{j+1}|^{\frac{1}{j+1}}(j+1)^{\frac{1}{j+1}}\right]^{\frac{j+1}{j}} = \lim_{j\to\infty}\left[|a_{j+1}|^{\frac{1}{j+1}}\right]^{\frac{j+1}{j}} = (\rho^{-1})^1 = \rho^{-1}$$

Thus, the radius of convergence of $\sum_{j=0}^{\infty} a_{j+1}(j+1)(x-x_0)^j$ is $\frac{1}{\rho^{-1}} = \rho$. □

Remark 32. $\forall x \in (x_0-\rho, x_0+\rho)$, $\frac{d^k}{dx^k}\sum_{j=0}^{\infty} a_j(x-x_0)^j = \sum_{j=0}^{\infty}\frac{d^k}{dx^k}a_j(x-x_0)^j$, $\forall k = 1, 2, ...$

Theorem: 7.11:

$\forall n \in \mathbb{N}$, define $c_n = (\int_{-1}^{1}(1-x^2)^n dx)^{-1} > 0$, $Q_n(x) = c_n(1-x^2)^n$. Then
1. $\forall n \in \mathbb{N}$, $\forall x \in [-1,1]$, $Q_n(x) \geq 0$
2. $\forall n \in \mathbb{N}$, $\int_{-1}^{1} Q_n(x)dx = 1$
3. $\forall \delta \in (0,1)$, $Q_n \to 0$ uniformly on $\{x : \delta \leq |x| \leq 1\}$

Remark 33. Q_n is like a delta function as $n \to \infty$

Proof. 1, 2 are immediate, we prove 3 only.
Firstly, we estimate c_n.
Let $g(x) = (1-x^2)^n - (1-nx^2)$, $x \in [0,1]$. Then $g(0) = 0$, $g'(x) = 2nx(1-(1-x^2)^{n-1}) \geq 0$ on $[0,1]$. Thus $g(x) \geq 0$ on $[0,1]$. *i.e.* $(1-x^2)^n \geq 1-nx^2$.

$$\begin{aligned}
\frac{1}{c_n} &= \int_{-1}^{1}(1-x^2)^n dx = 2\int_0^1 (1-x^2)^n dx \text{ (even function)}\\
&\geq 2\int_0^{\frac{1}{\sqrt{n}}}(1-x^2)^n dx \text{ (By Theorem 6.6)}\\
&\geq 2\int_0^{\frac{1}{\sqrt{n}}}(1-nx^2)dx\\
&= \frac{4}{3}\frac{1}{\sqrt{n}} > \frac{1}{\sqrt{n}}
\end{aligned}$$

Thus $c_n < \sqrt{n}$.

Let $\delta \in (0,1)$. Note that $\sqrt{n}(1-\delta^2)^n \to 0$ as $n \to \infty$ ($\lim_{n\to\infty} [\sqrt{n}(1-\delta^2)^n]^{1/n} = \lim_{n\to\infty} [n^{1/n}]^{1/2}(1-\delta^2) = 1-\delta^2 < 1$)

Let $\epsilon > 0$. Then $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $\sqrt{n}(1-\delta^2)^n < \epsilon$. Then $\forall n \geq M$, $\forall x$ s.t. $\delta \leq |x| \leq 1$, we have $Q_n(x) = c_n(1-x^2)^n \leq \sqrt{n}(1-\delta^2)^n < \epsilon$.

Thus $Q_n(x) \to 0$ uniformly. □

Theorem: 7.12: Weierstrass Approximation Theorem

If $f \in C([0,1])$, then $\exists$ sequence of polynomials $\{P_n(x)\}$ s.t. $P_n \to f$ uniformly on $[0,1]$.

Remark 34. We only consider the case $f(0) = 0$, $f(1) = 0$. If we prove this case, $\forall \tilde{f} \in C([0,1])$, $\exists \{P_n\}$ s.t. $P_n \to \tilde{f}(x) - \tilde{f}(0) - x[\tilde{f}(1) - \tilde{f}(0)]$ uniformly.
Then $P_n(x) + x[\tilde{f}(1) - \tilde{f}(0)] + \tilde{f}(0) \to \tilde{f}(x)$ uniformly and LHS is still a polynomial.

Proof. Suppose $f \in C([0,1])$, $f(0) = 0$, $f(1) = 0$. Extend f by 0 outside $[0,1]$. Then $f \in C(\mathbb{R})$.
Define

$$\begin{aligned} P_n(x) &= \int_0^1 f(t)Q_n(t-x)dt = \int_0^1 f(t)c_n(1-(x-t)^2)^n dt \\ &= \int_0^1 f(t)c_n \sum_{j=0}^{n} \binom{n}{j}(-1)^j(x-t)^{2j}dt \text{ (Binomial theorem on } (1-(x-t)^2)^n) \\ &= \int_0^1 f(t)c_n \sum_{j=0}^{n}\sum_{k=0}^{2j} \binom{n}{j}(-1)^j\binom{2j}{k}(-t)^k x^{2j-k}dt \text{ (Binomial theorem on } (x-t)^{2j}) \end{aligned}$$

Note $P_n(x) = \int_0^1 f(t)Q_n(t-x)dt = \int_{-x}^{1-x} f(x+t)Q_n(t)dt$ by change of variable $(t = t - x)$.

Since $f(x+t) = 0$ for $t \notin [-x, 1-x]$, we have $P_n(x) = \int_{-x}^{1-x} f(x+t)Q_n(t)dt = \int_{-1}^{1} f(x+t)Q_n(t)dt$

Since $Q_n(t)$ is approximately $\delta(t)$, $\int_{-1}^{1} f(x+t)Q_n(t)dt \to \int_{-1}^{1} f(x+t)\delta(t)dt = f(x+0) = f(x)$

We now prove $P_n \to f$ uniformly on $[0,1]$.
Let $\epsilon > 0$. Since $f \in C([0,1])$, f is uniformly continuous by Theorem 4.20, thus $\exists \delta > 0$ s.t. $\forall z, y$, $|z-y| < \delta$, $|f(z) - f(y)| < \frac{\epsilon}{2}$.
Since $f \in C([0,1])$, $\exists c > 0$ s.t. $|f(x)| \leq c$ for all $x \in [0,1]$. Then $|f(x+t) - f(x)| \leq 2c$ for $x, x+t \in [0,1]$ by triangle inequality.
Since $\sqrt{n}(1-\delta^2)^n \to 0$, then $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $\sqrt{n}(1-\delta^2)^n < \frac{\epsilon}{8C}$.

Thus, $\forall n \geq M$, $\forall x \in [0,1]$,

$$\begin{aligned}
|P_n(x) - f(x)| &= \left| \int_{-1}^{1} f(x+t)Q_n(t)dt - f(x) \right| \\
&= \left| \int_{-1}^{1} [f(x+t) - f(x)]Q_n(t)dt \right| \text{ (Because } \int_{-1}^{1} Q_n(x)dx = 1) \\
&\leq \int_{-1}^{1} |f(x+t) - f(x)|Q_n(t)dt \text{ (By Theorem 6.7 and that } Q_n(t) \geq 0) \\
&= \int_{-\delta}^{\delta} |f(x+t) - f(x)|Q_n(t)dt + \int_{\delta \leq |x| \leq 1} |f(x+t) - f(x)|Q_n(t)dt \text{ (By Theorem 6.6)} \\
&< \int_{-\delta}^{\delta} \frac{\epsilon}{2} Q_n(t) + \int_{\delta \leq |x| \leq 1} 2cQ_n(t)dt \\
&< \frac{\epsilon}{2} \int_{-\delta}^{\delta} Q_n(t) + \int_{\delta \leq |x| \leq 1} 2cc_n(1-\delta^2)^n dt \\
&< \frac{\epsilon}{2} + 2c\sqrt{n}(1-\delta^2)^n \int_{-1}^{1} dt \\
&= \frac{\epsilon}{2} + 4c\sqrt{n}(1-\delta^2)^n \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

Thus $P_n \to f$ uniformly on $[0,1]$. $\square$

8 Metric Spaces

8.1 Introduction

Definition: 8.1: Euclidean Distance

Given $x, y \in \mathbb{R}$, the Euclidean distance is

$$\|x-y\|_{\mathbb{R}^n} = \left(\sum_{i=1}^{n} |x_i - y_i|^2\right)^{1/2}$$

With the following properties:
1. Symmetric: $\|x-y\| = \|y-x\|$
2. Positive definite: $\|x-y\| \geq 0$ and $\|x-y\| = 0 \Leftrightarrow x = y$
3. Triangle inequality: $\|x-z\| \leq \|x-y\| + \|y-z\|$

Definition: 8.2: Metric Space

A metric space is a set X with function $d : X \times X \to [0, \infty)$ with the following properties:
1. Symmetric: $d(x, y) = d(y, x)$
2. Positive definite: $d(x, y) \geq 0$ and $d(x, y) = 0 \Leftrightarrow x = y$
3. Triangle inequality: $d(x, z) \leq d(x, y) + d(y, z)$

Example: $d_\infty \mathbb{R}^n \times \mathbb{R}^n \to [0, \infty)$, $d_\infty(x, y) = \max\limits_{1\leq i\leq n} |x_i - y_i|$ is a metric on $\mathbb{R}^n$.

Proof. Symmetric: $d_\infty(x, y) = \max\limits_{1\leq i\leq n} |x_i - y_i| = \max\limits_{1\leq i\leq n} |y_i - x_i| = d_\infty(y, x)$
Positive: because all terms in the sum is positive
If $d_\infty(x, y) = 0$, then $\forall i, |x_i - y_i| = 0$, by definition of absolute values, thus $x = y$. If $x = y$, then $\forall i, |x_i - y_i| = 0$, thus $d_\infty(x, y) = 0$
Triangle inequality: let $x, y, z \in \mathbb{R}^n$, $d_\infty(x, y) = \max\limits_{1\leq i\leq n} |x_i - y_i|$, $d_\infty(x, z) = \max\limits_{1\leq i\leq n} |x_i - z_i|$, $d_\infty(y, z) = \max\limits_{1\leq i\leq n} |y_i - z_i|$.
Since the dimension is finite, $\exists j$ s.t. $d_\infty(x, z) = |x_j - z_j| \leq |x_j - y_j| + |y_j - z_j| \leq \max_{1\leq i\leq n} |x_i - y_i| + \max_{1\leq i\leq n} |y_i - z_i| \leq d_\infty(x, y) + d_\infty(y, z)$. □

Definition: 8.3: l^p metrics

For $1 \leq p < \infty$, the l^p metrics are defined as:

$$d_p(x, y) = \left(\sum_{i=1}^{n} |x_i - y_i|^p\right)^{1/p}$$

Example: $d_X(x, y) = \begin{cases} 1, x \neq y \\ 0, x = y \end{cases}$ is a metric.

Proof. The tricky part is the triangle inequality. We consider the following three cases:

1. $x \neq y$, $y \neq z$, $z \neq y$: $d(x, z) = 1 \leq 2 = d(x, y) + d(y, z)$
2. $x = y \neq z$, $d(x, z) = 1 \leq 1 = d(x, y) + d(y, z)$

3. $x = y = z$, $d(x, z) = 0 \leq 0 = d(x, y) = d(y, z)$

□

Example: $x, y \in \mathbb{R}^2$. Then $d(x, y) = \begin{cases} \|x - y\|_{\mathbb{R}^2}, x, y \text{ colinear} \\ \|x\| + \|y\|, \text{ otherwise} \end{cases}$ is a metric.

Definition: 8.4: Convergent Sequence

Let $\{x_n\}$ be a sequence in a metric space (X, d) and let $x \in X$. $\{x_n\}$ converges to x if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(x_n, x) < \epsilon$.

Definition: 8.5: Cauchy Sequence

Let $\{x_n\}$ be a sequence in a metric space (X, d). $\{x_n\}$ is Cauchy if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$, $d(x_n, x_m) < \epsilon$.

Definition: 8.6: Open Sets

A set $A \subset X$ is open if $\forall a \in A$, $\exists \epsilon > 0$ s.t. $B_\epsilon(a) = B(a, \epsilon) = \{y \in X : d(x, y) < \epsilon\} \subset A$.

Example: $(0, 1)$ is open in $\mathbb{R}$.

Definition: 8.7: Continuous Functions

Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$. f is continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

Definition: 8.8: Set of Countinuous Functions

We define $C^0([a, b])$ to be the set of continuous functions on $[a, b]$.

Example: $f, g \in C^0([a, b])$, $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)|$ is a metric.

Proof. Symmetric: $d(f, g) = \sup_{x \in [a,b]} |f(x) - g(x)| = \sup_{x \in [a,b]} |g(x) - f(x)| = d(g, d)$
Positive: if $f = g$, then $f(x) = g(x)$, $\forall x$. Thus $d(f, g) = 0$
If $d(f, g) = 0$, then $f(x) - g(x) = 0$, $\forall x$, thus $f = g$.
Triangle inequality: Let $f, g, h \in C^0([a, b])$. $d(f, h) = \sup_{x \in [a,b]} |f(x) - h(x)| = |f(y) - h(y)|$ for some $y \in [a, b]$ by Theorem 4.15. Thus $d(f, h) = |f(y) - h(y)| \leq |f(y) - g(y)| + |g(y) - h(y)| \leq \sup_{x \in [a,b]} |f(x) - g(x)| + \sup_{x \in [a,b]} |g(x) - h(x)| = d(f, g) + d(g, h)$. □

Definition: 8.9: Set of k-differentiable Functions

We define $C^k([a, b])$ to be the set of continuous functions on $[a, b]$ s.t. the first k derivatives of f exists and are continuous.

Example: For $C^1([a,b])$, we can define the metric $d_{C^1}(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| + \sup_{x\in[a,b]} |f'(x)-g'(x)|$

Remark 35. The same can be applied to any $C^k([a,b])$ for finite ks. The first term $\sup |f-g|$ must be presented to ensure positive definite.

Definition: **8.10:**

We define $C^\infty([a,b])$ as the set of infinitely differentiable functions. The metric is

$$d_{C^\infty}(f,g) = \sum_{k=0}^{\infty} 2^{-k} \frac{d_{C^k}(f,g)}{1+d_{C^k}(f,g)}$$

Example: the map $\frac{d}{dx} : C^1([a,b]) \to C^0([a,b])$ is continuous as a function between metric spaces.

Proof. Let $f,g \in C^1([a,b])$, $\epsilon > 0$. We want to show $\exists \delta > 0$ s.t. $d_{C^1}(f,g) < \epsilon \Rightarrow d_{C^0}(f',g') < \epsilon$.
$d_{C^1}(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)| + \sup_{x\in[a,b]} |f'(x)-g'(x)|$
$d_{C^0}(f,g) = \sup_{x\in[a,b]} |f(x)-g(x)|$
Let $\delta = \epsilon$, then $d_{C^1}(f,g) < \delta \Rightarrow d_{C^0}(f',g') = \sup_{x\in[a,b]} |f'(x)-g'(x)| \leq d_{C^1}(f,g) < \delta = \epsilon$. $\square$

Definition: **8.11: L^p metrics**

For $1 \leq p < \infty$, $I_p(f,g) = \left(\int_0^1 |f-g|^p\right)^{1/p}$ defines a metric on $C^0([0,1])$ called the L^p metric.

Example: The map $I_1 : C^0([0,1]) \times C^0([0,1]) \to [0,\infty)$ s.t. $I_1(f,g) = \int_0^1 |f-g|$ is a metric.

Proof. Symmetric: $I_1(f,g) = \int_0^1 |f-g| = \int_0^1 |g-f| = I_1(g,f)$
Positive: If $f = g$, then $|f-g| = 0$, thus $I_1(f,g) = 0$. If $f \neq g$, then $I_1(f,g) \neq 0$ by continuity of f and g.
Triangle inequality: $\int_0^1 |f-h| \leq \int_0^1 (|f-g| + |g-h|) = \int_0^1 |f-g| + \int_0^1 |g-h|$ (By Theorem 6.7 and Theorem 6.6) $\square$

Definition: **8.12: Geodesics**

For spheres $S = \{x \in \mathbb{R}^n : \|x\| = 1\}$. We define a metric on the ball as the shortest line segment between two points on the sphere, which we call geodesics.

8.2 General Theory

Theorem: **8.1: Uniqueness of Limits**

Let $\{x_n\}$ be a sequence. Suppose $x_n \to x$, then x is unique.

Proof. Assume $\exists y \neq x \in X$ s.t. $x_n \to y$.
Let $\epsilon > 0$. Since $x_n \to x$, $\exists N_1 \in \mathbb{N}$ s.t. $\forall n \geq N_1$, $d(x_n, x) < \frac{\epsilon}{2}$.
Similarly, for y, $\exists N_2 \in \mathbb{N}$ s.t. $\forall n \geq N_2$, $d(x_n, y) < \frac{\epsilon}{2}$.

Take $N = \max(N_1, N_2)$, then $\forall n \geq N$, $d(x, y) \leq d(x_n, x) + d(x_n, y) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
Thus $x = y$. $\square$

Theorem: 8.2:

Let $\{x_n\}$ be a sequence, $x_n \to x$, $y \in X$. Then $d(x_n, y) \to d(x, y)$.

Proof. Let $\epsilon > 0$. Since $x_n \to x$, $\exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $d(x_n, x) < \epsilon$. Then $\forall n \geq N$ $d(x_n, y) \leq d(x_n, x) + d(x, y) < d(x, y) + \epsilon$.
On the other hand $d(x, y) = d(x, x_n) + d(x_n, y)$, so $d(x_n, y) = d(x, y) - d(x_n, x) > d(x, y) - \epsilon$.
Thus $d(x, y) - \epsilon < d(x_n, y) < d(x, y) + \epsilon$. *i.e.* $d(x_n, y) \to d(x, y)$. $\square$

Theorem: 8.3:

Let $\{x_n\}$, $\{y_n\}$ be sequences.

1. Suppose $x_n \to x$, $y_n \to y$, then $d(x_n, y_n) \to d(x, y)$
2. Suppose $\{x_n\}$, $\{y_n\}$ are Cauchy, then $d(x_n, y_n)$ converges.

Theorem: 8.4:

Convergent sequences are Cauchy.

Proof. Let $\{x_n\}$ be a sequence.
Let $\epsilon > 0$. Suppose $x_n \to x$, $\exists n \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(x_n, x) < \frac{\epsilon}{2}$.
Let $m \geq N$, $d(x_m, x_n) \leq d(x_m, x) + d(x_n, x) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. $\square$

Definition: 8.13: Cauchy Complete Space

A space is Cauchy complete $\Leftrightarrow$ all Cauchy sequences are convergent in the space.

Example: $C^0([0, 1])$ is Cauchy complete.

Definition: 8.14: Bounded Sequence and Sets

$\{x_n\}$ is bounded by $B > 0$ if $\exists p \in X$ s.t. $\forall n \in \mathbb{N}$, $d(x_n, p) < B$
A set $A \subset X$ is bounded by $B > 0$ if $\exists p \in X$ s.t. $\forall a \in A$, $d(a, p) < B$.

Theorem: 8.5:

If $x_n \to x$, then $\{x_n\}$ is bounded.

Proof. Let $\epsilon = 1 > 0$.
Since $x_n \to x$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $d(x_n, x) < \epsilon = 1$.
Let $B = \max\{d(x_1, x), ..., d(x_{N-1}, x), 1\}$. Then $d(x_n, x) < B$. $\square$

Theorem: 8.6:

Let $x_n \to x$, and $\{x_{n_k}\}$ a subsequence of $\{x_n\}$, then $\{x_{n_k}\}$ is convergent.

Proof. Let $\epsilon > 0$. We want to show that $\exists N \in \mathbb{N}$ s.t. $\forall n_k \geq N$, $d(x_{n_k}, x) < \frac{\epsilon}{2}$.

For $m \geq N$, $d(x_{n_k}, x) \leq d(x_{n_k}, x_m) + d(x_m, x) < \epsilon$ by Thereom 8.4 and triangle inequality. □

Theorem: 8.7: Topological Properties of Open Sets

Let X be a metric space and A_i be open sets in X. Then

1. $\emptyset$ and X are open in X
2. $\cup_{i=1}^{\infty} A_i$ is open in X (infinite union of open sets is open)
3. $\cap_{i=1}^{n} A_i$ is open in X (finite intersetion of open sets is open)

Proof.

1. $\emptyset$ is open sice $B_\epsilon(x) \subset \emptyset$ has no element. For X. $\forall x \in X$, we can pick $\epsilon > 0$ s.t. $B_\epsilon(x) \subset X$.
2. Pick $x \in \cup_{i=1}^{\infty} A_i$, then $\exists A_j \in \{A_i\}$ s.t. $x \in A_j$. Since A_j is open, $\exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset A_j \subset x \in \cup_{i=1}^{\infty} A_i$.
3. Pick $x \in \cap_{i=1}^{n} A_i$, then $x \in A_i$, $\forall i$. Thus $\exists \epsilon_i > 0$ s.t. $B_{\epsilon_i}(x) \subset A_i$.
   Choose $\epsilon = \min\{\epsilon_i\} > 0$. ϵ always exists because the intersection is finite. $B_\epsilon(x) \subset B_{\epsilon_i}(x) \subset A_i$, $\forall i$.
   Thus $B_\epsilon(x) \subset \cap_{i=1}^{n} A_i$.

□

Definition: 8.15: Closed Sets

$A \subset X$ is closed in X if $A^C = X \setminus A$ is open in X.

Note: Closed sets can be open as well.
Example: In $\mathbb{R}$, $\emptyset^C = \mathbb{R}$, so $\emptyset$ is closed, but $\emptyset$ is open at the same time as shown in Theorem 8.7.

Definition: 8.16: Limit Points

Suppose $A \subset X$. $x_0 \in X$ is a limit point of A if $\forall \epsilon > 0$, $B_\epsilon(x_0)$ contains infinitely many points in X.

Definition: 8.17: Connected Metric Space

A metric space X is disconnected if $\exists U_1, U_2$ that are disjoint, non-empty and open s.t. $X = U_1 \cup U_2$.
A metric space X is connected if it is not disconnected.

Note: A metric space X is connected $\Leftrightarrow$ The only sets that are both open and closed are $\emptyset$ and X.

Example: $X = (0,1) \cup (1,2)$ with usual metric on $\mathbb{R}$ is disconnected. $(0,1)$ is both open and closed in X.

Theorem: 8.8: Generalized De Morgan's Law

1. $(\cup_{i \in I} U_i)^C = \cap_{i \in I} U_i^C$
2. $(\cap_{i \in I} U_i)^C = \cup_{i \in I} U_i^C$

Theorem: 8.9: Topological Properties of Closed Sets

Let X be a metric space and A_i be closed sets in X. Then

1. $\emptyset$ and X are closed in X
2. $\cup_{i=1}^{n} A_i$ is closed in X (finite union of closed sets is closed)
3. $\cap_{i=1}^{\infty} A_i$ is open in X (infinite intersetion of closed sets is closed)

Theorem: 8.10:

Given $x \in X$, $\epsilon > 0$. The ball $B_\epsilon(x)$ is open.

Proof. Choose $y \in B_\epsilon(x)$. Notice $d(x, y) < \epsilon$ by definition.
Let $r = \epsilon - d(x, y)$. Then $B_r(y) \subset B_\epsilon(x)$.
Let $z \in B_\delta(y)$, $d(x, z) \leq d(x, y) + d(y, z) < d(x, y) + r < \epsilon$ ☐

Remark 36. Any open set $U \subset X$ can be written as a union of open balls.

Example: Suppose $x \in X$. Then $\{x\}$ is closed.

Proof. Consider $\{x\}^C = X \setminus \{x\}$. Let $y \in X \setminus \{x\}$. We want to show that $\{x\}^C$ is closed. *i.e.* $B_\epsilon(y) \subset \{x\}^C$. Choose $\epsilon = \frac{d(x,y)}{2}$. Suppose $x \in B_\epsilon(y)$. Then $d(x, y) < \epsilon = \frac{d(x,y)}{2}$. Contradiction. ☐

Theorem: 8.11:

Let $\{x_n\}$ be a sequence in $\mathbb{R}$, $x_n \to x \Leftrightarrow \forall \epsilon > 0$, all but finitely many x_i are in $(x - \epsilon, x + \epsilon)$.

Proof. ($\Rightarrow$) Suppose $x_n \to x$. Let $\epsilon > 0$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n \in \mathbb{N}$, $d(x_n, x) < \epsilon$. *i.e.* $x_n \in (x - \epsilon, x + \epsilon)$. Then all but finitely many $(\{x_1, ..., x_{N-1}\})$ are in $(x - \epsilon, x + \epsilon)$.
($\Leftarrow$) Suppose $\forall \epsilon > 0$, all but finitely many x are in $(x - \epsilon, x + \epsilon)$.
Let $\epsilon = \frac{1}{m}$. Choose $x_{n_m} \in (x - \frac{1}{m}, x + \frac{1}{m})$. Then $\forall \epsilon > 0$, we can choose M large enough s.t. $\forall n_m \geq M$, $\frac{1}{n_m} < \epsilon$. Then $x_{n_m} \in (x - \frac{1}{m}, x + \frac{1}{m}) \subset (x - \epsilon, x + \epsilon)$. Thus $|x_{n_m} - x| < \epsilon$. ☐

Theorem: 8.12:

$f : X \to Y$ is continuous at $c \in X \Leftrightarrow$ if $x_n \to c$, then $f(x_n) \to f(c)$.

Proof. ($\Rightarrow$) Suppose f is continuous at c and $x_n \to c$. Let $\epsilon > 0$, s.t. $d_X(x, c) < \delta \Rightarrow d_Y(f(x_n), f(c)) < \epsilon$. Since $x_n \to C$, $\exists N$ s.t. $\forall n \geq N$, $d_X(x_n, c) < \delta$. Therefore, $d_Y(f(x_n), f(c)) < \epsilon$. Thus $f(x_n) \to f(c)$.
($\Leftarrow$) Suppose f is not continuous at c. Let $\epsilon > 0$. Then $\forall n \in \mathbb{N}$, $\exists x_n$ s.t. $d_X(x_n, c) < \frac{1}{n}$, but $d_Y(f(x_n), f(c)) \geq \epsilon$. Then $x_n \to c$, but $f(x_n) \not\to f(c)$. ☐

Definition: 8.18: Neighborhood

Given a metric space (X, d). A neighborhood of a point y is an open set $U \subset X$ s.t. $y \in U$.

Theorem: 8.13: Open Mapping

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is continuous at $c \in X \Leftrightarrow$ for every neighborhood U of $f(c)$ in Y, the set $f^{-1}(U)$ is an open neighborhood of c in X.

Proof. ($\Rightarrow$) Since U is an open neighborhood of $f(c)$, $\exists \epsilon > 0$ s.t. $B_{\epsilon,d_Y}(f(c)) \subset U$.
Since f is continuous, $\exists \delta > 0$ s.t. $d_X(x,c) < \delta \Rightarrow d_Y(f(x), f(c)) < \epsilon$. Thus $f(B_\delta(c)) \subset B_\epsilon(f(c))$. Then $B_\delta(c) \subset f^{-1}(B_\epsilon(f(c))) \subset f^{-1}(U)$.
($\Leftarrow$) Let $\epsilon > 0$. Consider $B_\epsilon(f(c))$, $f^{-1}(B_\epsilon(f(c)))$ is an open neighborhood of c.
$\exists \delta > 0$ s.t. $B_\delta(c) \subset f^{-1}(B_\epsilon(f(c)))$, since $f^{-1}(B_\epsilon(f(c)))$ is open. Then $f(B_\delta(c)) \subset B_\epsilon(f(c))$. □

8.3 Compact Sets

In this section, we consider compact sets in $\mathbb{R}^n$.

Definition: 8.19: Vector Space

A vector space V over a field F is a set with addition ($+ : V \times V \to V$) and scalar multiplication ($\cdot : F \times V \to V$) with properties:

1. Commutativity: $u + v = v + u$ for $u, v \in V$
2. Associativity: $u + (v + w) = (u + v) + w$ for $u, v, w \in V$
3. Identity of addition: $\exists 0 \in V$ s.t. $v + 0 = v$, $\forall v \in V$
4. Inverse of addition: $\forall v \in V$, $\exists -v \in V$ s.t. $v + (-v) = 0$
5. Identity of multiplication: $1v = v$, $\forall v \in V$
6. Compatibility: $a(bv) = (ab)v$ for $a, b \in F$, $v \in V$
7. Distributivity: $a(u + v) = au + av$, $(a + b)v = av + bv$ for $a, b \in F$, $u, v \in V$.

Definition: 8.20: Norm

A norm on a vector space V over $\mathbb{R}$ is a map $\|\cdot\| : V \to [0, \infty)$ with the following properties:

1. Positive Definite: $\|v\| \geq 0 \ \forall v$ and $\|v\| = 0 \Leftrightarrow v = 0$
2. Homogeneity: $\|\lambda v\| = |\lambda| \|v\|$ for $\lambda \in \mathbb{R}$
3. Triangle Inequality: $\|v + w\| \leq \|v\| + \|w\|$

A vector space with a norm on it $(V, \|\cdot\|)$ is defined as a normed space.

Example: For $C^0([0,1])$, define $\|\cdot\| : C^0([0,1]) \to [0, \infty)$ with $\|f\| = \sup\limits_{x \in [0,1]} |f(x)|$.

1. Positive for sure. $\|f\| = 0 = \sup\limits_{x \in [0,1]} |f(x)| \Leftrightarrow f(x) = 0, \forall x$
2. $\|\lambda f\| = \sup\limits_{x \in [0,1]} |\lambda f(x)| = \sup\limits_{x \in [0,1]} |\lambda| |f(x)| = |\lambda| \sup\limits_{x \in [0,1]} |f(x)| = |\lambda| \|f\|$
3. $|f + g| \leq |f| + |g| \leq \sup |f| + \sup |g| = \|f\| + \|g\|$ for all x. Thus $\|f + g\| = \sup |f + g| \leq \|f\| + \|g\|$.

Example: L^1-norm. $\|\cdot\|_{L^1} : C^0([0,1]) \to [0, \infty)$ with $\|f\|_{L^1} = \int_0^1 |f|$.
Note: $\|f\|_{L^1}$ is always finite on $[0,1]$, but $L^1(\mathbb{R}) = \{f : f_{-\infty}^{\infty} |f| < \infty\}$ is finite if $f = 0$ outside of some interval $[-n, n]$. Note this is not if and only if. $e^{-x} \in L^1$.

Definition: 8.21: Support

Consider a function f on $\mathbb{R}$. The support of the function is the closure of $\{x : f(x) \neq 0\} = f^{-1}(\{0\}^C)$, *i.e.* $\overline{\{x : f(x) \neq 0\}}$, where $\{x : f(x) \neq 0\}$ is open and the *closure* is defined as the smallest closed set that contains the open set.

Definition: 8.22: Compact Support

A function $f \in C^0(\mathbb{R})$ has compact support if $f = 0$ outside of some interval $[-n, n]$.

Theorem: 8.14:

Let A be a finite set of metric space (X, d). Then

1. Every sequence in A has a convergent subsequence
2. A is closed and bounded
3. Given any function $f : A \to \mathbb{R}$, f achieves max and min on A and f is bounded.

Proof.

1. Let $\{x_i\}$ be a sequence in A. $\exists x_j \in A$ s.t. x_j appearing infinitely many times as A is finite, but $\{x_j\}$ has infinitely many terms. $x_{n_k} = x_j$ is a convergent subsequence.
2. A set of a single point is closed. A as a finite union of closed set is closed by Theorem 8.9. Fix $p \in A$. Define $B = \max\{d(p, x_i) : x_i \in A\} < \infty$, since max over finite set is well-defined. Thus A is bounded.
3. $f : A \to \mathbb{R}$ has a max and min since $f(A)$ is a finite set in $\mathbb{R}$

□

Definition: 8.23: Cover

A cover of a set A is a collection of sets $\{U_i\}_i$ s.t. $A \subset \cup U_i$. An *open cover* is a cover where all U_is are open.

Definition: 8.24: Compact Set

Let (X, d) be a metric space. $A \subset X$ is *compact* or *topologically compact* if every open cover of A has a finite subcover. $A \subset X$ is *sequentially compact* if every sequence of A has a convergent subsequence.
For A a compact subset of X, we write $A \Subset X$.

Example: $\mathbb{R} \subset \mathbb{R}$ is not compact.

Proof.

1. Choose $\{x_n\}$ s.t. $x_n = n$. x_n diverges and every subseqeunce of x_n diverges, thus $\mathbb{R}$ is not sequentially compact.
2. Consider the open cover $\cup_{n\in\mathbb{N}}(-n, n) \supset \mathbb{N}$. Suppose $\exists n_k$ s.t. $\cup_{k=1}^{m}(-n_k, n_k) \supset \mathbb{R}$. We know that $n_k \notin \cup_{k=1}^{m}(-n_k, n_k)$. Thus not compact.

□

Example: $(0, 1]$ is not compact

Proof.

1. Choose $x_n = \frac{1}{n}$. $x_n \to 0$, $x_{n_k} \to 0 \notin (0, 1]$. Thus it is not sequentially compact.
2. Consider open cover $\cup_{n=1}^{\infty}(\frac{1}{n}, 2) \supset (0, 1]$. If we choose a finite subcover $\cup_{k=1}^{m}(\frac{1}{n_k}, 2]$. Then $\frac{1}{2n_k} \in (0, 1]$ is not covered. Not compact.

□

Example: $[0, 1]$ is compact.

Proof.

1. Consider $\{x_n\} \subset [0,1]$. Theorem 3.18 tells us that there must exist a convergent subsequence
2. Let $[0,1] \subset \cup_{i=1}^{\infty} U_i$ open cover. $c' = \sup\{0 \le c < 1 : [0,c] \text{ has finite sub-cover}\}$ exists.
   Suppose $c' < 1$. Notice $c' \in \cup_{k=1}^{N} U_{i_k}$ is open. Then $\exists \epsilon > 0$ s.t. $B_\epsilon(c') \subset U_i$ for some i. Then there is an element $c' + \frac{\epsilon}{2} > c$, but $c' + + \frac{\epsilon}{2} \in U_i$. Thus $[0, c' + \frac{\epsilon}{2}]$ is covered by finitely many open sets from the cover, which is a contradiction. Thus $c' = 1$.

□

Remark 37. $[a,b]$ is compact in $\mathbb{R}$. $[a,b] \times [c,d]$ is compact in $\mathbb{R}^2$. In $\mathbb{R}^n$, the Cartesian products of closed intervals is compact.

***Theorem:* 8.15:**

Compact sets in (X,d) are closed and bounded.

Proof. Let $A \Subset X$.
Bounded: Fix $p \in X$ and consider $A \subset \cup_{i=1}^{\infty} B_i(p)$. Since A is compact, $A \subset \cup_{i=1}^{M} B_i(p) = B_M(p)$ is bounded.
Closed: we show that $X \setminus A$ is open.
Let $p \in X \setminus A$. Consider $\forall q \in A$. Define $V_q = B\left(p, \frac{d(p,q)}{2}\right)$, $W_q = B\left(q, \frac{d(p,q)}{2}\right)$ balls around p, q that don't intersect.
$A \subset \cup_{q \in A} W_q$. Since A is compact, $\exists$ a subcover s.t. $A \subset \cup_{i=1}^{M} W_{q_i} = \cup_{i=1}^{M} B\left(q_i, \frac{d(p,q_i)}{2}\right)$.
$\cap_{i=1}^{M} B\left(p, \frac{d(p,q_i)}{2}\right)$ does not interset A. Also $p \in \cap_{i=1}^{M} B\left(p, \frac{d(p,q_i)}{2}\right)$. Thus $X \setminus A$ is open. A is closed. □

***Theorem:* 8.16:**

If $F \subset K \Subset X$ is closed, then $F \Subset X$.

Proof. F closed $\Leftrightarrow F^C$ open. Let V_i be an open cover of F. $K \subset \cup_{i=1}^{\infty} U_i \cup F^C$.
$\cup_{i=1}^{\infty} U_i \cup F^C$ covers X, because $\cup_{i=1}^{\infty} U_i$ covers F.
Since K is compact, we can get a finite subcover $F \subset K \subset \cup_{i=1}^{M} U_i \cup F^c$.
Thus $F \subset \cup_{i=1}^{M} U_i$, F is compact in X. □

***Theorem:* 8.17: Heine-Borel**

Let $A \subset \mathbb{R}^n$, A is compact $\Leftrightarrow$ A is closed and bounded

Proof. In Theorem 8.15, we prove the $\Rightarrow$ direction.
($\Leftarrow$) Let $A \subset \mathbb{R}$ be closed and bounded. Then $A \subset [-n, n]$, A is compact by Theorem 8.16. □

Note: If a metric space has closed and bounded $\Rightarrow$ compact, we say the space has Heine-Borel property.

***Theorem:* 8.18: Bolzano-Weierstrass 2**

Let K be a subset of $\mathbb{R}^n$. K is sequentially compact $\Leftrightarrow$ K is closed and bounded.

Proof. (⇐) Let $\{x_n\}$ be sequence in K. Then $\{x_n\}$ is bounded as K is bounded. By Theorem 3.18, there exists a convergent subsequence of $\{x_n\}$. Since K is closed, $\exists x \in K$ s.t. $x_n \to x$. Therefore every sequence in K has a convergent subsequence in K. K is sequentially compact.
(⇒) Let $K \subset \mathbb{R}$ be sequentially compact. Let $\{x_n\}$ be a sequence in K s.t. $x_n \to x \in \mathbb{R}$. Then every subsequence of $\{x_n\}$ converges to x. Therefore $x \in K$. K contains all limit points thus is closed.
Suppose K is unbounded, $\exists \{x_n\}$ in K s.t. $|x_n| \to \infty$. Every subseqeunce of $\{x_n\}$ is unbounded. $\{x_n\}$ has no convergent subseqeunce. Contradiction to the fact that K is sequentially compact. □

8.4 Compact Metric Spaces

Lemma: 8.1: Lebesgue Number Lemma

Let (X, d) be a sequentially compact metric space and $\{U_i\}$ be an open cover of X. Then $\exists r > 0$ s.t. $\forall x \in X$, $B_r(x) \subset U_i$ for some i.

Proof. Assume towards a contradiction that $\forall r > 0$, $\exists x \in X$ s.t. $B_r(x) \not\subset U_i$.
For each $r = \frac{1}{n}$, choose x_n s.t. $B_r(x_n) \not\subset U_i$.
Since (X, d) is sequentially compact, then there exists a subsequence $x_{n_k} \to x \in U_{i_0}$ for some i_0, then $\exists r_0 > 0$ s.t. $B_{r_0}(x) \subset U_{i_0}$.
Choose N sufficiently large s.t. $\frac{1}{N} < \frac{r_0}{2}$ and $d(x_N, x) < \frac{r_0}{2}$.
Consider $B_{\frac{1}{N}}(x_n)$, $\forall y \in B_{\frac{1}{N}}(x_n)$, $d(x, y) \leq d(x_N, x) + d(x_N, y) < \frac{r_0}{2} + \frac{r_0}{2} = r_0$
Thus $B_{\frac{1}{N}}(x_n) \subset B_{r_0}(x) \subset U_{i_0}$. Contradiction. □

Definition: 8.25: Totally Bounded

A metric space (X, d) is totally bounded if $\forall \epsilon > 0$, $\exists y_1, ..., y_k$ s.t. $X \subset \cup_{i=1}^{k} B_\epsilon(y_i)$.

Theorem: 8.19:

If a metric space X is sequentially compact, then X is totally bounded.

Proof. Assume X is sequentially compact, but not totally bounded. *i.e.* $\exists \epsilon > 0$ s.t. there do no exist finitely many ϵ-balls that covers X. Then it takes infinitely many ϵ-balls to cover X.
Let $x_1 \in X$, $x_2 \in X \setminus B_\epsilon(x_1)$,...,$x_n \in X \setminus \cup_{i=1}^{n-1} B_\epsilon(x_i)$. Thus $d(x_i, x_j) > \epsilon$. Then there are no Cauchy subsequences for $\{x_i\}$. Contradiction to X being a sequentially compact metric space. □

Theorem: 8.20:

A metric space X is topologically compact ⇔ X is sequentially compact.

Proof. (⇐) If X is sequentially compact. Let $\{U_i\}$ be an open cover of X. Then by Lemma 8.1, $\exists r > 0$ s.t. there exist finitely many $y_1, ..., y_k$ s.t. $X^C \subset \cup_{i=1}^{k} B_r(y_i) \subset \cup_{j=1}^{k} U_{i_j}$.
(⇒) Suppose X is topologically compact. Assume X is not sequentially compact. *i.e.* $\exists$ sequence $\{x_n\}$ with no convergent subseqeunce. Then

1. None of the x_i can appear infinitely many times, otherwise $x_{n_k} \to x_i$
2. $\exists \epsilon_n > 0$ s.t. $B_{\epsilon_n}(x_j) = \{x_j\}$, otherwise, we get a convergent subsequence as well.

Let $U_0 = X \setminus \{X_i : i = 1, ..., \infty\}$, U_0 is open. $X \subset U_0 \cup \cup_{j=1}^{\infty} B_{\epsilon_n}(x_j)$.
Every finite subscover omits infinitely many points. Thus there is no finite subcover. □

Theorem: 8.21:

Let X, Y be metric spaces, $f : X \to Y$ continuous. Given $K \Subset X$, $f(K) \subset Y$ is compact.

Proof. Let $\{U_i\}$ be an open cover of $f(K) \subset Y$. Then $\{f^{-1}(U_i)\}$ is an open cover of K. Since K is compact in X, we have a finite subcover $\{f^{-1}(U_i)\}_{i=1}^m$. Then $\{U_i\}_{i=1}^m$ is an open cover of $f(K)$. □

Corollary 2. *Let X be a metric space, $K \Subset X$. Then if $f : X \to \mathbb{R}$ is continuous, f achieves a min and a max on K.*

Proof. $f(K) \subset \mathbb{R}$ is compact, thus closed and bounded by Theorem 8.17. We must have the max and min. □

Corollary 3. *Given a compact metric space X, every continuous function $f : X \to \mathbb{R}$ is bounded.*

Proof. $f(K) \subset \mathbb{R}$ is compact, thus bounded by Theorem 8.17. □

Theorem: 8.22: Cantor's Intersection Theorem

If $K_1 \supset K_2 \supset \cdots$ is a decreasing sequence of non-empty sequentially compact subsets of $\mathbb{R}^n$, then $\cap_{i\geq 1} K_i \neq \emptyset$.

Proof. Let $x_i \in K_i$, $\forall i$. x_i exists since K_i is non-empty. Notice $x_i \in K_1$, since $K_1 \supset K_i$ for $i \geq 1$. Then $\forall i$, $\exists$ a convergent subsequence $x_{n_k} \to a \in K_1$. Further $\{x_n\}_{n=2}^\infty$ is a sequence in K_2, thus it contains a convergent subsequence converging to $a \in K_2$. Iterative argument shows that $a \in \cap_{i\geq 1} K_i$. □

Definition: 8.26: Finite Intersection Property

A collection of closed sets $\{C_i\}_i$ has the *finite intersection property* if every finite subcollection has a non-empty intersection.

Theorem: 8.23:

Given a metric space (X, d), the followings are equivalent:

1. X is compact
2. X is sequentially compact
3. X is Cauchy Complete (Definition 8.13) and totally bounded (Definition 8.25)
4. Every collection of closed subsets of X with the finite intersection property has a non-empty intersection.

8.5 Complete Metric Spaces

8.5.1 The Banach Fixed Point Theorem

Definition: 8.27: Lipschitz

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is Lipschitz or k-Lipschitz if $\exists k \in \mathbb{R}$ s.t. $d_Y(f(x), f(y)) \leq k d_X(x, y)$, $\forall x, y \in X$.

Remark 38. If a function f is Lipschitz, then f is continuous. This is called Lipschitz continuous. In fact Lipschitz $\Rightarrow$ uniform continuous.

Proof. Let $\epsilon > 0$ and choose $\delta = \frac{\epsilon}{k}$, $d_Y(f(x), f(y)) \leq k d_X(x, y) < k\delta = \epsilon$. $\square$

Definition: 8.28: Uniform Continuous

Let (X, d_X) and (Y, d_Y) be metric spaces. A function $f : X \to Y$ is uniformly continuous if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$.

Theorem: 8.24:

Suppose $f : X \to Y$ is continuous and X is compact. Then f is uniformly continuous.

Proof. Let $\epsilon > 0$. Since f is continuous, $\forall c \in X$, exists δ_c s.t. $d_X(x, c) < \delta_c \Rightarrow d_Y(f(x), f(c)) < \frac{\epsilon}{2}$.
Since X is compact, the balls $B_{\delta_c}(c)$ convers X. By Lemma 8.1, $\exists \delta > 0$ s.t. $\forall x \in X$, $\exists c \in X$ s.t. $B_\delta(x) \subset B_{\delta_c}(c)$, then $d_X(x, y) < \delta \Rightarrow y \in B_{\delta_c}(c)$.
Then $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) \leq d_Y(f(x), f(c)) + d_Y(f(y), f(c)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$. $\square$

Theorem: 8.25:

Let $f : [a, b] \times [c, d] \to \mathbb{R}$ be continous, then $g(y) = \int_a^b f(x, y)dx$ is continous.

Proof. Let $\{y_n\}$ be a seqeunce in $[c, d]$ s.t. $y_n \to y$, we want to show $g(y_n) \to g(y)$ as by Theorem 8.12. $\lim_{n\to\infty} g(y_n) = \lim_{n\to\infty} \int_a^b f(x, y_n)dx$. Since f is uniformly continous, we can interchange the limits, and get $\lim_{n\to\infty} g(y_n) = \int_a^b \lim_{n\to\infty} f(x, y_n)dx = \int_a^b f(x, y)dx = g(y)$. $\square$

Definition: 8.29: Contraction

A function $f : X \to X$ is a contraction if it is k-Lipschitz for $0 \leq k < 1$. *i.e.* $\exists 0 \leq k < 1$ s.t. $d(f(x), f(y)) \leq k d(x, y)$.

Definition: 8.30: Fixed Point

$f : X \to X$, $x \in X$ is a fixed point for f if $f(x) = x$.

Theorem: 8.26: Banach Fixed Point Theorem/Contraction Mapping Theorem

Let (X, d) be a non-empty Cauchy complete metric space and $f : X \to X$ be a contraction. Then f has a unique fixed point.

Remark 39. This theorem also tells us how to find the fixed point.

Proof. Pick arbitrary $x_0 \in X$. Define $x_{n+1} = f(x_n)$. Then the seqence will be $x_0, f(x_0), f(f(x_0)), \ldots$
Note $d(x_{i+1}, x_i) = d(f(x_i), f(x_{i-1})) \leq k d(x_i, x_{i-1})$, since f is a contraction.
Then $d(x_{i+1}, x_i) \leq k^i d(x_1, x_0)$ by iteration.
By triangle inequality $d(x_n, x_m) \leq \sum_{i=n}^{m-1} d(x_{i+1}, x_i) \leq \sum_{i=n}^{m-1} k^i d(x_1, x_0) = k^n d(x_1, x_0) \sum_{i=0}^{m-n-1} k^i \leq \frac{k^n d(x_1, x_0)}{1-k}$.
Since $0 \leq k < 1$ as a contraction, $k^n \to 0$, then $d(x_m, x_n) \to 0$. Therefore $\{x_n\}$ is Cauchy.
Exsistence of fixed point (limit point): Since X is Cauchy complete and $\{x_n\}$ is Cauchy, $\exists x \in X$ s.t. $x_n \to x$

and $x = \lim_{n\to\infty} f(x_n) = f(\lim_{n\to\infty} x_n) = f(x)$ by definition of $\{x_n\}$.
Uniqueness: Let $y \in X$ s.t. $y = f(y)$. $d(x,y) = d(f(x), f(y)) \le kd(x,y)$ by Definition 8.30 and 8.27.
Then $(1-k)d(x,y) \le 0 \Rightarrow d(x,y) = 0$, *i.e.* $x = y$. □

Example: Let $\lambda \in \mathbb{R}$, $f, g \in C^0([a,b])$, $k \in C^0([a,b] \times [a,b])$. Consider the operator $T : C^0([a,b]) \to C^0([a,b])$ s.t.

$$T(f)(x) = g(x) + \lambda \int_a^b k(x,y) f(x) dx$$

For which λ is T a contraction?

Proof.

$$\begin{aligned}
d(T(f_1)(x), T(f_2)(x)) &= \sup_{x\in[a,b]} \left| \lambda \int_a^b k(x,y)(f_1(x) - f_2(y)) dx \right| \text{ (by Theorem 6.5)} \\
&\le |\lambda| \sup_{x\in[a,b]} \int_a^b |k(x,y)||f_1(x) - f_2(x)| dx \text{ (by Theorem 6.7)} \\
&\le |\lambda| \sup_{x\in[a,b]} |f_1(x) - f_2(x)| \sup_{x\in[a,b]} \int_a^b |k(x,y)| dx \\
&\le |\lambda| d(f_1, f_2) \sup_{x\in[a,b]} \int_a^b |k(x,y)| dx
\end{aligned}$$

Since k is continuous on a compact set, $|k|$ is bounded, $|k| < c$, and $\sup_{x\in[a,b]} \int_a^b |k(x,y)| dx \le c(b-a)$. Thus $d(T(f_1)(x), T(f_2)(x)) \le |\lambda| d(f_1, f_2) c(b-a)$
Therefore, if $|\lambda| < \frac{1}{c(b-a)}$. T is a contraction on a complete metric space.
Also by Theorem 8.26, there exists a unique $f \in C^0([a,b])$ s.t. $T(f)(x) = g(x) + \lambda \int_a^b k(x,y) f(x) dx$. □

Remark 40. If $g \in C^1([a,b])$, then $f \in C^1([a,b])$.

8.5.2 Completion of Metric Spaces

Example: $\mathbb{R}$ is a completion of $\mathbb{Q}$ in the following ways

1. Dedekind cuts (Rudin Ch1 Appendix)
2. Least upper bound property (Definition 2.3)
3. Equivalence classes of Cauchy sequences (we say $\{a_n\} \sim \{b_n\}$ if $|a_n - b_n| \to 0$)

Definition: **8.31: Equivalent Cauchy Sequences**

Two Cauchy sequences in a metric space are equivalent if $|a_n - b_n| \to 0$. The equivalent sequences have the following properties:

1. Reflexivity: $\{a_n\} \sim \{a_n\}$ or equivalently, $|a_n - a_n| = 0$
2. Symmetry: $\{a_n\} \sim \{b_n\} \Leftrightarrow \{b_n\} \sim \{a_n\}$
3. Transitivity: $\{a_n\} \sim \{b_n\}$ and $\{b_n\} \sim \{c_n\} \Rightarrow \{a_n\} \sim \{c_n\}$. ($d(a_n, c_n) \le d(a_n, b_n) + d(b_n, c_n) < \epsilon$ for large n)

Note: We have the same equivalence classes notion on metric spaces.

Lemma: 8.2:

The set $C_\infty(M) = \{f : M \to \mathbb{R} : f \text{ continuous and bounded}\}$ is a metric space with metric $d_\infty(f,g) = \sup_{m\in M} |f(m) - g(m)|$

Theorem: 8.27: Completion

Let (M,d) be a metric space. Then there exists a unique metric space $\bar{M}$ s.t.

1. $M \subset \bar{M}$
2. $d_{\bar{M}} = d_M$
3. $\bar{M}$ is Cauchy Complete
4. The closure of M is $\bar{M}$

Proof. Fix $m' \in M$ and define $g_m(p) = d(p,m) - d(p,m')$. $g_m(p)$ is continuous by Theorem 8.3.
$d(m_1,m_2) = \sup_{p\in M} |g_{m_1}(p) - g_{m_2}(p)| = d_\infty(g_{m_1}, g_{m_2})$, thus g is isometric (distance preserving) and bijective.
Thus $M \subset C_\infty(M)$. Let $\bar{M}$ be the closure of M in $C_\infty(M)$.
Since $\bar{M}$ is a closed subset of a Cauchy complete metric space, $\bar{M}$ is Cauchy complete. □

Example: The completion of Normed spaces is the Banach space. The completion of inner product space is the Hilbert space.
Many functions are not Riemann integrable. Consider $C_C^0(\mathbb{R})$ (compactly supported functions on $\mathbb{R}$ that are continuous). $\forall f \in C_C^0(\mathbb{R})$, $\int_{\mathbb{R}} |f(x)|dx < \infty$.
With $I_1(f,g) = \int_{\mathbb{R}} |f-g|$, $(C_C^0(\mathbb{R}), I_1) = L^1(\mathbb{R}) = \{f : \int |f| < \infty\}$ is the Lebesgue integrable functions.
For $I_p(f,g) = (\int_{\mathbb{R}} |f-g|^p)^{1/p}$, we can define $L^p(\mathbb{R}) = (C_C^0(\mathbb{R}), I_p) = \{f : \int |f|^p < \infty\}$. $L^p(\mathbb{R})$ is complete.

8.6 Relevant Topics

Definition: 8.32: Topology

A topology $\mathcal{T}$ on a set X is a collection of subsets of X s.t.

1. $\emptyset$ and X are in $\mathcal{T}$
2. For $T_i \subset \mathcal{T}$, $\cup_{i=1}^\infty T_i \in \mathcal{T}$
3. For $T_i \subset \mathcal{T}$, $\cap_{i=1}^n T_i \in \mathcal{T}$

A topological space is a set X with $\mathcal{T}$.

Definition: 8.33: open-close-topological-space

set $A \subset X$ is open if $A \in \mathcal{T}$ and closed if $X \setminus A \in \mathcal{T}$.

Note: The topology on a metric space is unions of ϵ-balls.

Definition: 8.34: Metrizable

A topological space X is metrizable if there exists a metric d on X s.t. the topology on X is the topology induced by d.

Definition: 8.35: Neighborhoods and Continuous Functions

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be topological spaces. Then

1. A neighborhood U of a point $x \in X$ is an open set $U \in \mathcal{T}_X$ s.t. $x \in U$.
2. $\{x_n\}$ in X converges to $x \in X$ if for every neighborhood U of x, $\exists N$ s.t. $\forall n \geq N$, $x_n \in U$
3. $f : X \to Y$ is continuous if for each open set $V \in \mathcal{T}_X$, $f^{-1}(V) \in T_X$

Definition: 8.36: Normed Space

Let $(X, \|\cdot\|)$ be a normed space and $\{x_n\}$ be a sequence in X. Let d be the metric induced by the norm. Then

1. $x_n \to x \Leftrightarrow \forall \epsilon > 0$, $\exists N$ s.t. $\forall n \geq N$, $d(x_n, x) = \|x_n - x\| < \epsilon$
2. $\{x_n\}$ is Cauchy $\Leftrightarrow \forall \epsilon > 0$, $\exists N$ s.t. $\forall n, m \geq N$, $d(x_n, x_m) = \|x_n - x_m\| < \epsilon$
3. A set A is open in X if $\forall x \in A$, $\exists \epsilon > 0$ s.t. $B_\epsilon(x) = \{y \in X : d(x, y) = \|x - y\| < \epsilon\} \subset A$

Definition: 8.37: Banach Space

A Banach space is a normed space that is Cauchy complete w.r.t. the norm

Example: $\mathbb{R}^n$, $\mathbb{C}^n$ and $C^0([a, b])$ are Banach spaces. The space

$$C_\infty(X) = \{f : X \to \mathbb{C} : f \text{ continuous and bounded}\}$$

is a Banach space w.r.t. the uniform norm on metric spaces.

Definition: 8.38: Functional

Let $(V, \|\cdot\|)$ be a normed space. A functional is a bounded linear map $f : V \to K$, where $K = \mathbb{R}$ or $\mathbb{C}$.

Example: The set of functionals T are Cauchy complete under operator norm $\|T_{op}\| = \sup\limits_{x \in V, \|x\|=1} |Tx|$ and is a Banach space.

Definition: 8.39: Inner Product Space

An inner product space is a vector space X with an inner product $\langle \cdot, \cdot \rangle : X \times X \to \mathbb{R}$ s.t.

1. Symmetry: $\langle x, y \rangle = \langle y, x \rangle$
2. Linearity: $\langle ax + by, z \rangle = a\langle x, z \rangle + b\langle y, z \rangle$
3. Positive definite: if $x \neq 0$, then $\langle x, x \rangle > 0$. $\langle x, x \rangle^{1/2}$ induces a norm $\|x\|$.

Definition: 8.40: Hilbert Space

A Hilbert space is a Cauchy complete inner product space.

8.7 Additional Definitions

These are some definitions and theorems covered in MIT 18.101, which are not covered in 18.190

Definition: 8.41: Interior

The interior of A is the set $\text{Int}(A) = (\overline{A^C})^C$. $x \in \text{Int}(A) \Leftrightarrow \exists \epsilon > 0$ s.t. $B_\epsilon(x) \subset A$. $\text{Int}(A)$ is open.

Definition: 8.42: Exterior

The exterior of A is the set $\text{Ext}(A) = (\text{Int}(A))^C$.

Definition: 8.43: Boundary

The boundary of A is $\text{Bd}(A) = X \setminus (\text{Int}(A) \cup \text{Ext}(A))$

Note: $X = \text{Int}(A) \cup \text{Ext}(A) \cup \text{Bd}(A)$ for any $A \subset X$.

Theorem: 8.28:

Let (X, d_X) and (Y, d_Y) be metric spaces, $f : X \to Y$ be continuous. If X is connected, then $f(X)$ is connected.

Proof. Assume that X is connected, but $f(X)$ is disconnected. $f(X) = U_1 \cup U_2$, where $U_1 \cap U_2 = \emptyset$ by Definition 8.17.
Then $X = f^{-1}(U_1) \cup f^{-1}(U_2)$ is disjoint union of open sets, since f is continuous. Contradition. □

Theorem: 8.29: Intermediate Value Theorem

Let (X, d) be connected, $f : X \to \mathbb{R}$ be continuous. If $a, b \in f(X)$, and $r \in (a, b)$, then $r \in f(X)$.

Proof. Assume that $r \notin f(X)$, then we can define $A = (-\infty, r)$, $B = (r, \infty)$ s.t. $X = f^{-1}(A) \cup f^{-1}(B)$ as disjoint union of open sets. Contradiction. □

9 Derivatives in Higher Dimensions

9.1 Differentiation in Higher Dimensions

***Definition:* 9.1: Directional Derivative**

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^m$ continuous at $a \in U$, $u \in \mathbb{R}^n$. The *directional derivative* of f in the direction of u at a is $D_u f(a) = \lim_{t\to 0} \dfrac{f(a+tu) - f(a)}{t}$.
For the standard basis vectors $e_1, ..., e_n$ of $\mathbb{R}^n$, we denote the directional derivatives by $D_i f(a) = D_{e_i} f(a) = \dfrac{\partial}{\partial x_i} f(a)$.

***Definition:* 9.2: Differentiable Functions in $\mathbb{R}^n$**

Let $U \subset \mathbb{R}^n$ be an open set, $f : U \to \mathbb{R}^m$ continuous at $a \in U$. f is differentiable at a if there exists a linear map $B : \mathbb{R}^n \to \mathbb{R}^m$ s.t. $\forall h \in \mathbb{R}^n \backslash \{0\}$, $\lim_{h\to 0} \dfrac{f(a+h) - f(a) - Bh}{|h|} = 0$. *i.e.* $f(a+h) - f(a) \approx Bh$ for small h.

Example: $f(x_1, x_2) = \begin{cases} 0, x_1 = 0 \text{ or } x_2 = 0 \\ 1, \text{ otherwise} \end{cases}$ is not differentiable at $(x_1, x_2) = (0, 0)$.

Proof. $\frac{\partial f}{\partial x_1}(0) = \frac{\partial f}{\partial x_2}(0) = 0$, but not differentiable along other directions at $(0, 0)$. □

Example: $f(x, y) = \begin{cases} \frac{xy^2}{x^2+y^4}, (x, y) \neq (0, 0) \\ 0, x = y = 0 \end{cases}$ is not differentiable at $(x, y) = (0, 0)$.

Proof. Let $u = (h, k)$.

$$\lim_{t\to 0} \frac{f(tu) - f(0)}{t} = \lim_{t\to 0} \frac{f(tu)}{t} = \lim_{t\to 0} \frac{t^3 hk^2}{t^2h^2 + t^4k^4} \frac{1}{t} = \begin{cases} 0, h = 0 \\ \frac{k^2}{h}, h \neq 0 \end{cases}$$

The function is a non-zero constant along (t^2, t), $f(t^2, t) = \frac{t^4}{2t^4} = \frac{1}{2}$, except that $f(0, 0) = 0$. Thus not differentiable. □

***Theorem:* 9.1:**

If f is differentiable at a, then for every u, the directional derivative of f in the direction of u at a exists.

Proof. Let $t \in \mathbb{R}$, $h = tu$. If f is differentiable at a, then $\frac{f(a+tu)-f(a)-Btu}{|tu|} \to 0$ as $t \to 0$ for $u \neq 0$.

$$\frac{f(a+tu) - f(a) - Btu}{|tu|} = \frac{t}{|tu|} \frac{f(a+tu) - f(a) - Btu}{t} = \frac{1}{|u|} \left(\frac{f(a+tu) - f(a)}{t} - Bu \right) \to 0 \text{ as } t \to 0$$

So $\frac{f(a+tu)-f(a)}{t} \to Bu$ as $t \to 0$. Also, B is unique, so the directional derivative exists. □

Definition: 9.3: Derivative

The derivative of f at a is $Df(a) = B$. $Df : \mathbb{R}^n \to \mathbb{R}^m$ is linear.

Theorem: 9.2:

If f is differentiable at a, then f is continuous at a.

Proof. If f is differentiable, $\forall h \in \mathbb{R}^n \setminus \{0\}$, $\frac{f(a+h)-f(a)-Bh}{|h|} \to 0$. Thus, $f(a+h) - f(a) - Bh \to 0$ as $h \to 0$. Choose h (in place of δ) s.t. $Bh < \epsilon$, f is continuous by Definition 8.7. $\square$

We now define some short hand notation for matrices and the Jacobian matrix.

Remark 41. Let $L : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and $a \in \mathbb{R}^n$. $a = \sum_{j=1}^{n} a_j e_j = (a_1, ..., a_n)$. The point $La \in \mathbb{R}^m$ can be written as $La = \sum a_j L e_j$. We can decompose $L = (L_1, ..., L_m)$, where $L_j : \mathbb{R}^n \to \mathbb{R}$ are linear. $Le_j = (L_1 e_j, ..., L_m e_j)$. Let $L_i e_j = l_{ij}$, They form a matrix $[l_{ij}] \in \mathbb{R}^{m \times n}$

Remark 42. Let $U \subset \mathbb{R}^n$, $f_1 : \mathbb{R}^n \to \mathbb{R}^{m_1}$, $f_2 : \mathbb{R}^n \to \mathbb{R}^{m_2}$ be differentiable. Let $m = m_1 + m_2$. Then $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} = \mathbb{R}^m$. Construct $f : \mathbb{R}^n \to \mathbb{R}^m$ by $f = (f_1, f_2)$. The derivative of f at a is $Df(a) = (Df_1(a), Df_2(a))$.

Definition: 9.4: Jacobian Matrix

Let $f : U \to \mathbb{R}^m$, $f = (f_1, ..., f_m)$, where $f_i : U \to \mathbb{R}$. $f(x) = (f_1(x), ..., f_m(x))$. Then $Df(a)e_j = (Df_1(a)e_j, ..., Df_m(a)e_j) = \left(\frac{\partial f_1}{\partial x_j}(a), ..., \frac{\partial f_m}{\partial x_j}(a)\right)$. The derivative $(Df)(a)$ can be represented by a $m \times n$ matrix

$$J_f(a) = Df(a) = \left[\frac{\partial f_i}{\partial x_j}(a)\right]$$

Theorem: 9.3:

Suppose all of the partial derivatives $\frac{\partial f_i}{\partial x_j}$ in the Jacobian matrix exist at all points $x \in u$ and all of the partial derivatives are continuous at $x = a$, then f is differentiable at a.

Proof. We consider $n = 2$, $m = 1$ cause. Let $f : U \to \mathbb{R}$ where $U \subset \mathbb{R}^2$, so $f = f(x_1, x_2)$.
Consider $a = (a_1, a_2) \in U$, $h \in \mathbb{R}^2 \setminus \{0\}$ s.t. $a + h \in U$.

$$\begin{aligned} f(a+h) - f(a) &= f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2) \\ &= f(a_1 + h_1, a_2 + h_2) - f(a_1, a_2 + h_2) + f(a_1, a_2 + h_2) - f(a_1, a_2) \\ &= \frac{\partial f}{\partial x_1}(c_1, a_2 + h_2)h_1 + \frac{\partial f}{\partial x_2}(a_1, d_2)h_2 \text{ for some } c_1 \in (a_1, a_1 + h_1), c_2 \in (a_2, a_2 + h_2). \end{aligned}$$

By Theorem 5.10.
Let $c = (c_1, a_2 + h_2)$, $d = (a_1, d_2)$.
We want to show $\frac{f(a+h)-f(a)-Df(a)h}{|h|} \to 0$ as $h \to 0$, where $Df(a) = \left[\frac{\partial f}{\partial x_1}(a), \frac{\partial f}{\partial x_2}(a)\right]$.

$$
\begin{aligned}
|f(a+h)-f(a)-Df(a)h| &= \left|\left(\frac{\partial f}{\partial x_1}(c)-\frac{\partial f}{\partial x_1}(a)\right)h_1+\left(\frac{\partial f}{\partial x_2}(d)-\frac{\partial f}{\partial x_2}(a)\right)h_2\right| \\
&\le \left|\frac{\partial f}{\partial x_1}(c)-\frac{\partial f}{\partial x_1}(a)\right||h_1|+\left|\frac{\partial f}{\partial x_2}(d)-\frac{\partial f}{\partial x_2}(a)\right||h_2| \\
&\text{(By Triangle inequality)}
\end{aligned}
$$

Note $|h| \ge \max(|h_1|,|h_2|)$, so $\left|\frac{f(a+h)-f(a)-Df(a)h}{h}\right| \le \left|\frac{\partial f}{\partial x_1}(c)-\frac{\partial f}{\partial x_1}(a)\right|+\left|\frac{\partial f}{\partial x_2}(d)-\frac{\partial f}{\partial x_2}(a)\right| \to 0$ as $c \to a$, $d \to a$.
By squeeze theorem, $\left|\frac{f(a+h)-f(a)-Df(a)h}{h}\right| \to 0$ □

***Definition:* 9.5:**

Let $U \subset \mathbb{R}^n$, $f : U \to \mathbb{R}$. Define $f \in C^1(U)$ if $\frac{\partial f}{\partial x_i}$, $i = 1, ..., n$ exist and are continuous at every point $x \in U$.
Similarly, we define $f \in C^k$ if $\frac{\partial f}{\partial x_i} \in C^{k-1}(U)$, $i = 1, ..., n$.
$f \in C^\infty(U)$ if $f \in C^k(U)$ for any $k \ge 1$.

***Theorem:* 9.4: Interchanging Partial Derivative**

$$
\frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j} = \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i} = \frac{\partial^2}{\partial x_i \partial x_j}
$$

Proof. Take $a \in U \subset \mathbb{R}^2$, $a = (a_1, a_2)$, $h = (h_1, h_2) \in \mathbb{R}^2 \setminus \{0\}$ s.t. $a + h \in U$.
Define $\Delta(h) = f(a_1+h_1, a_2+h_2) - f(a_1, a_2+h_2) - f(a_1+h_1, a_2) + f(a_1, a_2)$, $\phi(s) = f(a_1+h_1, s) - f(a_1, s)$.
Note that $\Delta(h) = \phi(a_2 + h_2) - \phi(a_2) = \phi'(c_2)h_2$ for $c_2 \in (a_2, a_2 + h_2)$ by Theorem 5.10.

$$
\begin{aligned}
\Delta(h) &= \left(\frac{\partial f}{\partial x_2}(a_1+h_1, c_2) - \frac{\partial f}{\partial x_2}(a_1, c_2)\right)h_2 \\
&= \left(\frac{\partial}{\partial x_1}\left(\frac{\partial f}{\partial x_2}(c_1, c_2)\right)h_1\right)h_2 \text{ for } c_1 \in (a_1, a_1+h) \text{ (By Theorem 5.10)} \\
&= \frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_2}(c)h_1h_2
\end{aligned}
$$

By symmetry, we get $\Delta(h) = \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_1}(d)h_1h_2$.
Thus $\frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_2}(c) = \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_1}(d)$. As $h \to 0$, $c \to a$ and $d \to a$, $\frac{\partial}{\partial x_1}\frac{\partial f}{\partial x_2}(a) = \frac{\partial}{\partial x_2}\frac{\partial f}{\partial x_1}(a)$ for any $a \in U$. □

9.2 Chain Rule

***Theorem:* 9.5: Multivariable Chain Rule**

Let $U, V \subset \mathbb{R}^n$ be open, $f : U \to V$, $g : V \to \mathbb{R}^k$. Choose $a \in U$ and $b = f(a) \in V$. Define $g \circ f : U \to \mathbb{R}^k$ s.t. $g \circ f(x) = g(f(x))$. If f is differentiable at a and g is differentiable at b, then $g \circ f$ is differentiable at a and $(Dg \circ f)(a) = (Dg)(b) \circ Df(a)$.

Proof. Let $h \in \mathbb{R}^n \setminus \{0\}$ s.t. $a + h \in U$.
Let $\Delta(h) = f(a+h) - f(a)$. $F(h) = \frac{f(a+h)-f(a)-Df(a)h}{|h|} = \frac{\Delta(h)-Df(a)h}{|h|}$.

Since f is differentiable at a, $F(h) \to 0$ as $h \to 0$ by Definition 9.2.
Firstly, we show taht $\frac{\Delta(h)}{|h|} = F(h) + \frac{Df(a)h}{|h|}$ is bounded.
Define $|Df(a)| = \sup_i \left|\frac{\partial f}{\partial x_i}(a)\right|$.
We can write $Df(a)h = \sum h_i Df(a)e_i = \sum h_i \frac{\partial f}{\partial x_i}(a)$
Then $|Df(a)h| = \left|\sum h_i \frac{\partial f}{\partial x_i}(a)\right| \le \sum |h_i| \left|\frac{\partial f}{\partial x_i}(a)\right| \le m|h||Df(a)|$ by triangle inequality and $|h_1| \le |h|$.
The sum is converted to multiplication of m because the terms are independent of i.
Thus $\frac{\Delta(h)}{|h|} \le F(h) + m|h||Df(a)|$ is bounded, since all values on the RHS are finite.

Now consider $g : V \to \mathbb{R}^k$ at $b = f(a) \in V$. Let $k \in \mathbb{R}^n \setminus \{0\}$ s.t. $b + k \in V$.
Define $G(k) = \frac{g(b+k)-g(b)-Dg(b)k}{|k|}$. Then $g(b+k) - g(b) = Dg(b)k + |k|G(k)$.

$$\begin{aligned} g \circ f(a+h) - g \circ f(a) &= g(f(a+h)) - g(f(a)) = g(b + \Delta(h)) - g(b) \\ &= Dg(b)\Delta(h) + |\Delta(h)|G(\Delta(h)) \text{ (Let } k = \Delta(h)) \\ &= Dg(b) \circ Df(a)h + |h|Dg(b)F(h) + |\Delta(h)|G(\Delta(h)) \\ &\text{(Substitute in } \Delta(h) = |h|F(h) + Df(a)h) \end{aligned}$$

Then $\frac{g\circ f(a+h) - g\circ f(a) - Dg(b)\circ Df(a)h}{|h|} = Dg(b)F(h) + \frac{\Delta(h)}{|h|}G(\Delta(h)) \to 0$ as $h \to 0$. □

Theorem: 9.6:

If $f : U \to V \subset \mathbb{R}^n$ is C^r and $g : V \to \mathbb{R}^p$ is C^r, then $g \circ f : U \to \mathbb{R}^p$ is C^r.

Proof. Consider $r = 1$, by Theorem 9.5, $Dg \circ f(x) = Dg(f(x)) \circ Df(x) = \left[\frac{\partial g_i}{\partial x_j} Df(x)\right]$. $g \in C^1$, so $\frac{\partial g_i}{\partial x_j}$ is continuous. Also, $Df(x) = \left[\frac{\partial f_i}{\partial x_j}\right]$ is continuous. Thus $Dg \circ f(x)$ is continuous. $g \circ f$ is C^1. Then we can prove by induction on r. □

Theorem: 9.7: Multivariable Mean Value Theorem

Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}$ and $f \in C^1$. For $a \in U$, $h \in \mathbb{R}^n$ s.t. $a + h \in U$, we have $f(a + h) - f(a) = Df(c)h$, where c is a point on the line segment $a + th$, $t \in [0, 1]$.

Proof. Define $\phi : [0, 1] \to \mathbb{R}$, $\phi(t) = f(a + th)$.
Then Theorem 5.10 and Theorem 9.5 implies $\phi(1) - \phi(0) = \phi'(c) = Df(c)h$ for $c \in \{a + th, t \in [0, 1]\}$ □

9.3 Inverse Function Theorem

Definition: 9.6: Euclidean and Supremum Ball

The Euclidean ball is $B_\delta(a) = \{x \in \mathbb{R}^n : \|x - a\| < \delta\}$. The supremum ball is $R_\delta(a) = \{x \in \mathbb{R}^n : |x - a| < \delta\}$. The supremum ball is a rectangular region and $B_\delta(a) \subset R_\delta(a)$.

Notation: If $a = 0$, we simply write B_δ and R_δ.

Definition: 9.7: Convex Set

$U \subset \mathbb{R}^n$ is convex if $a, b \in U \Rightarrow (1 - t)a + tb \in U$ for all $t \in [0, 1]$.

Definition: 9.8: Diffeomorphism

Let U and V be open sets in $\mathbb{R}^n$ and $f : U \to V$ a C^r map. The map f is a C^r diffeomorphism if it is bijective and $f^{-1} : V \to U$ is also C^r.

Lemma: 9.1:

Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}^k$ be C^1. Assume U is convex. If $|Df(a)| \leq c$ for all $a \in U$, then $\forall x, y \in U$, $|f(x) - f(y)| \leq nc|x - y|$.

Proof. Let $x, y \in U$, since U is convex, we can apply Theorem 9.7 to a point d on the line joining x to y in U.
For each component f_i,

$$\begin{aligned}
|f_i(x) - f_i(y)| &= \left| \sum_j \frac{\partial f_i}{\partial x_j}(d)(x_j - y_j) \right| \\
&\leq \sum_j \left| \frac{\partial f_i}{\partial x_j}(d) \right| |x_j - y_j| \text{ (By Triangle inequaltiy)} \\
&\leq \sum_j c|x_j - y_j| \text{ since } |Df(a)| \leq c \\
&\leq nc|x - y|
\end{aligned}$$

This is true for all i, thus $|f(x) - f(y)| \leq nc|x - y|$. □

Lemma: 9.2:

Let $U \subset \mathbb{R}^n$ be open and $f : U \to \mathbb{R}$ be C^1. Suppose f achieves min at $b \in U$. Then $\frac{\partial f}{\partial x_i}(b) = 0$ for $i = 1, ..., n$

Proof. We can reduce to one dimension in each direction.
Let $b = (b_1, ..., b_n)$, $\phi(t) = f(b_1, ..., b_{i-1}, t, b_{i+1}, ..., b_n)$. $\phi(t)$ is C^1 near b_i and has a min at b_i, then $\frac{\partial f}{\partial x_i}(b) = 0$. □

Theorem: 9.8: Inverse Function Theorem

Let $U, V \subset \mathbb{R}^n$ be open, and $f : U \to V$ be C^1. Suppose $g = f^{-1} : V \to U$ is also C^1. *i.e.* $g(f(x)) = x$. Then

$$Dg(b) \circ Df(a) = I, Dg(b) = (Df(a))^{-1}$$

Remark 43. To prove the theorem, we consider a local diffeomorphism. If $Df(a) : U \subset \mathbb{R}^n \to \mathbb{R}^n$ is bijective, then there exists a neighborhood U_1 of a in U and a neighborhood V of $f(a)$ in $\mathbb{R}^n$ s.t. $f(U_1) \subset V$ is a C^r diffeomorphism of U_1.

Proof. We want to show f is a locally diffeomorphism at a, so we need to show that f is bijective and f^{-1} is C^r.

Firstly, we show that f is bijective:

1. (**Injective**) Assume for simplicity $a = 0$, $f(a) = 0$ and $Df(0) = I$.
   Define $g : U \to \mathbb{R}^n$ by $g(x) = x - f(x)$, $Dg(0) = I - Df(0) = 0$, *i.e.* $\exists \delta > 0$ s.t. $\forall x \in R_\delta(0)$ $|Dg(x)| < \frac{\epsilon}{n}$.
   By Lemma 9.1, $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. $\forall x, y \in R_\delta(0)$, $|g(x) - g(y)| \le n|Dg(x)||x - y| < \epsilon|x - y|$.
   Take $x, y \in R_\delta(0)$.
   $|x-y| = |x-f(x)+f(x)-f(y)+f(y)-y| = |g(x)-g(y)+f(x)-f(y)| \le |g(x)-g(y)|+|f(x)-f(y)|$
   $\Rightarrow |x - y| \le \epsilon|x - y| + |f(x) - f(y)| \Rightarrow (1 - \epsilon)|x - y| \le |f(x) - f(y)|$
   Choose δ s.t. $\epsilon > \frac{1}{2}$. $|f(x) - f(y)| \ge \frac{1}{2}|x - y|$. Thus $\forall x \ne y$, $f(x) \ne f(y)$. f is injective.

2. (**Surjective**) Since $Df(0) = I$, $\det\left[\frac{\partial f_i}{\partial x_j}(0)\right] = 1$, we can choose δ s.t. $\forall x \in R_\delta$, $\det\left[\frac{\partial f_i}{\partial x_j}(0)\right] > \frac{1}{2}$.
   Let $y \in B_{\delta/4}$.
   Consider $h : \overline{R_\delta} \to \mathbb{R}$ with $h(x) = \|f(x) - y\|^2$. Since $\overline{R_\delta}$ is compact, h has a minimum at $c \in \overline{R_\delta}$ by Corollary of Theorem 8.21. We now want to show that $c \in \text{Int}(R_\delta)$.
   Consider the boundary points, $x \in \overline{R_\delta}$ s.t. $|x| = \delta$, $|f(x) - f(0)| = |f(x)| \ge \frac{\delta}{2}$.
   Then $\|f(x)\| \ge \frac{\delta}{2}$ and $\|f(x) - y\| \ge \frac{\delta}{4}$, since $y \in B_{\delta/4}$. Then $h(x) \ge (\delta/4)^2$.
   $h(0) = \|f(0) - y\|^2 = \|y\|^2 < (\delta/4)^2$. Then $h(0) \le h(x)$, $\forall x \in \text{Bd}(R_\delta)$. Thus c cannot be in the boundary, and $c \in R_\delta$.
   By Lemma 9.2, $\frac{\partial h}{\partial x_j}(x) = 0$, $\forall j = 1, ..., n$.
   Since $h(x) = \displaystyle\sum_{i=1}^{n}(f_i(c) - y_i)^2$, $\dfrac{\partial h}{\partial x_j}(c) = 2\displaystyle\sum_{i=1}^{n}(f_i(c) - y_1)\dfrac{\partial f_i}{\partial x_j}(c)$.
   However, $\det\left[\frac{\partial f_i}{\partial x_j}(0)\right] > \frac{1}{2} \ne 0$, we must have $f_i(c) - y_i = 0$. *i.e.* $\forall y \in B_{\delta/4}$, $\exists c \in R_\delta$ s.t. $f(c) = y$, so f is surjective.

Thus $f : U_1 = R_\delta \to V = B_{\delta/4}$ is bijective.

Then, we show that $f^{-1} : V \to U_1$ is continuous: Let $a, b \in V$. Define $x = f^{-1}(a)$, $y = f^{-1}(b)$. Then $a = f(x)$ and $b = f(y)$, $|a - b| = |f(x) - f(y)| \ge \frac{1}{2}|x - y|$
Thus $|a - b| \ge \frac{1}{2}|f^{-1}(a) - f^{-1}(b)|$, so f^{-1} is continuous on $V = B_{\delta/4}$.

We show that $f^{-1} : V \to U_1$ is differentiable at 0 and $Df^{-1}(0) = I$.
Let $k \in \mathbb{R}^n \setminus \{0\}$ s.t. $k \to 0$. We want to show $\frac{f^{-1}(0+k)-f^{-1}(0)-Df^{-1}(0)k}{|k|} \to 0$ as $k \to 0$.
Note $f^{-1}(0) = 0$, this simplifies to $\frac{f^{-1}(k)-k}{|k|}$.
Define $h = f^{-1}(k)$ s.t. $f(h) = k$ and $|k| = |k - 0| \ge \frac{1}{2}|f^{-1}(k) - f^{-1}(0)| = \frac{1}{2}|h|$.
Thus $\frac{f^{-1}(k)-k}{|k|} \le \frac{h-f(h)}{\frac{1}{2}|h|}$.
Note that $\frac{h-f(h)}{|h|} = \frac{f(h)-f(0)-Df(0)h}{|h|}$. Since f is differentiable, $\frac{h-f(h)}{|h|} = \frac{f(h)-f(0)-Df(0)h}{|h|} \to 0$.
By squeeze theorem, $\frac{f^{-1}(0+k)-f^{-1}(0)-Df^{-1}(0)k}{|k|} \to 0$ as $k \to 0$. Thus f^{-1} is differentiable.

Now we have shown that there exists a neighborhood $U_1 = R_\delta$ of 0 in U and a neighborhood $V = B_{\delta/4}$ in $\mathbb{R}^n$ s.t. $f : U_1 \to V$ is bijective, $f^{-1} : V \to U_1$ is continuous and f^{-1} is differentiable at 0.

We can shift f from 0 to any arbitrary point $a \in U$ by defining $U' = U - a = \{x - a : x \in U\}$.
Define $f_1 : U' \to \mathbb{R}$ by $f_1(x) = f(x + a) - b$ s.t. $f_1(0) = 0$ and $Df_1(0) = Df(a)$.
Let $A = Df_1(0) = Df(a)$. A is invertible.
Define $f_2 : U' \to \mathbb{R}^n$ by $f_2 = A^{-1}f_1$ s.t. $f_2(0) = 0$ and $Df_2(0) = I$. Thus the results around 0 also hold for f_2.
Because $f_1 = A \circ f_2$, the results also apply to f_1.
Finally, $f(x) = f_1(x - a) + b$, thus the theorem holds for f.

We have now shown that $f : U \to V$ is bijective for any U, V.
Let $c \in U$, $Df(c) = \left[\frac{\partial f_i}{\partial x_j}(c)\right] = J_f(c)$.
Since $Df(c)$ is bijective, $\det \left[\frac{\partial f_i}{\partial x_j}(c)\right] \neq 0$.
$f \in C^1 \Rightarrow \frac{\partial f_i}{\partial x_j}$ continuous on U. If $\det \left[\frac{\partial f_i}{\partial x_j}(c)\right] \neq 0$, then $\det J_f(c) \neq 0$ for $c \to a$.

We can shrink U s.t. $\det J_f(c) \neq 0 \ \forall c \in U$.
Then $\forall c \in U$, f^{-1} is differentiable at $f(c)$.
Let $g = f^{-1}$ s.t. $g \circ f = I$. Suppose $p \in U$ and $q = f(p)$. Then by chain rule, $Dg(q) \circ Df(p) = 1$ and $Dg(q) = Df(p)^-1$, $J_q(q) = J_f(p)^-1$.
i.e. $\forall x \in V$, $\left[\frac{\partial g_i}{\partial x_j}(x)\right] = \left[\frac{\partial f_i}{\partial x_j} g(x)\right]^{-1}$.

Also $g \in C^1$ since $f \in C^1$. And we can show by induction that $g \in C^k$ if $f \in C^k$. $\square$