Differential Forms

This is mainly from introductory level Youtube Video by Michael Penn `https://www.youtube.com/watch?v=PaWjOWxUxGg&list=PL22w63XsKjqzQZtDZO_9s2HEMRJnaOTX7&index=2`.

1 Introduction

Definition: 1.1: Tangent Space

Suppose $C \subset \mathbb{R}^2$ is a curve and $p \in C$. The tangent space to C at p T_pC is the set of all vectors tangent to C at p.

Example: $y = f(x)$, $p = (a, f(a))$.
The tangent vector is $v = \langle 1, f'(a) \rangle$. $T_pC = \text{span}\{\langle 1, f'(a) \rangle\} = \langle c, f'(a) \rangle$, $c \in \mathbb{R}$.

To distinguish between points on $C \subset \mathbb{R}^2$ and vectors in $T_pC \subset \mathbb{R}^2$, we use the following coordinate systems:

Definition: 1.2: Coordinate Systems

On $C \subset \mathbb{R}^2$, $(x:y) : C \to \mathbb{R}^2$, $(x,y)(p) = (x(p), y(p))$. Here $x : C \to \mathbb{R}$, $y : C \to \mathbb{R}$.
On $T_pC \subset \mathbb{R}^2$, $\langle dx, dy \rangle : T_pC \to \mathbb{R}^2$, $\langle dx, dy \rangle(v) = \langle dx(v), dy(v) \rangle$. Here $dx : T_pC \to \mathbb{R}$, $dy : T_pC \to \mathbb{R}$.

Example: $y = x^2$, $(x,y)(p) = (a, a^2)$, $\langle dx, dy \rangle(v) = \langle 1, 2a \rangle$.
Notation: $(x,y) = (a,a^2) \in C$, $\langle dx, dy \rangle = \langle 1, 2a \rangle \in T_pC$.

Example: $\mathbb{R}^2 = \text{span}\{(1,0),(0,1)\} = \{(x,y) : x, y \in \mathbb{R}\}$
$T_p\mathbb{R}^2 = \text{span}\{\langle 1,0 \rangle, \langle 0,1 \rangle\} = \{\langle dx, dy \rangle_p : dx, dy \in \mathbb{R}\}$
$T_q\mathbb{R}^2 = \text{span}\{\langle 1,0 \rangle, \langle 0,1 \rangle\} = \{\langle dx, dy \rangle_q : dx, dy \in \mathbb{R}\}$
We can use subscripts p, q to show the base points.

Definition: 1.3: 1-form

A 1-form is a linear function $\omega : T_p\mathbb{R}^n \to \mathbb{R}$. *i.e.* $\omega : (T_p\mathbb{R}^n)^*$ (dual space of the tangent space)

Example: For $\mathbb{R}^2$ and $T_p\mathbb{R}^2$, $\omega : T_p\mathbb{R}^2 \to \mathbb{R}$ and linear.
Then $\omega(\langle dx, dy \rangle) = adx + bdy = \langle a, b \rangle \cdot \langle dx, dy \rangle = \|\langle a,b \rangle\| \text{proj}_{\langle a,b \rangle} \langle dx, dy \rangle$.

Example: On $\mathbb{R}^n$, $\omega : T_p\mathbb{R}^n \to \mathbb{R}$ gives $\omega(\langle dx_1, ..., dx_n \rangle) = a_1dx_1 + \cdots a_ndx_n$.

Remark 1. A 1-form is a multiple of the scalar projection of $\langle dx, dy \rangle$ onto some line $\langle a, b \rangle$. A line integral is an integral on a 1-form.

Example: Define $\omega(\langle dx, dy\rangle) = 3dx + 2dy.\omega$ projects vectors onto a line with direction $\langle 3, 2\rangle$, *i.e.* $dy = \frac{2}{3}dx$.

Example: Suppose ω scalar projects onto the line $dy = 2dx$ with length 3. Find ω.
$\omega(\langle dx, dy\rangle) = \langle a, b\rangle\langle dx, dy\rangle$, and we need $\langle a, b\rangle \parallel \langle 1, 2\rangle$, so $\langle a, b\rangle = \langle a, 2a\rangle$.
Also, $\|\langle a, 2a\rangle\| = 3$, so $a = \frac{3}{\sqrt{5}}$, $b = \frac{6}{\sqrt{5}}$. $\omega(\langle dx, dy\rangle) = \frac{3}{\sqrt{5}}dx + \frac{6}{\sqrt{5}}dy$.

1.1 Wedge Product and m-forms

Now we want to define a wedge product of 1-forms $\omega_1 \wedge \omega_2$, which is a linear function $\omega_1 \wedge \omega_2 : T_p\mathbb{R}^n \times T_p\mathbb{R}^n \to \mathbb{R}$ that has a meaningful geometric interpretation.

Let $v_1, v_2 \in T_p\mathbb{R}^n$, if we have ω_1 act on v_1 we just get a scalar. Similarly, ω_1 acting on v_2 gives a different scalar. We can now create a vector $\langle \omega_1(v_1), \omega_2(v_1)\rangle$ using these two scalars. We can also create a vector $\langle \omega_1(v_2), \omega_2(v_2)\rangle$ using v_2.

***Definition:* 1.4: Wedge Product**

Define $\omega_1 \wedge \omega_2(v_1, v_2)$ to be the signed area of the parallelogram spanned by $\langle \omega_1(v_1), \omega_2(v_1)\rangle$ and $\langle \omega_1(v_2), \omega_2(v_2)\rangle$. *i.e.*,

$$\omega_1 \wedge \omega_2(v_1, v_2) = \det \begin{bmatrix} \omega_1(v_1) & \omega_2(v_1) \\ \omega_1(v_2) & \omega_2(v_2) \end{bmatrix}$$

Example: $\omega_1 = 3dx - 2dy - dz$, $\omega_2 = dx + 4dy$, $v_1 = \langle 1, 2, -5\rangle$, $v_2 = \langle 0, 3, -2\rangle$.
$\omega_1(v_1) = 3 \cdot 1 - 2 \cdot 2 - (-5) = 4$, $\omega_2(v_1) = 1 + 4 \cdot 2 = 9$
$\omega_1(v_2) = 3 \cdot 0 - 2 \cdot 3 - (-2) = -4$, $\omega_2(v_2) = 00 + 4 \cdot 3 = 12$
$\omega_1 \wedge \omega_2(v_1, v_2) = \det \begin{bmatrix} \omega_1(v_1) & \omega_2(v_1) \\ \omega_1(v_2) & \omega_2(v_2) \end{bmatrix} = \det \begin{bmatrix} 4 & 9 \\ -4 & 12 \end{bmatrix} = 84$

***Theorem:* 1.1: Properties of Wedge Products**

Let $\omega_1, \omega_2, \omega_3, \omega$ be 1-forms.
1. $\omega_1 \wedge \omega_2 = -\omega_2 \wedge \omega_1$
2. $\omega_1 \wedge \omega_2(v_1, v_2) = -\omega_1 \wedge \omega_2(v_2, v_1)$
3. $\omega \wedge \omega = 0$
4. Distributive: $\omega_1 \wedge (\omega_2 + \omega_3) = \omega_1 \wedge \omega_2 + \omega_1 \wedge \omega_3$.

Proof.
1. Suppose $v_1, v_2 \in T_p\mathbb{R}^n$, $\omega_1 \wedge \omega_2(v_1, v_2) = \det \begin{bmatrix} \omega_1(v_1) & \omega_2(v_1) \\ \omega_1(v_2) & \omega_2(v_2) \end{bmatrix} = -\det \begin{bmatrix} \omega_2(v_1) & \omega_1(v_1) \\ \omega_2(v_2) & \omega_1(v_2) \end{bmatrix} = -\omega_2 \wedge \omega_1(v_1, v_2)$
2. $\omega_1 \wedge \omega_2(v_1, v_2) = \det \begin{bmatrix} \omega_1(v_1) & \omega_2(v_1) \\ \omega_1(v_2) & \omega_2(v_2) \end{bmatrix} = -\det \begin{bmatrix} \omega_1(v_2) & \omega_2(v_2) \\ \omega_1(v_1) & \omega_2(v_1) \end{bmatrix} = -\omega_1 \wedge \omega_2(v_2, v_1)$
3. By 1, we know that $\omega \wedge \omega = -\omega \wedge \omega$, so $\omega \wedge \omega = 0$.
4. Suppose $v, w \in T_p\mathbb{R}^n$

$$\omega_1 \wedge (\omega_2 + \omega_3)(v, w) = \det \begin{bmatrix} \omega_1(v) & (\omega_2 + \omega_3)(v) \\ \omega_1(w) & (\omega_2 + \omega_3)(w) \end{bmatrix} = \det \begin{bmatrix} \omega_1(v) & \omega_2(v) + \omega_3(v) \\ \omega_1(w) & \omega_2(w) + \omega_3(w) \end{bmatrix}$$
$$= \det \begin{bmatrix} \omega_1(v) & \omega_2(v) \\ \omega_1(w) & \omega_2(w) \end{bmatrix} + \det \begin{bmatrix} \omega_1(v) & \omega_2(v) \\ \omega_1(w) & \omega_3(w) \end{bmatrix} = (\omega_1 \wedge \omega_2)(v, w) + (\omega_1 \wedge \omega_3)(v, w)$$

□

Theorem: 1.2:

For all 1-forms, $\omega_1, \omega_2 : T_p\mathbb{R}^2 \to \mathbb{R}$, $\omega_1 \wedge \omega_2 = cdx \wedge dy$ for some $c \in \mathbb{R}$.

Proof. Let $\omega_1 = Adx + Bdy$, $\omega_2 = Cdx + Ddy$.

$$\begin{aligned}\omega_1 \wedge \omega_2 &= (Adx + Bdy) \wedge (Cdx + Ddy)\\ &= ACdx \wedge dx + ADdx \wedge dy + BCdy \wedge dx + BDdy \wedge dy\\ &= (AD - BC)dx \wedge dy\end{aligned}$$

Since $dx \wedge dx = dy \wedge dy = 0$ and $dx \wedge dy = -dy \wedge dx$ by Theorem 1.1.

And $AD - BC$ is a constant $c \in \mathbb{R}$. □

***Definition:* 1.5: m-form**

An m-form on $T_p\mathbb{R}^n$ is a function $\omega : (T_p\mathbb{R}^n)^m \to \mathbb{R}$ s.t. ω is multilinear and alternating.

1. Multilinear: Let $u_j \in T_p\mathbb{R}^n$, $v, w \in T_p\mathbb{R}^n$, $a, b \in \mathbb{R}$, $\omega(u_1, ..., u_{i-1}, av + bw, u_{i+1}, ..., u_m) = a\omega(u_1, ..., u_{i-1}, v, u_{i+1}, ..., u_m) + b\omega(u_1, ..., u_{i-1}, w, u_{i+1}, ..., u_m)$
2. Alternating: Suppose $\sigma \in S_m$ (Symmetric group of order m, *i.e.* all permutations on m elements.). Then $\omega(u_{\sigma(1)}, ..., u_{\sigma(m)}) = (-1)^{\text{sgn}(\sigma)}\omega(u_1, ..., u_m)$

Example: $dx \wedge dy$ is a 2-form.

Suppose $v = \langle a_1, a_2\rangle$, $w = \langle b_1, b_2\rangle$, $dx \wedge dy(v, w) = \det \begin{bmatrix} dx(v) & dy(v) \\ dx(w) & dy(w) \end{bmatrix} = \det \begin{bmatrix} v \\ w \end{bmatrix} =$ signed area of parallelogram defined by v and w.
Note: $dx(\langle a_1, a_2\rangle) = a_1$, $dy(\langle a_1, a_2\rangle) = a_2$ for any vector $\langle a_1, a_2\rangle$.

Example: (Alternating) $\omega(u_3, u_2, u_1) = -\omega(u_1, u_2, u_3)$, because $(1, 3) \in S_3$ is an odd permutation (transposition).
$\omega(u_2, u_3, u_1) = \omega(u_1, u_2, u_3)$, because $(1, 2, 3) \in S_3$ is an even permutation (3-cycle).

***Theorem:* 1.3: Construction of m-forms**

Let $\omega_1, ..., \omega_n$ be 1-forms. We can construct a m-form by

$$(\omega_1 \wedge \cdots \wedge \omega_m)(v_1, ..., v_m) = \det \begin{bmatrix} \omega_1(v_1) & \cdots & \omega_m(v_1) \\ \vdots & \ddots & \vdots \\ \omega_1(v_m) & \cdots & \omega_m(v_m) \end{bmatrix}$$

Example: $\omega = 2dx \wedge dy \wedge dz$, $v_1 = \langle 2, -1, 0\rangle$, $v_2 = \langle 1, 2, -1\rangle$, $v_3 = \langle 0, 1, 2\rangle$
$\omega = \omega_1 \wedge \omega_2 \wedge \omega_3$, where $\omega_1 = 2dx$, $\omega_2 = dy$, $\omega_3 = dz$
Then $\omega(v_1, v_2, v_3) = \det \begin{bmatrix} 4 & -1 & 0 \\ 2 & 2 & -1 \\ 0 & 1 & 2 \end{bmatrix} = 4 \cdot 5 - (-1)^2 4 + 0 = 24$

Example: $\omega_1 = dx + 2dy$, $\omega_2 = dx - dz$, $\omega_3 = dx + dy + dz$
$v_1 = \langle 2, 1, 0\rangle$, $v_2 = \langle -1, 3, -2\rangle$, $v_3 = \langle 1, 0, 1\rangle$

$$\omega_1 \wedge \omega_2 \wedge \omega_3(v_1, v_2, v_3) = \det \begin{bmatrix} 2+2\cdot 1 & 2-0 & 2+1+0 \\ -1+2\cdot 3 & -1-(-2) & -1+3+(-2) \\ 1+2\cdot 0 & 1-1 & 1+0+1 \end{bmatrix} = \det \begin{bmatrix} 4 & 2 & 3 \\ 5 & 1 & 0 \\ 1 & 0 & -2 \end{bmatrix}$$
$$= 1(0-3) + 2(4-10) = -3 - 12 = -15$$

Note: The distributive rule from Theorem 1.1 still holds.
Example:

$$\begin{aligned}
&(dx + dy + dz) \wedge (2dx - 3dy) \wedge (dx + 2dz) \\
&= (2dx \wedge dx + 2dy \wedge dx + 2dz \wedge dx - 3dx \wedge dy - 3dy \wedge dy - 3dz \wedge dy) \wedge (dx + 2dz) \\
&= (-5dx \wedge dy - 3dz \wedge dy + 2dz \wedge dx) \wedge (dx + 2dz) \\
&= -5dx \wedge dy \wedge dx - 3dz \wedge dy \wedge dx + 2dz \wedge dx \wedge dx - 10dx \wedge dy \wedge dz \\
&- 6dz \wedge dy \wedge dz + 4dz \wedge dx \wedge dz \\
&= -7dx \wedge dy \wedge dz
\end{aligned}$$

***Theorem:* 1.4:**

Every m-form on $T_p\mathbb{R}^n$ can bet written as

$$\omega = \sum_{1 \le i_i < \cdots < i_m \le n} a_{i_1 \cdots i_m} dx_{i_1} \wedge dx_{i_2} \wedge \cdots \wedge dx_{i_m}$$

Sometimes, we write $I = (i_1, ..., i_m)$, $dx_{i_1} \wedge \cdots dx_{i_m} = dx_I$.

***Definition:* 1.6: Space of m-forms**

The space of m-forms has a basis given by $\{dx_{i_1} \wedge \cdots dx_{i_m} : 1 \le i_1 < \cdots < i_m \le n\}$
The space is denoted as $\bigwedge^m \mathbb{R}^n$.
$dx_I(v^{(1)}, ..., v^{(m)}) = \det[v^{(j)}_{i_k}]_{1 \le j,k \le m}$.

***Theorem:* 1.5: Dimension of Space of m-forms**

The dimension of the space of m-forms on $T_p\mathbb{R}^n$ is $\binom{n}{m} = \frac{n!}{m!(n-m)!}$

Proof. The basis of $T_p\mathbb{R}^n$ is $\{dx_I\}$. To construct an m-form, we choose m elements from $\{dx_1, ..., dx_n\}$. There are exactly $\binom{n}{m}$ ways. □

Example: On $T_p\mathbb{R}^4$, there are one 0-form, four 1-forms, six 2-forms, four 3-forms, one 4-forms.
0-forms: $\mathbb{R}$
1-forms: $\{dx_1, dx_2, dx_3, dx_4\}$
2-forms: $\{dx_1 \wedge dx_2, dx_1 \wedge dx_3, dx_1 \wedge dx_4, dx_2 \wedge dx_3, dx_2 \wedge dx_4, dx_3 \wedge dx_4\}$
3-forms: $\{dx_1 \wedge dx_2 \wedge dx_3, dx_1 \wedge dx_2 \wedge dx_4, dx_1 \wedge dx_3 \wedge dx_4, dx_2 \wedge dx_3 \wedge dx_4\}$ (dual of 1-forms)
4-forms: $\{dx_1 \wedge dx_2 \wedge dx_3 \wedge dx_4\}$ (dual of 0-forms)

Theorem: 1.6:

If α is a k-form and β is an l-form. Then $\beta \wedge \alpha = (-1)^{kl} \alpha \wedge \beta$.

Proof. Any permutation on $\{1, ..., m\}$ can be written as a product of transpositions $(j, j+1)$. Consider the following swap of j with $j+1$.

$$dx_{i_1} \wedge \cdots \wedge dx_{i_{j+1}} \wedge dx_{i_j} \wedge \cdots \wedge dx_{i_m}(v^{(1)}, ..., v^{(m)}) = \det \begin{bmatrix} v_{i_1}^{(1)} & \cdots & v_{i_{j+1}}^{(1)} & v_{i_j}^{(1)} & \cdots & v_{i_m}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{i_1}^{(m)} & \cdots & v_{i_{j+1}}^{(m)} & v_{i_j}^{(m)} & \cdots & v_{i_m}^{(m)} \end{bmatrix}$$

$$= -\det \begin{bmatrix} v_{i_1}^{(1)} & \cdots & v_{i_j}^{(1)} & v_{i_{j+1}}^{(1)} & \cdots & v_{i_m}^{(1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ v_{i_1}^{(m)} & \cdots & v_{i_j}^{(m)} & v_{i_{j+1}}^{(m)} & \cdots & v_{i_m}^{(m)} \end{bmatrix}$$

$$= -dx_{i_1} \wedge \cdots \wedge dx_{i_j} \wedge dx_{i_{j+1}} \wedge \cdots \wedge dx_{i_m}(v^{(1)}, ..., v^{(m)})$$

For the k-form, $\alpha = \sum a_{i_1 \cdots i_k} dx_{i_1} \wedge \cdots \wedge dx_{i_k}$
For the l-form $\beta = \sum b_{j_1 \cdots j_l} dx_{j_1} \wedge \cdots \wedge dx_{j_l}$

We need to move k elements each passing l elements. And passing l elements gives a $(-1)^l$. k times makes it $(-1)^{kl}$

$$\begin{aligned} \beta \wedge \alpha &= \sum_I \sum_J a_I b_J dx_{j_1} \wedge \cdots dx_{j_l} \wedge dx_{i_1} \wedge \cdots dx_{i_k} \\ &= (-1)^l \sum_I \sum_J a_I b_J dx_{i_1} \wedge dx_{j_1} \wedge \cdots dx_{j_l} \wedge dx_{i_2} \wedge \cdots dx_{i_k} \\ &= (-1)^{kl} \sum_I \sum_J a_I b_J dx_{i_1} \wedge \cdots dx_{i_k} \wedge dx_{j_1} \wedge \cdots dx_{j_l} \\ &= (-1)^{kl} \alpha \wedge \beta \end{aligned}$$

□

Corollary 1. *If k is odd, $\alpha \wedge \alpha = 0$, but for k even, not necessarily true.*

Example: $v^{(1)} = \langle 1, -1, 3, 5 \rangle$, $v^{(2)} = \langle 0, 1, -1, 4 \rangle$
$dx \wedge dy(v^{(1)}, v^{(2)}) = \det \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix} = 1$
$dz \wedge dw(v^{(1)}, v^{(2)}) = \det \begin{bmatrix} 3 & 5 \\ -1 & 4 \end{bmatrix} = 17$
$dx \wedge dz(v^{(1)}, v^{(2)}) = \det \begin{bmatrix} 1 & 3 \\ 0 & -1 \end{bmatrix} = -1$
Let $\omega = 2dx \wedge dy + 3dz \wedge dw - 5dx \wedge dz$, then $\omega(v^{(1)}, v^{(2)}) = 2 \cdot 1 + 3 \cdot 17 - 5(-1) = 58$

2 Integration on Forms

2.1 Differential m-forms

Definition: 2.1: Differential m-forms

A differential m-form on $\mathbb{R}^n$ is given by $\omega = \sum_I f_I dx_I$, $I = (i_1, ..., i_m)$, where $f_I : \mathbb{R}^n \to \mathbb{R}$ are differentiable.

Example: $\omega = x^2 dx \wedge dy - x^3 z dy \wedge dz$. For full evaluation, we need three inputs:

- 1 base point $p \in \mathbb{R}^3$
- 2 vectors $v^{(1)}, v^{(2)} \in T_p\mathbb{R}^3$

Suppose $p = (2, 1, -1)$, $\omega_p = 4dx \wedge dy + 8dy \wedge dz$.
Now, suppose $v^{(1)} = \langle 1, -2, 3\rangle$, $v^{(2)} = \langle 2, 0, 1\rangle$,
$\omega_p(v^{(1)}, v^{(2)}) = 4 \det \begin{bmatrix} 1 & -2 \\ 2 & 0 \end{bmatrix} + 8 \det \begin{bmatrix} -2 & 3 \\ 0 & 1 \end{bmatrix} = 4 \cdot 4 + 8(-2) = 0$

Remark 2. Generally, a differential m-form on $\mathbb{R}^n$ ω takes in m vector fields on $\mathbb{R}^n$ and outputs a function $\mathbb{R}^n \to \mathbb{R}$.

Example: $\omega = x^2 dx \wedge dy - x^3 dy \wedge dz$, $p = (x, y, z)$, $v^{(1)} = \langle x, 2yz, xy\rangle$, $v^{(2)} = \langle y, xz, y^2\rangle$.

$$\begin{aligned} \omega_p(v^{(1)}, v^{(2)}) &= x^2 \det \begin{bmatrix} x & 2yz \\ y & xz \end{bmatrix} - x^3 \det \begin{bmatrix} 2yz & xy \\ xz & y^2 \end{bmatrix} \\ &= x^2(x^2 z - 2y^2 z) - x^3(2y^3 z - x^2 yz) \end{aligned}$$

Example: $\omega = xy dx \wedge dy \wedge dz - 2dx \wedge dy \wedge dw$,
$v^{(1)} = \langle x, y, w, z\rangle$, $v^{(2)} = \langle x^2 y, yz, x, x^2\rangle$, $v^{(3)} = \langle w, z, x, y\rangle$.

$$\omega(v^{(1)}, v^{(2)}, v^{(3)}) = xy \det \begin{bmatrix} x & y & w \\ x^2 y & yz & x \\ w & z & x \end{bmatrix} - 2 \det \begin{bmatrix} x & y & z \\ x^2 y & yz & x^2 \\ w & z & y \end{bmatrix}$$

2.2 Integrating 2-forms

Let $\phi : D \to \mathbb{R}^n$, $D \subset \mathbb{R}^2$ be a smooth C^∞ function parametrizing a surface S in $\mathbb{R}^n$. We want to calculate $\int_S \omega$ where ω is a differential 2-form on $\mathbb{R}^n$.

Consider three points (u_i, v_j), (u_{i+1}, v_j), $(u_i, v_{j+1}) \in D$.
We can get a point $p = \phi(u_i, v_j) \in \mathbb{R}^n$ and two vectors $\phi(u_{i+1}, v_j) - \phi(u_i, v_j), \phi(u_i, v_{j+1}) - \phi(u_i, v_j) \in T_{\phi(u_i,v_j)}\mathbb{R}^n$.

Let $\Delta u = u_{i+1} - u_i$, $\Delta v = v_{j+1} - v_j$

$$\begin{aligned}\int_S \omega &= \lim_{\Delta u, \Delta v \to 0} \sum_{i,j} \omega_{\phi(u_i,v_j)}(\phi(u_{i+1},v_j) - \phi(u_i,v_j), \phi(u_i,v_{j+1}) - \phi(u_i,v_j)) \\ &= \lim_{\Delta u, \Delta v \to 0} \sum_{i,j} \omega_{\phi(u_i,v_j)} \left(\frac{\phi(u_{i+1},v_j) - \phi(u_i,v_j)}{\Delta u}, \frac{\phi(u_i,v_{j+1}) - \phi(u_i,v_j)}{\Delta v} \right) \Delta u \Delta v \\ &= \iint_D \omega_{\phi(u,v)} \left(\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v} \right) dA\end{aligned}$$

Example: $\omega = xydx \wedge dy + x^2 dx \wedge dz$. $\phi(u,v) = \langle u, v, u^2 + v^2 \rangle$, $D = \{(u,v) : u^2 + v^2 = 1\}$
$\frac{\partial \phi}{\partial u} = \langle 1, 0, 2u \rangle$, $\frac{\partial \phi}{\partial v} = \langle 0, 1, 2v \rangle$

$$\begin{aligned}\int_S \omega &= \iint_D uvdx \wedge dy(\langle 1,0,2u \rangle, \langle 0,1,2v \rangle) + u^2 dx \wedge dz(\langle 1,0,2u \rangle, \langle 0,1,2v \rangle)dV \\ &= \iint_D uv \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + u^2 \det \begin{bmatrix} 1 & 2u \\ 0 & 2v \end{bmatrix} dV \\ &= \iint_D uv + 2u^2 vdV \\ &= \int_0^{2\pi} \int_0^1 (r^2 \sin\theta \cos\theta + 2r^3 \cos^2\theta \sin\theta) rdrd\theta = 0\end{aligned}$$

Example: $\omega = yzdx \wedge dy + \frac{z}{x} dy \wedge dz$, $S = \{(x,y,z) : x^2 + y^2 = 1, 0 \le z \le 2\}$ a cylinder.
We can parametrize S by $\phi(\theta, z) = \langle \cos\theta, \sin\theta, z \rangle$, $\theta \in [0, 2\pi]$, $z \in [0,2]$.
$\frac{\partial \phi}{\partial \theta} = \langle -\sin\theta, \cos\theta, 0 \rangle$, $\frac{\partial \phi}{\partial z} = \langle 0, 0, 1 \rangle$.

$$\omega_{\phi(\theta,z)} \left(\frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right) = \sin\theta z \det \begin{bmatrix} -\sin\theta & \cos\theta \\ 0 & 0 \end{bmatrix} + \frac{z}{\cos\theta} \det \begin{bmatrix} \cos\theta & 0 \\ 0 & 1 \end{bmatrix} = z$$

$$\int_S \omega = \int_0^{2\pi} \int_0^2 \omega_{\phi(\theta,z)} \left(\frac{\partial \phi}{\partial \theta}, \frac{\partial \phi}{\partial z} \right) dzd\theta = \int_0^{2\pi} \int_0^2 zdzd\theta = 4\pi$$

2.3 Integrating m-forms

Let ω be a differential m-form on $\mathbb{R}^n$, $\omega = \sum_I f_I dx_I$, where $I = (i_1, ..., i_m)$. Let $\phi : D \to \mathbb{R}$, $D \subset \mathbb{R}^m$ be a smooth C^∞ function parametrizing a m-dimensional hypersurface in $\mathbb{R}^n$.

$$\int_S \omega = \int \cdots \int_D \omega_{\phi(u_1,...,u_m)} \left(\frac{\partial \phi}{\partial u_1}, ..., \frac{\partial \phi}{\partial u_m} \right) dV_m$$

Example: $\omega = x_1 dx_1 + (x_1^2 + x_2)dx_2 + x_3 x_4 dx_4$. $\phi : [0, 3\pi] \to \mathbb{R}^4$, $\phi(t) = \langle \cos t, \sin t, t, -t \rangle$.
$\frac{\partial \phi}{\partial t} = \langle -\sin t, \cos t, 1, -1 \rangle$, $dx_1 = -\sin t$, $dx_2 = \cos t$, $dx_4 = -1$.

$$\int_C \omega = \int_0^{3\pi} \cos t(-\sin t) + (\cos^2 t + \sin t)\cos t + (-t^2)(-1)dt = 9\pi^3$$

Example: $\omega = x_3 dx_1 \wedge dx_3 \wedge dx_4$. $\phi : D = [0,1]^3 \to \mathbb{R}^4$, $\phi(u_1, u_2, u_3) = \langle u_1 u_2, u_1^2 + u_3, u_2 u_3, u_1 + 2u_2 + u_3 \rangle$
$\frac{\partial \phi}{\partial u_1} = \langle u_2, 2u_1, 0, 1 \rangle$, $\frac{\partial \phi}{\partial u_2} = \langle u_1, 0, u_3, 2 \rangle$, $\frac{\partial \phi}{\partial u_3} = \langle 0, 1, u_2, 1 \rangle$

$$\begin{aligned}
\int_S \omega &= \iiint_{[0,1]^3} u_2 u_3 dx_1 \wedge dx_3 \wedge dx_4 \left(\frac{\partial \phi}{\partial u_1}, \frac{\partial \phi}{\partial u_2}, \frac{\partial \phi}{\partial u_3} \right) dV \\
&= \iiint_{[0,1]^3} u_2 u_3 \det \begin{bmatrix} u_2 & 0 & 1 \\ u_1 & u_3 & 2 \\ 0 & u_2 & 1 \end{bmatrix} dV \\
&= \iiint_{[0,1]^3} u_2^2 u_3^2 - 2u_2^3 u_3 + u_1 u_2^2 u_3 dV = 0
\end{aligned}$$

2.4 Change of Variables

Example: Integrate $\omega = x^2 dx$ over $[0,5] \subset \mathbb{R}$

1. $\phi(t) = t$, $t \in [0,5]$,

$$\int_{[0,5]} \omega = \int_0^5 \omega_\phi(\phi'(t)) = \int_0^5 t^2 dx(1) dt = \int_0^5 t^2 dt = \frac{125}{3}$$

2. $\phi(t) = 5t - 5$, $t \in [1,2]$,

$$\int_{[1,2]} \omega = \int_1^2 \omega_\phi(5) = \int_1^2 (5t-5)^2 dx(5) dt = 5 \int_1^2 (5t-5)^2 dt = \frac{125}{3}$$

This example shows that u-substitution is just a change of parametrization in a differential 1-form.

Consider $\phi : [a,b] \to [\phi(a), \phi(b)]$, $\omega = f(x)dx$

1. Trivial parametrization:

$$\int_{[\phi(a),\phi(b)]} \omega = \int_{\phi(a)}^{\phi(b)} f(x) dx$$

2. ϕ-parametrization:

$$\int_{[\phi(a),\phi(b)]} \omega = \int_a^b \omega_{\phi(t)}(\phi'(t)) dt = \int_a^b f(\phi(t)) dx(\phi'(t)) dt = \int_a^b f(\phi(t)) \phi'(t) dt$$

Example: Calculate $\int_0^1 \frac{1}{\sqrt{1-x^2}} dx$ with differential 1-form.
$\omega = \frac{1}{\sqrt{1-x^2}} dx$, $[\phi(a), \phi(b)] = [0,1]$
Let $\phi : [0, \frac{\pi}{2}] \to [0,1]$, $\phi(t) = \sin t$, $\phi'(t) = \cos t$

$$\int_{[0,1]} \omega = \int_0^{\pi/2} \omega_{\phi(t)}(\phi'(t)) dt = \int_0^{\pi/2} \frac{1}{\sqrt{1 - \sin^2 t}} dx(\cos t) dt = \int_0^{\pi/2} \frac{\cos t}{\cos t} dt = \frac{\pi}{2}$$

We can apply the same technique to 2-forms and m-Forms

To integrate $\omega f(x,y) dx \wedge dy$ over $D \subset \mathbb{R}^2$

1. Trivial parametrization:

$$\int_D \omega = \iint_D \omega_{id}\left(\frac{\partial id}{\partial x}, \frac{\partial id}{\partial y}\right) dA = \iint_D f(x,y)dx \wedge dy(\langle 1,0\rangle, \langle 0,1\rangle)dA$$
$$= \iint_D f(x,y) \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} dA = \iint_D f(x,y)dA$$

2. ϕ-parametrization: Let $\phi(u,v) = \langle x(u,v), y(u,v)\rangle$, $\frac{\partial \phi}{\partial u} = \langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\rangle$, $\frac{\partial \phi}{\partial v} = \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\rangle$

$$\int_D \omega = \iint_D \omega_\phi\left(\frac{\partial \phi}{\partial u}, \frac{\partial \phi}{\partial v}\right) dA' = \iint_D f(x(u,v), y(u,v))dx \wedge dy\left(\langle \frac{\partial x}{\partial u}, \frac{\partial y}{\partial u}\rangle, \langle \frac{\partial x}{\partial v}, \frac{\partial y}{\partial v}\rangle\right) dA'$$
$$= \iint_D f(x(u,v), y(u,v)) \det \begin{bmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} \end{bmatrix} dA'$$

Example: Compute $\iint_D (x^2+y^2)dA$ with $D = \{(x,y) : x^2+y^2 \leq 4\}$
Let $\omega = (x^2+y^2)dx \wedge dy$. Define $\phi : [0,2] \times [0,2\pi] \to D$ with $\phi(r,\theta) = \langle r\cos\theta, r\sin\theta\rangle$.
$\frac{\partial \phi}{\partial r} = \langle \cos\theta, \sin\theta\rangle$, $\frac{\partial \phi}{\partial \theta} = \langle -r\sin\theta, r\cos\theta\rangle$.

$$\int_D \omega = \int_0^2 \int_0^{2\pi} \omega_\phi\left(\frac{\partial \phi}{\partial r}, \frac{\partial \phi}{\partial \theta}\right) d\theta dr$$
$$= \int_0^2 \int_0^{2\pi} r^2 \det \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} d\theta dr$$
$$= \int_0^2 \int_0^{2\pi} r^3 d\theta dr = 8\pi$$

3 Exterior Derivative

The goal here is to define a derivative d that takes differential m-forms on $\mathbb{R}^n$ and outputs differential (m+1)-forms on $\mathbb{R}^n$.

Starting point d : 0-forms $\to$ 1-forms. *i.e.* Given a 0-form (a function $f(x_1, ..., x_n)$) on $\mathbb{R}^n$, what is df?

1. df is a 1-form
2. To evaluate df, we set a point $p \in \mathbb{R}^n$, $v \in T_p\mathbb{R}^n$,

$$(df)_p(v) = D_v f(p) = \nabla f(p) \cdot v = \frac{\partial f}{\partial x_1} v_1 + \cdots \frac{\partial f}{\partial x_n} v_n = \frac{\partial f}{\partial x_1} dx_1(v) + \cdots + \frac{\partial f}{\partial x_n} dx_n(v)$$

Thus $df = \frac{\partial f}{\partial x_1} dx_1 + \cdots \frac{\partial f}{\partial x_n} dx_n$.

***Definition:* 3.1: Exterior Derivative**

Given a differential m-form $d(f dx_I) = \frac{\partial f}{\partial x_1} dx_1 \wedge dx_I + \cdots + \frac{\partial f}{\partial x_n} dx_n \wedge dx_I$.

Define $d\omega = \sum_I \sum_{j=1}^{n} \frac{\partial f_I}{\partial x_j} dx_j \wedge dx_I$. This is a differential (m+1)-form.

Example: In $\mathbb{R}^3$
0-form: a function f, $df = f_x dx + f_y dy + f_z dz$
1-form: $\alpha = f dx + g dy + h dz$,

$$\begin{aligned} d\alpha &= f_x dx \wedge dx + f_y dy \wedge dx + f_z dz \wedge dx + g_x dx \wedge dy + g_y dy \wedge dy + g_z dz \wedge dy \\ &\quad + h_x dx \wedge dz + h_y dy \wedge dz + h_z dz \wedge dz \\ &= (g_x - f_y) dx \wedge dy + (h_y - g_z) dy \wedge dz + (h_x - f_z) dx \wedge dz \end{aligned}$$

2-form: $\beta = F dx \wedge dy + G dx \wedge dz + H dy \wedge dz$,

$$\begin{aligned} d\beta &= F_z dz \wedge dx \wedge dy + G_y dy \wedge dx \wedge dz + H_x dx \wedge dy \wedge dz \\ &= (F_z - G_y + H_x) dx \wedge dy \wedge dz \end{aligned}$$

3-form: $\gamma = R dx \wedge dy \wedge dz$, $d\gamma = 0$

Example: $\omega = x^3 dx + 2xy dy + xyz dz$

$$\begin{aligned} d\omega &= 3x^2 dx \wedge dx + 2y dx \wedge dy + 2x dy \wedge dy + yz dx \wedge dz + xz dy \wedge dz + xy dz \wedge dz \\ &= 2y dx \wedge dy + yz dx \wedge dz + xz dy \wedge dz \end{aligned}$$

Example: $\omega = x^2y^2 dx \wedge dz + 2x^3yz dy \wedge dz$

$$\begin{aligned} d\omega &= 2xy^2 dx \wedge dx \wedge dz + 2x^2y dy \wedge dx \wedge dz \\ &\quad + 6x^2yz dx \wedge dy \wedge dz + 2x^3z dy \wedge dy \wedge dz + 2x^3y dz \wedge dy \wedge dz \\ &= (2x^2y + 6xyz) dx \wedge dy \wedge dz \end{aligned}$$

***Theorem:* 3.1: Product Rule**

Given ω a m-form, μ a k-form,

$$d(\omega \wedge \mu) = (d\omega) \wedge \mu + (-1)^m \omega \wedge (d\mu)$$

Proof. Let $\omega = \sum_I f_I dx_I$, $\mu = \sum_J g_J dx_J$, where $I = (i_1, ..., i_m)$, $J = (j_1, ..., j_k)$.

$$\omega \wedge \mu = \sum_I \sum_J f_I g_J dx_I \wedge dx_J$$

$$\begin{aligned} d(\omega \wedge \mu) &= \sum_{I,J} \sum_{r=1}^n \frac{\partial}{\partial x_r}(f_I g_J) dx_r \wedge dx_I \wedge dx_J \\ &= \sum_{I,J} \sum_{r=1}^n \left(\frac{\partial f_I}{\partial x_r} g_J + f_I \frac{\partial g_J}{\partial x_r} \right) dx_r \wedge dx_I \wedge dx_J \\ &= \sum_{I,J} \sum_{r=1}^n \frac{\partial f_I}{\partial x_r} g_J dx_r \wedge dx_I \wedge dx_J + \sum_{I,J} \sum_{r=1}^n f_I \frac{\partial g_J}{\partial x_r} dx_r \wedge dx_I \wedge dx_J \\ &= \sum_{I,J} \sum_{r=1}^n \frac{\partial f_I}{\partial x_r} g_J dx_r \wedge dx_I \wedge dx_J + (-1)^m \sum_{I,J} \sum_{r=1}^n f_I \frac{\partial g_J}{\partial x_r} dx_I \wedge dx_r \wedge dx_J \\ &= (d\omega) \wedge \mu + (-1)^m \omega \wedge (d\mu) \end{aligned}$$

□

***Theorem:* 3.2:**

Suppose ω is a differential m-form on $\mathbb{R}^n$. Then $d^2(\omega) = d(d\omega) = 0$

Proof. Set $\omega = f dx_I$, $d\omega = \sum_{j=1}^n \frac{\partial f}{\partial x_j} dx_j \wedge dx_I$.

$$\begin{aligned} d^2\omega &= \sum_{k=1}^n \sum_{j=1}^n \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I \\ &= \sum_{k=1}^n \left(\sum_{j=1}^{k-1} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I + \sum_{j=k+1}^n \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I \right) \\ &= \sum_{k=1}^n \sum_{j=1}^{k-1} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I - \sum_{k=1}^n \sum_{j=1}^{k-1} \frac{\partial^2 f}{\partial x_k \partial x_j} dx_k \wedge dx_j \wedge dx_I = 0 \end{aligned}$$

For the general m-form $\omega = \sum_I f_I dx_I$, by linearity, we have $d^2\omega = 0$. □

Example: Given a 2-form $\omega = f dx \wedge dy$ on $\mathbb{R}^4$.
$d\omega = f_z dz \wedge dx \wedge dy + f_w dw \wedge dx \wedge dy$
$d^2\omega = f_{zw} dw \wedge dz \wedge dz + f_{wz} dz \wedge dw \wedge dx \wedge dy = 0$

Note that for gradients, curl and div, we have the following relationship:
Functions on $\mathbb{R}^3 \xrightarrow{\text{grad}}$ vector fields on $\mathbb{R}^3 \xrightarrow{\text{curl}}$ vector fields on $\mathbb{R}^3 \xrightarrow{\text{div}}$ Functions on $\mathbb{R}^3$

$$\text{curl}(\text{grad}(f)) = 0 = d^2 f$$
$$\text{div}(\text{curl}(F)) = 0 = d^2 F$$

3.1 Hodge Operator

Recall that $\bigwedge^m(\mathbb{R}^n)$ is the vector space of m-forms on $\mathbb{R}^n$. The dimension is $\dim \bigwedge^m(\mathbb{R}^n) = \binom{n}{m}$. Note that $\binom{n}{m} = \binom{n}{n-m}$. We want to know if there is a relation between $\bigwedge^m(\mathbb{R}^n)$ and $\bigwedge^{n-m}(\mathbb{R}^n)$

Definition: 3.2: Space of Differential Forms

The space of differential forms is a module over the space of forms

$$\bigoplus_{m=0}^{n} \bigwedge^{m}(\mathbb{R}^n)$$

Definition: 3.3: Hodge Operator

The hodge operator is $\star : \bigwedge^m(\mathbb{R}^n) \to \bigwedge^{n-m}(\mathbb{R}^n)$.
$\star dx_I = dx_J$ s.t. $dx_I \wedge dx_J = dx_1 \wedge \cdots dx_n$

Example: On $\mathbb{R}^2$.
0-form: $\star 1 = dx \wedge dy$
1-forms: $\star dx = dy$, since we need $dx \wedge (\star dx) = dx \wedge dy$
$\star dy = -dx$, since $dy \wedge (\star dy) = dy \wedge (-dx) = dx \wedge dy$
2-forms: $\star dx \wedge dy = 1$

Example: On $\mathbb{R}^3$, $\star : \bigwedge^1(\mathbb{R}^3) \to \bigwedge^2(\mathbb{R}^3)$ and $\star : \bigwedge^2(\mathbb{R}^3) \to \bigwedge^1(\mathbb{R}^3)$.
1-forms: $\star dx = dy \wedge dz$,
$\star dy = -dx \wedge dz$, since $dy \wedge (\star dy) = dy \wedge dx \wedge dz = -dx \wedge dy \wedge dz$,
$\star dz = dx \wedge dy$, since $dz \wedge (\star dz) = dz \wedge dx \wedge dy = dx \wedge dy \wedge dz$.
2-forms: by symmetry, $\star dx \wedge dy = dz$, $\star dx \wedge dz = -dy$, $\star dy \wedge dz = dx$

Example: $\star : \bigwedge^2(\mathbb{R}^5) \to \bigwedge^3(\mathbb{R}^5)$, $\omega = dx_1 \wedge dx_2 + 2dx_3 \wedge dx_4 + 7dx_1 \wedge dx_5$
$\star\omega = dx_3 \wedge dx_4 \wedge dx_5 + 2dx_1 \wedge dx_2 \wedge dx_5 - 7dx_2 \wedge dx_3 \wedge dx_4$

Remark 3. If $\omega = \sum_I f_I dx_I$, then $\star\omega = \sum_I f_I(\star dx_I)$.

Definition: 3.4: Grad, Curl, Div

For $f : \mathbb{R}^3 \to \mathbb{R}$, $\text{grad}(f) = \langle f_x, f_y, f_z \rangle$
For $F = \langle P, Q, R \rangle$,
$\text{curl}(F) = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$
$\text{div}(F) = P_x + Q_y + R_z$
Identity: $F = \langle P, Q, R \rangle = Pdx + Qdy + Rdz = \omega_F$

Theorem: 3.3:

1. $\text{grad}(f) = df$
2. $\text{curl}(F) = \star d\omega_F$
3. $\text{div}(F) = \star d(\star\omega_F)$

Proof. 1. $df = \sum_{j=1}^{3} \frac{\partial f}{\partial x_j} dx_j = f_x dx + f_y dy + f_z dz = \langle f_x, f_y, f_z \rangle = \text{grad}(f)$

2.

$$\begin{aligned}\star d\omega_F &= \star(P_x dx\wedge dx + Q_x dx\wedge dy + R_x dx\wedge dz + P_y dy\wedge dx + Q_y dy\wedge dy + R_y dy\wedge dz \\ &\quad + P_z dz\wedge dx + Q_z dz\wedge dy + R_z dz\wedge dz) \\ &= \star[(Q_x - P_y)dx\wedge dy + (R_x - P_z)dx\wedge dz + (R_y - Q_z)dy\wedge dz] \\ &= (Q_x - P_y)dz + (R_x - P_z)dy + (R_y - Q_z)dx \\ &= \langle R_y - Q_z, P_z - R_x, Q_x - P_y\rangle = \text{curl}(F)\end{aligned}$$

3.

$$\begin{aligned}\star d(\star\omega) &= \star d(\star(Pdx + Qdy + Rdz)) \\ &= \star d[Pdy\wedge dz - Qdx\wedge dz + Rdx\wedge dy] \\ &= \star(P_x dx\wedge dy\wedge dz + Q_y dx\wedge dy\wedge dz + R_z dx\wedge dy\wedge dz) \\ &= P_x + Q_y + R_z\end{aligned}$$

Also, since $d^2 = 0$,
$\text{curl}(\text{grad}(f)) = \star d(df) = \star d^2 f = 0$
$\text{div}(\text{curl}(F)) = \star d\star\star d\omega_F = \star d^2\omega_F = 0$ □

3.2 Hodge Product via Inner Product

***Definition:* 3.5: Hodge Product via Inner Product**

Let $\langle\cdot,\cdot\rangle : (\bigwedge^k(\mathbb{R}^n))^2 \to \mathbb{R}$ be a bilinear form (inner product) on the space of k-forms.
Define $\star\alpha$ by the unique (n-m)-form s.t. $\forall\beta\in\bigwedge^{n-m}(\mathbb{R})^n$,

$$\beta\wedge(\star\alpha) = \langle\alpha,\beta\rangle dx_1\wedge\cdots\wedge dx_n$$

Example: Assume $\langle dx_I, dx_J\rangle = \begin{cases}1, I = J\\ 0, I\neq J\end{cases}$, $dx, dy, dz \in \bigwedge^1(\mathbb{R}^3)$ 1-forms on $\mathbb{R}^3$
Let $\star dx = Adx\wedge dy + Bdy\wedge dz + Cdx\wedge dz$.

$$Adx\wedge dx\wedge dy + Bdx\wedge dy\wedge dz + Cdx\wedge dx\wedge dz = dx\wedge(\star dx) = \langle dx, dx\rangle dx\wedge dy\wedge dz$$

Since $dx\wedge dx = 0$, $\langle dx, dx\rangle = 1$, we have $B = 1$.
From $dy\wedge(\star dx)$, we get $C = 0$. From $dz\wedge(\star dx)$, we get $A = 0$.
Thus $\star dx = dy\wedge dz$.

***Definition:* 3.6: Matrix Representation of Inner Product**

On $\bigwedge^1(\mathbb{R}^n) = \text{span}\{dx_1, ..., dx_n\}$. The inner product can be given by $\begin{bmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{bmatrix}$ where $a_{ij} = \langle dx_i, dx_j\rangle$.

Example: $\bigwedge^1(\mathbb{R}^2) = \text{span}\{dx, dy\}$

$$\begin{aligned}\langle Adx + Bdy, Cdx + Ddy\rangle &= AC\langle dx, dx\rangle + AD\langle dx, dy\rangle + BC\langle dy, dx\rangle + BD\langle dy, dy\rangle \\ &= \begin{bmatrix}A & B\end{bmatrix}\begin{bmatrix}\langle dx, dx\rangle & \langle dx, dy\rangle\\ \langle dy, dx\rangle & \langle dy, dy\rangle\end{bmatrix}\begin{bmatrix}C\\ D\end{bmatrix}\end{aligned}$$

Definition: 3.7:

Suppose we know $\langle dx_i, dx_j\rangle$, $1 \le i, j \le n$. We can list to an inner product $\langle\cdot,\cdot\rangle : (\bigwedge^m(\mathbb{R}^n))^2 \to \mathbb{R}$, $I = (i_1, ..., i_m)$, $J = (j_1, ..., j_m)$

$$\langle dx_I, dx_J\rangle = \sum_{\sigma\in S_m} \operatorname{sgn}(\sigma)\langle dx_{i_1}, dx_{j_{\sigma(1)}}\rangle \cdots \langle dx_{i_m}, dx_{j_{\sigma(m)}}\rangle$$

Example: Let $\langle\cdot,\cdot\rangle : (\bigwedge^1(\mathbb{R}^3))^2 \to \mathbb{R}$ given by $\begin{bmatrix} 1 & 3 & -2 \\ 3 & 2 & 0 \\ -2 & 0 & -1 \end{bmatrix}$

1. Compute $\langle 2dx_1 + dx_2, dx_1 + dx_3\rangle$:

$$\begin{aligned}\langle 2dx_1 + dx_2, dx_1 + dx_3\rangle &= 2\langle dx_1, dx_1\rangle + 2\langle dx_1, dx_3\rangle + \langle dx_2, dx_1\rangle + \langle dx_2, dx_3\rangle \\ &= 2\cdot 1 + 2(-2) + 3 + 0 = 1\end{aligned}$$

2. Compute $\langle dx_1 \wedge dx_2, dx_2 \wedge dx_3\rangle$:
   Here $I = (1, 2)$, $J = (2, 3)$, $i_1 = 1$, $i_2 = 2$, $j_1 = 2$, $j_2 = 3$, $S_2 = \{(1), (1\ 2)\}$

$$\begin{aligned}\langle dx_I, dx_J\rangle &= \langle dx_1, dx_2\rangle\langle dx_2, dx_3\rangle - \langle dx_1, dx_3\rangle\langle dx_2, dx_2\rangle \\ &= 3\cdot 0 - (-2)2 = 4\end{aligned}$$

Example: Let $\langle\cdot,\cdot\rangle : (\bigwedge^1(\mathbb{R}^4))^2 \to \mathbb{R}$ given by $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & -1 & 0 & -2 \\ 3 & 0 & 1 & 2 \\ 4 & -2 & 2 & 3 \end{bmatrix}$

Note $S_3 = \{(1), (1\ 2), (1\ 3), (2\ 3), (1\ 2\ 3), (1\ 3\ 2)\}$, where $(1), (1\ 2\ 3), (1\ 3\ 2)$ are even permutations with $\operatorname{sgn} = 1$, and $(1\ 2), (1\ 3), (2\ 3)$ are odd permutations with $\operatorname{sgn} = -1$

$$\begin{aligned}\langle dx_1 \wedge dx_2 \wedge dx_3, dx_2 \wedge dx_3 \wedge dx_4\rangle &= \langle dx_1, dx_2\rangle\langle dx_2, dx_3\rangle\langle dx_3, dx_4\rangle - \langle dx_1, dx_3\rangle\langle dx_2, dx_2\rangle\langle dx_3, dx_4\rangle \\ &- \langle dx_1, dx_4\rangle\langle dx_2, dx_3\rangle\langle dx_3, dx_2\rangle - \langle dx_1, dx_2\rangle\langle dx_2, dx_4\rangle\langle dx_3, dx_3\rangle \\ &+ \langle dx_1, dx_3\rangle\langle dx_2, dx_4\rangle\langle dx_3, dx_2\rangle + \langle dx_1, dx_4\rangle\langle dx_2, dx_2\rangle\langle dx_3, dx_3\rangle \\ &= 2\cdot 0\cdot 2 - 3(-1)2 - 4\cdot 0\cdot 0 - 2(-2)1 + 3(-2)0 + 4(-1)1 \\ &= 6 + 4 - 4 = 6\end{aligned}$$

4 Applications

4.1 Maxwell's equations

Definition: 4.1: Minkowski Inner Product

The Minkowski Inner Product (Metric) on 1-form is defined as $\text{diag}(1, -1, -1, -1)$. *i.e.* $\langle dt, dt \rangle = 1$, $\langle dx, dx \rangle = \langle dy, dy \rangle = \langle dz, dz \rangle = -1$, and all other entries as 0.

We can compute the 2-forms to be $\text{diag}(\langle dt \wedge dx \rangle, \langle dt \wedge dy \rangle, \langle dt \wedge dz \rangle, \langle dx \wedge dy \rangle, \langle dx \wedge dz \rangle, \langle dy \wedge dz \rangle) = \text{diag}(-1, -1, -1, 1, 1, 1)$

Proof. $\langle dt \wedge dx, dt \wedge dx \rangle = \langle dt, dt \rangle \langle dx, dx \rangle - \langle dt, dx \rangle \langle dx, dt \rangle = -1$.
Similar for $\langle dt \wedge dy, dt \wedge dy \rangle$ and $\langle dt \wedge dz, dt \wedge dz \rangle$.

$\langle dx \wedge dy, dx \wedge dy \rangle = \langle dx, dx \rangle \langle dy, dy \rangle - \langle dx, dy \rangle \langle dy, dx \rangle = 1$.
Similar for $\langle dx \wedge dz, dx \wedge dz \rangle$ and $\langle dy \wedge dz, dy \wedge dz \rangle$.

For the off-diagonal elements. *e.g.* $\langle dt \wedge dx, dt \wedge dy \rangle = \langle dt, dt \rangle \langle dx, dy \rangle - \langle dt, dy \rangle \langle dx, dt \rangle = 0$. □

The 3-forms can be computed as
$\text{diag}(dx \wedge dy \wedge dz, dt \wedge dx \wedge dy, dt \wedge dx \wedge dz, dt \wedge dy \wedge dz) = \text{diag}(-1, 1, 1, 1)$

The dual of dt is $\star dt = dx \wedge dy \wedge dz$

The electromagnetic 2-forms is $E = \langle E_1, E_2, E_3 \rangle$ (electric field), $B = \langle B_1, B_2, B_3 \rangle$ (magnetic field).
The EM-field can be defined as

$$F = E_1 dx \wedge dt + E_2 dy \wedge dt + E_3 dz \wedge dt + B_1 dy \wedge dz + B_2 dx \wedge dz + B_3 dx \wedge dy$$

The current 1-form is ρ (charge density), $j = \langle J_1, J_2, J_3 \rangle$ (current density),

$$J = \rho dt - J_1 dx - J_2 dy - J_3 dz$$

The Maxwell equations can be defined in the following 2 equivalent ways:

$$\begin{cases} \nabla \cdot E = \rho \\ \nabla \cdot B = 0 \\ \nabla \times E = -\frac{\partial B}{\partial t} \\ \nabla \times B = j + \frac{\partial E}{\partial t} \end{cases} \Leftrightarrow \begin{cases} dF = 0 \\ \star d \star F = J \end{cases}$$

4.2 Integrals of m-forms over m-chains

Definition: 4.2: m-cell

An m-cell, σ, is the image of a differentiable map $\phi : [0, 1]^m \to \mathbb{R}^n$ with a specific orientation.

Definition: 4.3: m-chain

An m-chain is a linear combination of m-cells

$$\Sigma = \sum_i n_i \sigma_i$$

Example: 0-cell: $\phi : [0,1]^0 \to \mathbb{R}^n$ is a single point.
1-cell: $\phi : [0,\pi] \to \mathbb{R}^2$, $\phi(t) = \langle \cos t, \sin t \rangle$ is a curve
2-cell: $\phi : [0,2\pi] \times [0, \frac{\pi}{2}] \to \mathbb{R}^3$, $\phi(u,v) = \langle \cos u \sin v, \sin u, \sin v, \cos v \rangle$
3-cells: $x_1^2 + x_2^2 \leq 1$, $x_3 \in [0,1]$, $x_4 = 2$, $\phi : [0,1] \times [0,2\pi] \times [0,1] \to \mathbb{R}^4$,
$\phi(r,\theta,x_3,x_4) = \langle r\cos\theta, r\sin\theta, x_3, 2 \rangle$.

Remark 4. The closed rectangles $[a_1,b_1] \times \cdots \times [a_n,b_n]$ are diffeomorphic to $[0,1]^n$.

***Definition:* 4.4: Integrals on m-chain**

Given $\Sigma = \sum_i n_i \sigma_i$ a m-chain, ω a differential m-form,

$$\int_\Sigma \omega = \sum_i n_i \int_{\sigma_i} \omega$$

Example: 0-form $f(x,y) = 3x + xy^2$ on 0-cells $p = (0,3), q = (5,2)$

$$\int_{2p-q} f = 2\int_p f - \int_q f = 2f(p) - f(q) = 2(0+0) - (3 \cdot 5 + 5 \cdot 2^2) = -35$$

Example: 1-form $\omega = xydx + (x^2+y)dy$ on 1-cells $\sigma_1(t) = \langle \cos t, \sin t \rangle$, $t \in [\pi, 0]$, $\sigma_2(t) = \langle t, t-1 \rangle$, $t \in [1,2]$

$$\begin{aligned}\int_\Sigma f &= \int_{2\sigma_1 - 3\sigma_2} f = 2\int_{\sigma_1} xydx + (x^2+y)dy - 3\int_{\sigma_2} xydx + (x^2+y)dy \\ &= 2\int_\pi^0 \cos t \sin t(-\sin t)dt + (\cos^2 t + \sin t)\cos t dt - 3\int_1^2 t(t-1)dt + (t^2+t-1)dt \\ &= -11\end{aligned}$$

Example: 2-form $\omega = xzdx \wedge dy + dy \wedge dz$ on 2-cell $\phi : [0,1] \times [0,\pi] \to \mathbb{R}^3$, $\phi(r,\theta) = \langle r\cos\theta, r\sin\theta, 2 \rangle$

$$\begin{aligned}\int_\sigma \omega &= \int_\sigma xzdx \wedge dy + dy \wedge dz \\ &= \int_0^1 \int_0^\pi 2r\cos\theta \det \begin{bmatrix} \cos\theta & \sin\theta \\ -r\sin\theta & r\cos\theta \end{bmatrix} d\theta dr + \det \begin{bmatrix} \sin\theta & 0 \\ r\cos\theta & 0 \end{bmatrix} d\theta dr \\ &= \int_0^1 \int_0^\pi 2r^2 \cos\theta d\theta dr = 0\end{aligned}$$

***Definition:* 4.5: Boundary of m-cell**

The boundary $\partial\sigma$ of an m-cell σ is the image:

$$\sum_{i=1}^m (-1)^{i+1}(\phi(x_1,...,x_{i-1},1,x_{i+1},...,x_m) - \phi(x_1,...,x_{i-1},0,x_{i+1},...,x_m))$$

The boundary of an m-cell is a (m-1)-chain.

Example: 1-cell $\phi : [0,1] \to \mathbb{R}^n$, $\phi(0) = p$, $\phi(1) = q$, $\partial\sigma = q - p$ is a 0-chain.

2-cell $\phi : [0,1] \times [0, \frac{\pi}{2}] \to \mathbb{R}^2$, $\phi(r,\theta) = \langle r\cos\theta, r\sin\theta \rangle$

$$\begin{aligned}\partial\sigma &= (-1)^2[\phi(1,\theta) - \phi(0,\theta)] + (-1)^3[\phi(r,\frac{\pi}{2}) - \phi(r,0)] \\ &= \phi(1,\theta) - \phi(0,\theta) - \phi(r,\frac{\pi}{2}) + \phi(r,0)\end{aligned}$$

This is the counter-clockwise traversal of the boundary of the circle in the first quadrant.

3-cell $\phi : [0,1] \times [0,2\pi] \times [0, \frac{\pi}{2}] \to \mathbb{R}^3$, $\phi(r,u,v) = \langle r\cos u \sin v, r\sin u \sin v, r\cos v \rangle$

$$\begin{aligned}\partial\sigma &= (\phi(1,u,v) - \phi(0,u,v)) - (\phi(r,2\pi,v) - \phi(r,0,v)) + (\phi(r,u,\frac{\pi}{2} - \phi(r,u,0))) \\ &= \phi(1,u,v) + \phi(r,u,\frac{\pi}{2})\end{aligned}$$

The first term is the top-half of the sphere. The second term is the disk at the bottom.

***Definition:* 4.6: Boundary of m-chain**

If Σ is an m-chain where $\Sigma = \sum_i n_i\sigma_i$, σ_i are m-cells. Then $\partial\Sigma = \sum_i n_i \partial\sigma_i$

4.3 Generalized Stokes Theorem

***Theorem:* 4.1: Generalized Stokes Theorem**

Suppose $\omega = f dx_2 \wedge \cdots \wedge dx_n$ is an (n-1)-form on $\mathbb{R}^n$ and $R = [0,1]^n$. Then

$$\int_{\partial R} \omega = \int_R d\omega.$$

$\int_{\partial R} \omega$ is the integral of an (n-1)-form on an (n-1)-chain.
$\int_R \partial\omega$ is the integral of an n-form on an n-chain.

Note: An (n-1)-form on $\mathbb{R}^n$ looks like $\eta = \sum_{i=1}^{n} f_i dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n$, with $\widehat{dx_i}$ removed.

Proof. Fix $N \in \mathbb{N}$. For $I = (i_1, ..., i_n)$, $x_I = \left(\frac{i_1}{N}, ..., \frac{i_n}{N}\right)$. $\frac{1}{N}e_j = (0, ..., \frac{1}{N}, ..., 0)$.
$d\omega = d(f dx_2 \wedge \cdots \wedge dx_n) = \frac{\partial f}{\partial x_1} dx_1 \wedge \cdots \wedge dx_n$.

Let $x_I^* \in [\frac{i_1}{N}, \frac{i_1+1}{N}] \times [\frac{i_2}{N}, \frac{i_2+1}{N}] \times \cdots$ be a point in the region.

$$\begin{aligned}
\int_R d\omega &= \lim_{N\to\infty} \sum_{i_j=1, j\in[1,n]}^{N} d\omega_{x_I^*}\left(\frac{e_1}{N}, ..., \frac{e_n}{N}\right) \\
&= \lim_{N\to\infty} \sum_{i_j=1, j\in[1,n]}^{N} \frac{\partial f}{\partial x_1}(x_I^*) dx_1 \wedge \cdots \wedge dx_n \left(\frac{e_1}{N}, ..., \frac{e_n}{N}\right) \\
&\text{(By MVT and evaluation of m-forms)} \\
&= \lim_{N\to\infty} \sum_{i_j=1, j\in[1,n]}^{N} \frac{f\left(\frac{i_1+1}{N}, i_2, ..., i_n\right) - f(x_I)}{\frac{1}{N}(i_1+1-i_1)} \frac{dx_2 \wedge \cdots dx_n}{N} \left(\frac{e_1}{N}, ..., \frac{e_n}{N}\right) \\
&= \lim_{N\to\infty} \sum_{i_j=1, j\in[1,n]}^{N} \left(f\left(1, \frac{i_2}{N}, ..., \frac{i_n}{N}\right) - f\left(0, \frac{i_2}{N}, ..., \frac{i_n}{N}\right)\right) dx_2 \wedge \cdots \wedge dx_n \left(\frac{e_1}{N}, ..., \frac{e_n}{N}\right) \\
&= \int_{\{1\}\times[0,1]^{n-1}} \omega - \int_{\{0\}\times[0,1]^{n-1}} \omega = \int_{\{0^-,1^+\}\times[0,1]^{n-1}} \omega = \int_{\partial R} \omega
\end{aligned}$$

Note: $\partial R = \{0^-, 1^+\} \times [0,1]^{n-1} \cup [0,1]\{0^-, 1^+\} \times [0,1]^{n-2} \cup \cdots$, but the following terms will contribute zero to the results. □