ECE1647 Introduction to Nonlinear Systems

Introduction

- Goal: to analyze (not model or design/control) a nonlinear system
- Caveate: to perform a rigorous mathematical analysis, generally the system has to have a reasonable dimension and/or be a structured model

Dynamic systems of the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, ..., x_n) \\ &\cdots \\ \dot{x}_n &= f(x_1, ..., x_n), \end{aligned}$$

where $x_1(t), ..., x_n(t) \in \mathbb{R}$ are the states and each $f_i : \mathbb{R}^n \to \mathbb{R}$ is a nonlinear map. Each equation is an autonomous ODE. More generally, $\dot{x}_i = f_i(t, x_1, ..., x_n)$ is a non-autonomous nonlinear ODE.

Introduce the state vector $x(t) = \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$, $f(t,x) = \begin{bmatrix} f_1(t, x_1, ..., x_n) \\ \vdots \\ f_n(t, x_1, ..., x_n) \end{bmatrix}$. Then we write $\dot{x} = f(x)$ or $\dot{x} = f(t,x)$. At times, we consider $\dot{x} = f(x,u)$, where $u \in \mathbb{R}^m$ is the control input.

A special form of nonlinear control system is called **control affine system**

$$\dot{x} = f(x) + g(x)u,$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$, $x \in \mathbb{R}^n$, $g : \mathbb{R}^m \to \mathbb{R}^n$, $u \in \mathbb{R}^m$. Even more ubiquitous is **LTI systems**:

$$\dot{x} = Ax + Bu.$$

A third class of models is for mechanical systems (*e.g.* robotic manipulation):

$$m(q)\ddot{q} + c(q, \dot{q})\dot{q} + g(q) = u$$

Note: this model is nonlinear, but not presented as a state model. To convert to a state model, we must define states. The states for a mechanical system are $x = \begin{bmatrix} q \\ \dot{q} \end{bmatrix}$, $q \in \mathbb{R}^N$. The system is

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= m(q)^{-1}[-c(q,\dot{q})\dot{q} - g(q) + u] = m(x_1)^{-1}[-c(x_1,x_2)x_2 - g(x_q) + u] \end{aligned}$$

Why nonlinear models?

- Nonlinear models arise in significant applications
- Nonlinear phenomena (distinct from linear systems)
  - Finite escape time
  - Multiple isolated equilibrium
  - Limit cycles
  - Chaos

0.1 Nonlinear Analysis

A. Existence and Uniqueness of Solutions
Example: $\dot{x} = -x$, $x(t) = x(0)e^{-t}$ is exponentially stable.
$\dot{x} = -x^2$ has finite escape time.
Key concept: Lipschitz continuity.

B. Invariant Sets
Key concept: Nagumo Theorem

C. Stability
Type of stability: Globally Exponential Stability, Asymptotic Stability, Stability

Key concept: Lyapunov analysis
Suppose for $\dot{x} = f(x), f(0) = 0$, then $x = 0$ is an equilibrium. We want to study stability of the equilibrium.

Define a Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$, Compute the Lie derivative: $\frac{dV(x(t))}{dt} = \frac{\partial V}{\partial x}\big|_{x=x(t)} \cdot \frac{dx}{dt} = \frac{\partial V}{\partial x}(x(t))f(x(t))$. We want $\dot{V} \leq 0$ or more preferrably $\dot{V} \leq \alpha V$.

0.2 Examples

Pendulum Model
Using Newton's 2nd law:

$$ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$$

Define the state $x_1 = \theta, x_2 = \dot{\theta}$. Then the state model is

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

It has the form: $\dot{x} = f(x)$ where $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. Let $f(x) = 0$ and solve for x, we get $(x_1, x_2) = (k\pi, 0)$.

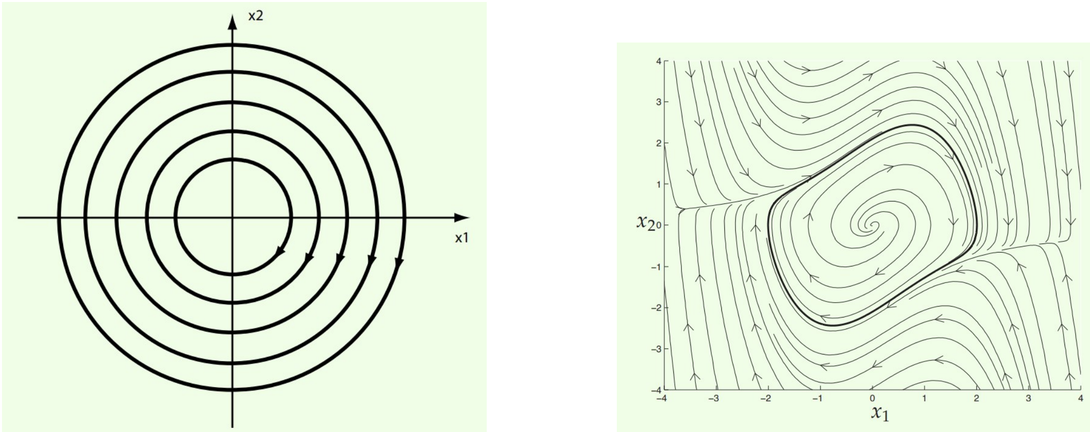
Van der Pol Equation

$$\ddot{V} - \epsilon(1 - V^2)\dot{V} + V = 0, \epsilon > 0$$

Define $x_1 = V$, $x_2 = \dot{V}$, then

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$

If $\epsilon = 0$, then $\dot{x} = Ax$, with $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$. It is an oscillator.
Otherwise, $\epsilon > 0$, we get a limit cycle.



(a) $\epsilon = 0$ (b) $\epsilon > 0$

Figure 1: Phase Plots for Van der Pol Equation

Adaptive Control

$$\begin{aligned} \dot{x} &= Ax + Bu - Bd \\ e &= cx, \end{aligned}$$

where $d = \phi^T w$, where w is a known regressor, ϕ is an unknown parameter.
The standard solution in adaptive control is

$$u = kx + \hat{\phi}^T w$$

The gradient law for parameter adaptation is

$$\dot{\hat{\phi}} = -\gamma e w.$$

Then we get $\dot{x} = (A + Bk)x + B(\hat{\phi}^T w - \phi^T w)$.

Let $\tilde{\phi} = \hat{\phi} - \phi$, $\dot{x} = (A + Bk)x + B\tilde{\phi}w$, with $\dot{\tilde{\phi}} = -\gamma e w$.

0.3 Phase Portraits

Consider a second-order linear system $\dot{x} = Ax$, $x \in \mathbb{R}^2$. Depending on the eigenvalues of A, the real Jordan has one of these forms

1. $\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, $\lambda_1, \lambda_2 \in \sigma(A)$.
2. $\begin{bmatrix} \lambda & * \\ 0 & \lambda \end{bmatrix}$, $* \in \{0, 1\}$.
3. $\begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$, $\lambda_1, \lambda_2 = \alpha \pm i\beta$.

Case 1: Two real distinct eigenvalues $\lambda_1 \neq \lambda_2 \neq 0$.
If $\lambda_1 < 0, \lambda_2 > 0$. This is a **saddle point**.
If $\lambda_1 < 0, \lambda_2 < 0$. This is a **stable node**. It is exponentially stable equilibrium
If $\lambda_1 > 0, \lambda_2 > 0$. This is unstable.

Case 2: $\lambda_{1,2} = \alpha \pm i\beta$.
If $\alpha > 0$, unstable focus
If $\alpha = 0$, center (oscillator)
If $\alpha < 0$, stable focus.

1 Mathematical Background

Definition: 1.1: Norm

A norm of $\mathbb{R}^n$ is a function $\|\cdot\| : \mathbb{R}^n \to \mathbb{R}$ satisfying:
- $\|x\| \geq 0$, $\|x\| = 0 \Leftrightarrow x = 0, \forall x \in \mathbb{R}^n$
- $\|\lambda x\| = |\lambda| \, \|x\|$, $\forall x \in \mathbb{R}^n, \lambda \in \mathbb{R}$
- $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^n$

Examples:

- Euclidean Norm: $\|x\|_2 = (x^T x)^{1/2}$
- p-Norm: $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p}$
- ∞-Norm: $\|x\|_\infty = \max_i |x_i|$

The notion of a norm is a generalization of the length of a vector.

If we take a vector space X, equipped with a norm $\|\cdot\|$, denoted $(X, \|\cdot\|)$ is called a *normed vector space*.

1.1 Sequences

Definition: 1.2: Limit of a Sequence

Consider a sequence $\{x_n\}$ of vectors in $(X, \|\cdot\|)$. We say $\{x_n\}$ converges to an element $x^* \in X$ if $\|x_n - x^*\| \to 0$ as $n \to \infty$. Equivalently, $\forall \epsilon > 0$, $\exists N(\epsilon) > 0$ s.t. if $n \geq N(\epsilon)$, then $\|x_n - x^*\| < \epsilon$. Notation: $\lim_{n \to \infty} x_n = x^*$ or $x_n \to x^*$ as $n \to \infty$.

This definition is hard to work with in practice, because we have to already know x^*.

Definition: 1.3: Monotonic Sequence

A sequence $\{x_n\}$ of real numbers is monotonically increasing if $x_n \leq x_{n+1}, \forall n$, monotonically decreasing if $x_n \geq x_{n+1}, \forall n$.

Theorem: 1.1: Convergence of Monotonic Sequence

Suppose $\{x_n\}$ is monotonic. Then $\{x_n\}$ converges if and only if it is bounded.

Definition: 1.4: Cauchy Sequence

A sequence $\{x_n\}$ in a normed linear space $(X, \|\cdot\|)$ is said to be a Cauchy sequence if $\forall \epsilon > 0$, $\exists N(\epsilon) > 0$ s.t. if $n, m \geq N(\epsilon)$, then $\|x_n - x_m\| < \epsilon$.

Lemma 1. *Every convergent sequence in a normed linear space is a Cauchy sequence.*

Proof. Suppose $\{x_n\}$ is a convergent sequence with a limit x^*. To prove it is Cauchy, suppose $\epsilon > 0$ is given.
By definition of a convergent sequence, we can select $N(\epsilon) > 0$ s.t. $n \geq N(\epsilon)$, then $\|x_n - x^*\| < \frac{\epsilon}{2}$.

Consider $p, q \geq N(\epsilon)$, $\|x_p - x_q\| \leq \|x_p - x^*\| + \|x_q - x^*\| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$.
Hence, $\{x_n\}$ is Cauchy. □

Definition: 1.5: Banach Space

A normed linear space is called a Banach space if every Cauchy sequence converges.

Similarly for functions $f : X \to Y$, we write $\lim_{x \to x_0} f(x) = y_0$, if $\forall \epsilon > 0$, $\exists \delta > 0$, s.t. $\|x - x_0\| < \delta \Rightarrow \|f(x) - y_0\| < \epsilon$

1.2 Continuous Functions

Definition: 1.6: Continuous Functions

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at $x_0 \in \mathbb{R}^n$ if $\forall \epsilon > 0$, $\exists \delta(\epsilon, x_0) > 0$ s.t. $\|x - x_0\| < \delta \Rightarrow \|f(x) - f(x_0)\| < \epsilon$. Equivalently, $\lim_{x \to x_0} f(x) = f(x_0)$.

Definition: 1.7: Uniform Continuous

$f : \mathbb{R}^n \to \mathbb{R}^m$ is uniformly continuous if it is continuous and $\delta(\epsilon)$ does not depend on x_0.

Often, continuity is characterized in terms of sequences

Theorem: 1.2:

Let $f : X \to Y$. Then $\lim_{x \to x_0} f(x) = y_0$ if and only if $\lim_{n \to \infty} f(x_n) = y_0$ for every sequence $\{x_n\}$ s.t. $x_n \neq x_0$, $\lim_{n \to \infty} x_n = x_0$.

It shows the close relationship between sequences and continuity.
Let $\{x_n\}$ s.t. $\lim_{k \to \infty} x_k = x_0$. Let $f : X \to Y$ be continuous at x_0, $y_k = f(x_k)$, $y_0 = f(x_0)$.
Then we can write $y_0 = f(x_0) = f(\lim_{k \to \infty} x_k) = \lim_{k \to \infty} f(x_k) = \lim_{k \to \infty} y_k = y_0$.

A key property of continuous functions requres their boundedness properties on certain sets.

Definition: 1.8: Compact Set

$\Omega \subset X$ is compact if it is closed and bounded.

Theorem: 1.3: Bounded Functions

Let $f : X \to Y$ be continuous at every $x \in X$. Let $\Omega \subset X$ be a compact set in X. Then f is bounded on Ω. *i.e.* $\exists M > 0$ s.t. $\forall x \in \Omega$, $\|f(x)\| \leq M$.

Theorem: 1.4:

Let $f : X \to \mathbb{R}$ be continuous and $\Omega \subset X$ is compact. Then $\exists x_{min}, x_{max}$ s.t. $f(x_{min}) = \inf_{x \in \Omega} f(x)$, $f(x_{max}) = \sup_{x \in \Omega} f(x)$

Definition: 1.9: Lipschitz Continuous

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is Lipschitz continuous at $x_0 \in \mathbb{R}^n$ if $\exists \delta, L > 0$ s.t. $\forall x, y \in \mathbb{R}^n$ with $x, y \in B_\delta(x_0)$, $\|f(x) - f(y)\| \leq L \|x - y\|$.
Let $\Omega \subset \mathbb{R}^n$, f is *locally Lipschitz* on Ω if f is Lipschitz at every $x \in \Omega$.
Let $\Omega \subset \mathbb{R}^n$, f is *globally Lipschitz* on Ω if $\exists L > 0$ s.t. $\forall x, y \in \Omega$, $\|f(x) - f(y)\| \leq L \|x - y\|$.

Note that for locally Lipschitz, the choice of L depends on x_0.

Theorem: 1.5:

Let $\Omega \subset \mathbb{R}^n$ be compact. If $f : \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz on Ω, then it is globally Lipschitz on Ω.

Example: $f(x) = x^2$ is locally Lipschitz on $\mathbb{R}$, but it is not globally Lipschitz on $\mathbb{R}$.

Proof. Locally Lipschitz:
Let $x_0 \in \mathbb{R}$, choose $\delta > 0$, $L = 2(\delta + \|x_0\|)$.
Let $x, y \in B_\delta(x_0)$, then we have $\left\|x^2 - y^2\right\| = \|x - y\| \|x + y\| \leq \|x - y\| [\|x\| + \|y\|]$.
Since $x \in B_\delta(x_0)$, $\|x\| \leq \|x - x_0\| + \|x_0\| \leq \delta + \|x_0\|$. Similarly, $\|y\| \leq \delta + \|x_0\|$.
Therefore, $\left\|x^2 - y^2\right\| \leq \|x - y\| 2(\delta + \|x_0\|) = L \|x - y\|$.

Globally Lipschitz:
Assume $\exists L > 0$ s.t. $\forall x, y \in \mathbb{R}$, $\|f(x) - f(y)\| \leq L \|x - y\|$.
Choose $x = 0$, $y = 2L$, we get $\left\|4L^2\right\| = 2L \|x - y\| > L \|x - y\|$. Contradiction. □

1.3 Matrix Norms

Consider a norm $\|\cdot\|$ on $\mathbb{R}^n$. Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. Define $\|A\| = \sup_{x \neq 0, x \in \mathbb{R}^n} \frac{\|Ax\|}{\|x\|} = \sup_{\|x\|=1} \|Ax\|$.

Lemma 2. $\|A\|$ *is a norm.*

Proof. The first two conditions are trivial.
For the triangle inequality,

$$\|A + B\| = \sup_{\|x\|=1} \|(A + B)x\| = \sup_{\|x\|=1} \|Ax + Bx\| \leq \sup_{\|x\|=1} \|Ax\| + \sup_{\|x\|=1} \|Bx\| = \|A\| + \|B\|$$

□

Note: $\|Ax\| = \|Ax\| \frac{\|x\|}{\|x\|} = \left\|A \frac{x}{\|x\|}\right\| \|x\| \leq \|A\| \|x\|$.

1.4 Existence and Uniqueness of Solutions of ODEs

Definition: 1.10: Fixed Point

Let $(X, \|\cdot\|)$ be a Banach space. Let $P : X \to X$ be a map on X. An element $x^* \in X$ is a fixed point of P if $P(x^*) = x^*$.

Theorem: 1.6: Contraction Mapping Thereom

Let $P : X \to X$ be a map for which there exists $\rho \in (0,1)$ s.t. $\|P(x) - P(y)\| \leq \rho \|x - y\|, \forall x, y \in X$. Then

1. There exists a unique x^* s.t. $P(x^*) = x^*$
2. $\forall x \in X$, the sequence defined by $x_0 = x$, $x_{n+1} = P(x_n)$ converges to x^*
3. Moreover, $\|x^* - x_n\| \leq \frac{\rho^n}{1-\rho} \|P(x_0) - x_0\|$

Proof. Let $x \in X$, we show that $\{x_n\}$ forms a Cauchy sequence.
For each $n \geq 0$, we have $\|x_{n+1} - x_n\| \leq \rho \|x_n - x_{n-1}\| \leq \cdots \leq \rho^n \|x_1 - x_0\|$.
Let $m = n + r, r \geq 0$. Then

$$\begin{aligned}
\|x_m - x_n\| = \|x_{n+r} - x_n\| &\leq \sum_{i=0}^{r-1} \|x_{n+i-1} - x_{n+i}\| \text{ (Triangle Inequality)} \\
&\leq \sum_{i=0}^{r-1} \rho^{n+i} \|x_1 - x_0\| \leq \sum_{i=0}^{\infty} \rho^{n+i} \|x_1 - x_0\| \\
&= \frac{\rho^n}{1-\rho} \|x_1 - x_0\|
\end{aligned}$$

Since $\rho \in (0,1)$, we can make $\|x_m - x_n\|$ small by choosing n sufficiently large. Therefore $\{x_n\}$ is a Cauchy sequence. Because X is Banach, $\|x_n\|$ converges to some $x^k \in X$.
Apply Definition 1.7, $\|P(x) - P(y)\| \leq \rho \|x - y\|$ to show $P(x)$ is uniformly continuous.
Then $P(x^*) = P(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} P(x_n) = \lim_{n\to\infty} x_{n+1} = x^*$. Hence x^* is a fixed point.

To show that x^* is unique, suppose $y^* \neq x^*$ is a fixed point.

$$\|x^* - y^*\| = \|P(x^*) - P(y^*)\| \leq \rho \|x^* - y^*\| \Rightarrow \|x^* - y^*\| = 0$$

Therefore, $x^* = y^*$.

To prove 3, we use the fact that $\|\cdot\|$ is a continuous function. We have

$$\|x^* - x_n\| = \left\| \lim_{m\to\infty} x_m - x_n \right\| = \lim_{m\to\infty} \|x_m - x_n\| \leq \frac{\rho^n}{1-\rho} \|x_1 - x_0\|$$

□

Consider a non-linear ODE

$$\begin{aligned}
\dot{x} &= f(x) \\
x(0) &= x_0
\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state and $f : \mathbb{R}^n \to \mathbb{R}^n$. Consider a time interval $[0,T]$ and denote $C^n[0,T]$ the set of all continuous functions mapping $[0,T]$ to $\mathbb{R}^n$. Define a norm $\|\cdot\|_C = \max_{t\in[0,T]} \|x(t)\|$.

Proof. Triangle Inequaliy: Let $x, y \in C^n[0,T]$. Then

$$\begin{aligned}
\|x + y\|_C = \max_{t\in[0,T]} \|x(t) + y(t)\| &\leq \max_{t\in[0,T]} (\|x(t) + y(t)\|) \\
&\leq \max_{t\in[0,T]} \|x(t)\| + \max_{t\in[0,T]} \|y(t)\| = \|x\|_C + \|y\|_C
\end{aligned}$$

□

Fact: $(C^n[0,T], \|\cdot\|_C)$ is a Banach space.

Definition: 1.11: Solution of ODE

A solution of a nonlinear ODE over $[0,T]$ is an element $x(\cdot) \in C^n[0,T]$ s.t.

1. $\dot{x}(t)$ is defined almost everywhere (a.e.)
2. Equation (1) holds at every t where $\dot{x}$ is defined.

Remark 1. If $x(t)$ is a solution of (1) over $[0,T]$, then $x(t)$ also satisfies

$$x(t) = x_0 + \int_0^t f(x(\tau))d\tau \tag{2}$$

Conversely, if $x(\cdot) \in C^n[0,T]$ satisfies (2), then x is differentiable and satisfies (1). Every solution of (1) is a solution of (2) and vice versa.

Theorem: 1.7: Peano

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuous, then for each $x_0 \in \mathbb{R}^n$, there exists at least one solution.

Example: $\dot{x} = x^{1/3}, x(0) = 0$ has two solutions: $x(t) = 0$ and $x(t) = \left(\frac{2}{3}t\right)^{3/2}$

Example: $\dot{x} = \sqrt{|x|}$ has infinitely many solutions $x(t) = \begin{cases} 0, t \le c \\ -\frac{(c-t)^2}{2}, t > c \end{cases}$

Lemma: 1.1: Bellman-Gronwall

Let $y : [0,T] \to \mathbb{R}$ be a nonnegative continuous function. Let $C \ge 0$ and $L \ge 0$ s.t. $y(t) \le C + \int_0^t Ly(\tau)d\tau$. Then $y(t) \le C\exp(Lt), \forall t \in [0,T]$.

Proof. Let $r(t) = C + \int_0^t Ly(\tau)d\tau$, then $y(t) \le r(t), \forall t \in [0,T]$.
Also $\dot{r}(t) = Ly(t) \le Lr(t)$. $\dot{r}(t) - Lr(t) \le 0, \forall t \in [0,T]$.
Using integration factor, $r(t)\exp(-Lt) \le r(0) = C$.
Therefore, $y(t) \le r(t) \le C\exp(Lt), \forall t \in [0,T]$. □

Theorem: 1.8: Picard-Lindelof

Consider $\dot{x} = f(x), x(0) = x_0$. Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz, *i.e.* $\exists L > 0$ s.t. $\|f(x) - f(y)\| \le L\|x - y\|$. Then the ODE has exactly one solution over $[0,T]$ for any $T \in [0,\infty]$ and $x_0 \in \mathbb{R}^n$.
Note: this version uses a more restrictive globally Lipschitz condition, which can be replaced by locally Lipschitz.

Proof. Fix $T < \infty$. Define a mapping $P : C^n[0,T] \to C^n[0,T]$ (Picard iteration) s.t. $(Px)(t) = x_0 + \int_0^t f(x(\tau))d\tau$.
Define $x_k = (P^k x_0)(\cdot)$ We show that $\{x_k\}_k$ is a Cauchy sequence.

Note $x_1(t) - x_0(t) = \int_0^t f(x_0(\tau))d\tau$.

$$\begin{aligned}
\|x_1(t) - x_0(t)\| &\le \int_0^t \|f(x_0(\tau))\| \, d\tau \text{ (Triangle Inequality of Integrals)} \\
&\le \int_0^t L \|x_0(\tau)\| \, d\tau \text{ (Lipschitz)} \\
&\le mt \text{ for some constant } m \\
\|x_2(t) - x_1(t)\| &\le L \int_0^t \|x_1(\tau) - x_0(\tau)\| \, d\tau \\
&\le Lm\frac{t^2}{2} \\
&\vdots \\
\|x_{k+1}(t) - x_k(t)\| &\le \int_0^t \|f(x_k(\tau)) - f(x_{k-1}(\tau))\| \, d\tau \\
&\le L \int_0^t \|x_k(\tau) - x_{k-1}(\tau)\| \, d\tau
\end{aligned}$$

Iteratively, we get $\|x_{k+1}(t) - x_k(t)\| \le L^{k-1} M \frac{t^k}{k!}$, where $k!$ comes from the interation of $t, t^2, \ldots$. Therefore,

$$\|x_{k+p}(t) - x_k(t)\| \le \sum_{i=0}^{p-1} \|x_{k+i+1}(t) - x_{k+i}(t)\| \le \sum_{i=0}^{p-1} ML^{k+i} \frac{t^{k+i+1}}{(k+i+1)!}$$

$$\|x_{k+p} - x_k\|_C = \max_{t \in [0,T]} \|x_{k+p}(t) - x_k(t)\| \le \sum_{i=k+1}^{k+p} ML^{i-1}\frac{T^i}{i!} \le \sum_{i=k+1}^{\infty} ML^{i-1}\frac{T^i}{i!}$$

Consider the sequence $\left\{ \sum_{i=0}^{k} ML^{i-1}\frac{T^i}{i!} \right\}_{k=1}^{\infty} = \left\{ \sum_{i=0}^{k} \frac{M}{L}\frac{(LT)^i}{i!} \right\}_{k=1}^{\infty} \to \frac{M}{L}\exp(LT)$ as $k \to \infty$.

Then $\sum_{i=k+1}^{\infty} ML^{i-1}\frac{T^i}{i!} = \frac{M}{L}\exp(LT) - \sum_{i=0}^{k} ML^{i-1}\frac{T^i}{i!} \to 0$ as $k \to \infty$.

Hence x_k is Cauchy in $C^n[0,T]$, so it converges to $x^* \in C^n[0,T]$.

Now we show that x^* is a solution of (2).

Let $z_1, z_2 \in C^n[0,T]$. Then $(Pz_1)(t) - (Pz_2)(t) = \int_0^t f(z_1(\tau)) - f(z_2(\tau))d\tau$. By Lipschitz and bounded time:

$$\|(Pz_1)(t) - (Pz_2)(t)\| \le \int_0^t \|f(z_1(\tau)) - f(z_2(\tau))\| \, d\tau \le LT \|z_1 - z_2\|_C$$

Therefore, $\|Pz_1 - Pz_2\|_C \le LT \|z_1 - z_2\|_C$.

If $\{x_k\}$ converges to x^*, then $(Px^*) = P(\lim_{k\to\infty} x_k) = \lim_{k\to\infty}(Px_k) = \lim_{k\to\infty} x_{k+1} = x^*$. Therefore x^* satisfies (2).

Next we show x^* is unique. Let y^* satisfy (2).

$$\|x^*(t) - y^*(t)\| \le \int_0^t \|f(x^*(\tau)) - f(y^*(\tau))\| \, d\tau \le L \int_0^t \|x^*(\tau) - y^*(\tau)\| \, d\tau$$

By Lemma 1.1, $\|x^*(t) - y^*(t)\| = 0$, $x^* = y^*$ $\square$

1.5 Differentiability

***Definition:* 1.12: Differentiability (Scalar)**

A function $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $x_0 \in \mathbb{R}$ if the limit exists

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Notation: $df_{x_0} h = f'(x_0)h$.

Rewrite as $\lim_{h \to 0} \frac{|f(x_0 + h) - f(x_0) - df_{x_0} h|}{h} = 0$.

***Definition:* 1.13: Differentiability (General)**

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at $x_0 \in \mathbb{R}^n$ if

$$\lim_{\|v\| \to 0} \frac{\|f(x_0 + v) - f(x_0) - df_{x_0} v\|}{\|v\|} = 0$$

f is differentiable if f is differentiable at every $x_0 \in \mathbb{R}^n$. The matrix $df_{x_0} \in \mathbb{R}^{m \times n}$ is called the *derivative, differential, or Jacobian.*
A function $f \in C^1$ (continuously differentiable) if f is differentiable and df_{x_0} is continuous as a function of x_0.

***Theorem:* 1.9:**

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at x_0, then the partial derivatives $\left.\frac{\partial f_i}{\partial x_j}\right|_{x_0}$ exist and moreover are the elements of df_{x_0}, $(df_{x_0})_{ij} = \left.\frac{\partial f_i}{\partial x_j}\right|_{x_0}$.

***Theorem:* 1.10:**

$f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1 if and only if $\frac{\partial f_i}{\partial x_j}$ exist and are continuous functions.

***Theorem:* 1.11:**

If $f : \mathbb{R}^n \to \mathbb{R}^m$ is C^1, then f is locally Lipschitz.

1.6 Comparison, Continuity and Finite Escape Time

***Lemma:* 1.2: Comparison Lemma**

Consider the ODE $\dot{x} = f(x), x(0) = x_0$, where $f : \mathbb{R} \to \mathbb{R}$ and f is locally Lipschitz on $\mathbb{R}$. Suppose we have a solution $x(t)$ on time interval $[0, T]$. Let $w(t)$ be a C^1 function s.t. $\dot{w}(t) \leq f(w(t))$, $w(0) \leq x_0$, $\forall t \in [0, T]$. Then $w(t) \leq x(t), \forall t \in [0, T]$.

Example: Consider the ODE $\dot{y} = -(1 + y^2)y, y(0) = y_0$. We can verify that it has a unique solution on $[0, \delta]$ since $f(y) = -(1 + y^2)y$ is locally Lipschitz.
Define $w(t) = y^2(t)$. Then $\dot{w}(t) = 2y\dot{y} = -2y^2(1 + y^2) = -2w(1 + w) = -2w - 2w^2 < -2w$, $w(0) = y_0^2$.
Consider $\dot{x} = -2x, x(0) = y_0^2$.

It has solution: $x(t) = y_0^2 \exp(-2t)$.
By Lemma 1.2, $y^2(t) = w(t) \leq x(t) = y_0^2 \exp(-2t)$, then $|y(t)| \leq |y_0| \exp(-t)$.

Remark 2. Lemma 1.2 is often used in stability proof.

***Theorem:* 1.12: Continuity w.r.t. Initial Condition**

Suppose $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz, with Lipschitz constant $L > 0$. Consider $x(t), x'(t)$ two solutions of $\dot{x} = f(x)$ for $t \in [0, T]$. Then $\forall t \in [0, T]$, $\|x(t) - x'(t)\| \leq \|x(0) - x'(0)\| \exp(Lt)$.

Proof. Define $y(t) = \|x(t) - x'(t)\| \geq 0$ for $t \in [0, T]$. We know that

$$x(t) - x'(t) = x(0) - x'(0) + \int_0^t (f(x(\tau)) - f(x'(\tau)))d\tau$$

$$\begin{aligned}
\text{Then, } y(t) = \left\|x(t) - x'(t)\right\| &\leq \left\|x(0) - x'(0) + \int_0^t \left\|f(x(\tau)) - f(x'(\tau))\right\| d\tau\right\| \\
&\leq y(0) + \int_0^t L \left\|x(\tau) - x'(\tau)\right\| d\tau \text{ (By Lipschitz)} \\
&= y_0 + \int_0^t L \left\|y(\tau)\right\| d\tau.
\end{aligned}$$

By Lemma 1.1, $y(t) \leq y(0) \exp(Lt)$. □

***Theorem:* 1.13: Finite Escape Time**

If $f : \mathbb{R}^n \to \mathbb{R}^n$ is globally Lipschitz, then solutions exist for all $t \geq 0$ for each $x_0 \in \mathbb{R}^n$.
If $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, then Theorem 1.8 gives a solution $x(t)$ for $t \in (-\delta, \delta)$.

Example: $\dot{x} = x^2$, $f(x) = x^2$ is not globally Lipschitz, but locally Lipschitz.
$x(t) = \frac{x_0}{1 - x_0 t}$. It has maximum existence time of $T_{x_0} = [0, \frac{1}{x_0})$.

Note: globally Lipschitz is not a necessary condition for solution to exist on $T_{x_0} = [0, \infty)$

2 Dynamic Systems

For this section, we always consider $\dot{x} = f(x), x \in \mathbb{R}^n$, where f is Lipschitz.

2.1 Invariant Set

Equilibria and closed orbits (periodic solutions) are examples of invariant sets.

***Definition:* 2.1: Invariant**

A set $\Omega \subset \mathbb{R}^n$ is invariant under $\dot{x} = f(x)$ if $\forall x_0 \in \Omega$, the solution starting at $x(0) = x_0$ satisfies $x(t) \in \Omega$, $\forall t \in T_{x_0}$ (domain on which the solution exists)
Notation: Write $\phi(t, x_0)$ to denote the solution $x(t)$ starting at x_0.

***Definition:* 2.2: Positively/Negatively Invariant**

A set $\Omega \subset \mathbb{R}^n$ is positively invariant if $\forall x_0 \in \Omega$, $t \in T^+_{x_0}$ =forward time domain, $\phi(t, x_0) \in \Omega$. Similarly, $\Omega \subset \mathbb{R}^n$ is negatively invariant if $\forall x_0 \in \Omega$, $t \in T^-_{x_0}$, $\phi(t, x_0) \in \Omega$.

Example: For the Van der Pol oscillator, the limit cycle Ω_0, equilibrium point $\Omega_1 = \{(0,0)\}$, $\Omega_2 = \{\text{region enclosed by } \Omega_0\}$ and $\Omega_3 = \{\text{region outside } \Omega_0\}$ are invariant.

Example: $\dot{x}_1 = x_1, \dot{x}_2 = -x_2$. $\Omega_1 = \{(x, 0) : x \in \mathbb{R}\}$, $\Omega_2 = \{(0, y) : y \in \mathbb{R}\}$, and the span of any eienvector of $A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ are invariant.

2.2 Nagumo Theorem

Problem: Given a non-empty and closed set $\Omega \subset \mathbb{R}^n$ and an ODE $\dot{x} = f(x)$. Find conditions of $f(x)$ s.t. Ω is positively invariant.

Intuition: In order for $\phi(t, x_0)$ to stay inside Ω, $f(x)$ should point inside Ω at $x \in \partial\Omega$.

Technical difficulty arise in defining the correct notion of pointing inside.

Special case: suppose $\Omega = \{x \in \mathbb{R}^n : \psi(x) \leq c\}$ where $c \in \mathbb{R}$ is a constant and $\psi : \mathbb{R}^n \to \mathbb{R}$. Assume ψ is a C^1 function.

Example: $\psi(x) = x_1^2 + x_2^2$, $\Omega = \left\{x \in \mathbb{R}^2 : \psi(x) \leq 1\right\}$ is the unit ball.

We want $\psi(x)$ to decrease along solutions of $\dot{x} = f(x)$. Thus we want $\frac{d}{dt}\psi(\phi(t, x_0))|_{t=0} \leq 0$ for $x_0 \in \partial\Omega$.

By chain rule, $\frac{d}{dt}\psi(\phi(t, x_0)) = \frac{\partial\psi}{\partial x}\Big|_{\phi(t,x_0)} \cdot \frac{d\phi(t,x_0)}{dt}$, where $\frac{\partial\psi}{\partial x} = d\psi(x) = \left(\frac{\partial\psi}{\partial x_1}, ..., \frac{\partial\psi}{\partial x_n}\right)$ is the differential or derivative of ψ. Also $\nabla\psi(x) = \left(\frac{\partial\psi}{\partial x}\right)^T$ is the gradient of ψ. Since $\phi(t, x_0)$ satisfies $\dot{x} = f(x)$, $\frac{d}{dt}\phi(t, x_0) = f(\phi(t, x_0))$.

Hence $\frac{d}{dt}\phi(\phi(t, x_0)) = \frac{\partial\psi}{\partial x}\Big|_{\phi(t,x_0)} f(\phi(t, x_0))$. Evaluating at $t = 0$, $\frac{d}{dt}\phi(\phi(t, x_0))\Big|_{t=0} = \frac{\partial\psi}{\partial x}(x_0)f(x_0) = \nabla\psi(x_0)^T f(x_0) \leq 0$.

Theorem: 2.1: Special Nagumo I

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz, $\psi : \mathbb{R}^n \to \mathbb{R}$ be C^1. Define $\Omega = \{x \in \mathbb{R}^n : \psi(x) \leq 0\} \neq \emptyset$. Suppose $d\psi(x) = \frac{\partial \psi}{\partial x} \neq 0, \forall x \in \partial\Omega$. Then Ω is positively invariant under $\dot{x} = f(x)$ if and only if $\frac{\partial \psi}{\partial x} f(x) \leq 0$, $\forall x \in \partial\Omega$.

Notation: We write $L_f\psi(x) = \frac{\partial \psi}{\partial x}(x) \cdot f(x)$, the Lie derivative of ψ along f.

Theorem: 2.2: Special Nagumo II

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be locally Lipschitz, $\psi : \mathbb{R}^n \to \mathbb{R}^m$ be C^1 with $m \leq n$. Define $\Omega = \{x \in \mathbb{R}^n : \psi(x) = 0\} \neq \emptyset$. Suppose $\text{rank}(d\psi(x)) = m, \forall x \in \partial\Omega$. Then Ω is positively invariant under $\dot{x} = f(x)$ if and only if $L_f\psi(x) = 0$, $\forall x \in \partial\Omega$.

Example: Consider $\dot{x} = Ax$ with $x \in \mathbb{R}^n$. Given $V \subset \mathbb{R}^n$ a subspace. Suppose V is A-invariant. *i.e* if $x \in V$, then $Ax \in V$. (Notation $AV \subset V$). Claim: V is an invariant set.

$V = \{x \in \mathbb{R}^n : h_1 x = h_2 x = \cdots = h_m x = 0\} = \{x \in \mathbb{R}^n : \psi(x) = 0\}$, where $\psi(x) = \begin{bmatrix} h_1 x \\ \vdots \\ h_m x \end{bmatrix} = \begin{bmatrix} h_1^T \\ \vdots \\ h_m^T \end{bmatrix} x = Hx$. $d\psi(x) = \frac{\partial \psi}{\partial x} = H$.
Then $L_f\psi(x) = d\psi(x) \cdot f(x) = HAx$. If $x \in V$, then $Ax \in V$ and $HAx = 0$.

Example: $\dot{x}_1 = 1, \dot{x}_2 = 1$, $\psi(x) = x_1^2 + x_2^2$.
$d\psi(x) = (2x_1, 2x_2)$. Clearly, $\Omega = \left\{x \in \mathbb{R}^2 : \psi(x) \leq 0\right\} = \{(0,0)\}$ is not positively invariant. However, $L_f\psi(x) = d\psi(x)f(x) = 0$. The problem is that $d\psi(x) = 0$ for $x \in \Omega$.

Definition: 2.3: Bouligand Tangent Cone

Given a set $\Omega \subset \mathbb{R}^n$, define the point to set distance $d_\Omega(x) = \inf_{z \in \Omega} \|x - z\|$. This function is globally Lipschitz but not differentiable.
Let $\Omega \subset \mathbb{R}^n$ be a nonempty closed set. Let $x \in \mathbb{R}^n$. The Bouligand tangent cone to Ω at x is

$$T_\Omega(x) = \left\{ v \in \mathbb{R}^n : \liminf_{\epsilon \searrow 0} \frac{d_\Omega(x + \epsilon v)}{\epsilon} = 0 \right\}$$

For $x \in \Omega$, $T_\Omega(x) = \mathbb{R}^n$. For $x \notin \Omega$, $T_\Omega(x) = \emptyset$. For $x \in \partial\Omega$, $T_\Omega(x) = \{\text{vectors pointing into } \Omega\}$.

Let $\Omega \subset \mathbb{R}^n$ be closed and non-empty. Consider $\dot{x} = f(x)$. We want if $x_0 \in \Omega$, then $\phi(t, x_0) \in \Omega$, $\forall t \geq 0$. That is $d_\Omega(\phi(t, x_0)) = 0$, $\forall t \geq 0$.

Theorem: 2.3:

Let $h : \mathbb{R} \to \mathbb{R}$ be a continuous function, define the lower right Dini derivative:

$$Dh(t) = \liminf_{\epsilon \searrow 0} \frac{h(t + \epsilon) - h(t)}{\epsilon}$$

The continuous function $h : \mathbb{R} \to \mathbb{R}$ is decreasing if and only if $Dh(t) \leq 0$, $\forall t \in \mathbb{R}$.

Apply this to our problem, $h(t) = d_\Omega(\phi(t, x_0))$ is decreasing if and only if

$$Dh(t)|_{t=0} = \liminf_{\epsilon \searrow 0} \frac{d_\Omega(\phi(t, x_0)) - d_\Omega(x_0)}{\epsilon} \leq 0$$

Note if $x_0 \in \Omega$, $d_\Omega(x_0) = 0$. Also by Taylor expansion, $\phi(\epsilon, x_0) = x_0 + \epsilon f(x_0) + o(\epsilon)$, where $\lim_{\epsilon \to 0} \frac{o(\epsilon)}{\epsilon} = 0$, we get

$$Dh(t)|_{t=0} = \liminf_{\epsilon \searrow 0} \frac{d_\Omega(x_0 + \epsilon f(x_0) + o(\epsilon))}{\epsilon}$$

Since d_Ω is globally Lipschitz with Lipschitz constant $L > 0$,

$$|d_\Omega(x_0 + \epsilon f(x_0) + o(\epsilon)) - d_\Omega(x_0 + \epsilon f(x_0))| \leq Lo(\epsilon)$$

Therefore,

$$\begin{aligned} Dh(t)|_{t=0} &= \liminf_{\epsilon \searrow 0} \frac{d_\Omega(x_0 + \epsilon f(x_0) + o(\epsilon)) - d_\Omega(x_0 + \epsilon f(x_0)) + d_\Omega(x_0 + \epsilon f(x_0))}{\epsilon} \\ &= \liminf_{\epsilon \searrow 0} \frac{d_\Omega(x_0 + \epsilon f(x_0))}{\epsilon} \leq 0 \end{aligned}$$

Since the distance function can never go negative, it is equivalent to $\liminf_{\epsilon \searrow 0} \frac{d_\Omega(x_0 + \epsilon f(x_0))}{\epsilon} = 0$.

***Theorem:* 2.4: Nagumo**

Consider $\dot{x} = f(x)$, where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz. Let $\Omega \subset \mathbb{R}^n$ be a closed non-empty set. Then the following are equivalent:

1. $f(x) \in T_\Omega(x), \forall x \in \Omega$
2. Ω is positively invariant.

2.3 Poincare Bendixson Theorem

Limit sets are a special type of invariant sets that capture the steady-state response of a non-linear system.

***Definition:* 2.4: Limit Sets**

Let $x_0 \in \mathbb{R}^n$. A point $p \in \mathbb{R}^n$ is a positive limit point of x_0 if $T_{x_0} = [0, \infty)$ and there exists a sequence of times $\{t_i\}$, $t_i > 0$ with $t_i \to \infty$ such that $\phi(t_i, x_0) \to p$. The set of all positive limit points of x_0 is the positive limit set of x_0, denoted $L^+(x_0)$. Analogously, we can define the negative limit set of x_0, $L^-(x_0)$.

Example: Van der Pol oscillator. Let $x_0 \in \mathbb{R}^n$ and $p \in \Omega$, the limit cycle. $L^+(x_0) = \Omega$.

Notation: positive orbit through x_0, $O^+(x_0) = \left\{\phi(t, x_0) : t \in T^+_{x_0}\right\}$

***Theorem:* 2.5: Birkhoff's Theorem**

Consider $\dot{x} = f(x)$. Assume $f : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. For any $x_0 \in \mathbb{R}^n$, $L^+(x_0)$ and $L^-(x_0)$ are closed invariant sets. Moreover, $O^+(x_0) \subset K \subset \mathbb{R}^n$, where K is a compact, then $L^+(x_0)$ is non-empty, compact, connected, invariant and $d(\phi(t, x_0), L^+(x_0)) \to 0$ as $t \to \infty$, $t \geq 0$. An analogues statement can be made about $L^-(x_0)$.

As an application, we consider limit sets of planar nonlinear systems. This gives the Poincare-Bendixson theory. Consider

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2), \end{aligned}$$

$x \in \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2$. Assume $f : \mathbb{R}^2 \to \mathbb{R}^2$ is C^1. We know from Theorem 2.5 that if $O^+(x_0)$ is bounded, then $L^+(x_0) \neq \emptyset$. Moreover, $L^+(x_0)$ is compact, connected, invariant and $\phi(t, x_0) \to L^+(x_0)$.

Q: When is $L^+(x_0)$ a closed orbit?
A: In $\mathbb{R}^2$, the answer is easy by below. In $\mathbb{R}^n$ for $n \geq 3$, one of Hilbert's problem.

***Theorem:* 2.6: Poincare-Bendixson**

A non-empty compact positive or negative limit set of $\dot{x} = f(x)$, which contains no equilibrium is a closed orbit.

Example: Show that the annulus $\Omega = \left\{x \in \mathbb{R}^2 : \frac{1}{2} \leq x_1^2 + x_2^2 \leq \frac{3}{2}\right\}$ contains a closed orbit.

$$\begin{aligned} \dot{x}_1 &= x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ \dot{x}_2 &= -2x_2 + x_2 - x_2(x_1^2 + x_2^2) \end{aligned}$$

Proof. Note that Ω is compact and it contains no equilibria (The only equilibria is origin).
We can write $\Omega = \Omega_1 \cap \Omega_2$, where $\Omega_1 = \left\{x : x_1^2 + x_2^2 - \frac{3}{2} \leq 0\right\}$, and $\Omega_2 = \left\{x : \frac{1}{2} - x_1^2 - x_2^2 \leq 0\right\}$.
Apply Theorem 2.1 to Ω_1.

$$\begin{aligned} L_f\psi = \frac{\partial \psi}{\partial x} f(x) &= (2x_1, 2x_2) \begin{bmatrix} x_1 + x_2 - x_1(x_1^2 + x_2^2) \\ -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \end{bmatrix} \Bigg|_{x_1^2 + x_2^2 = \frac{3}{2}} \\ &= 2(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) - 2x_1x_2|_{x_1^2 + x_2^2 = \frac{3}{2}} \end{aligned}$$

Apply Young's inequality to the cross term, $-x_1^2 - x_2^2 \leq 2x_1x_2 \leq x_1^2 + x_2^2$. Then

$$L_f\psi \leq 2(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) + (x_1^2 + x_2^2)|_{x_1^2 + x_2^2 = \frac{3}{2}} = 2\frac{3}{2}\left(-\frac{1}{2}\right) + \frac{3}{2} = -\frac{3}{2} + \frac{3}{2} = 0 \leq 0$$

Similarly for Ω_2,

$$L_f\psi_2 = \frac{\partial \psi_2}{\partial x} = -2(x_1^2 + x_2^2)(1 - x_1^2 - x_2^2) + 2x_1x_2|_{\partial\Omega_2} \leq 0$$

Then we apply Theorem 2.5 to show $L^+(x_0)$ is compact, non-empty. By Theorem 2.6, we can deduce that $L^+(x_0)$ is a closed orbit. □

2.3.1 Non-trivial Consequences

***Theorem:* 2.7:**

Let Ω be a compact positively invariant set for $\dot{x} = f(x)$. If Ω contains no equilibrium, then $\forall x_0 \in \Omega$, $O^+(x_0)$ is either a closed orbit or a curve spiralling towards a closed orbit.

***Theorem:* 2.8:**

Let γ be a closed orbit of $\dot{x} = f(X)$, and let Ω be the bounded open set whose boundary is γ. Then Ω contains an equilibrium.

Example: Limit cycles in glycolysis biochemical process used by living cells to extract energy by burning sugar.

$$\begin{aligned} \dot{x}_1 &= -x_1 + ax_2 + x_1^2x_2 \\ \dot{x}_2 &= b - ax_2 + x_1^2x_2, \end{aligned}$$

where x_1, x_2 are concentrations. We study the nullclines where $\dot{x}_1 = 0$ or $\dot{x}_2 = 0$.
$\dot{x}_1 = 0 \Rightarrow x_2 = \frac{x_1}{a+x_1^2}$, $\dot{x}_2 = 0 \Rightarrow x_2 = \frac{b}{a+x_1^2}$.

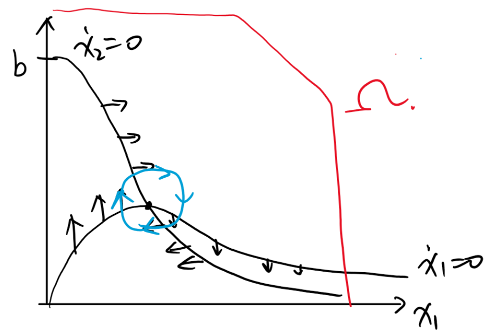


Figure 2: Limit Cycles

***Theorem:* 2.9: Bendixson Criterion**

If $\text{div} f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero and does not change sign on a simply connected set D, then there are no closed orbits of $\dot{x} = f(x)$ entirely in D.

Example: Consider the Van der Pol oscillator:

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -x_1 + \epsilon(1 - x_1^2)x_2$$

. Take $D = \{x \in \mathbb{R}^2 : \|x\| < 1\}$. $\frac{\partial f_1}{\partial x_1} = 0$, $\frac{\partial f_2}{\partial x_2} = \epsilon(1 - x_1^2)$. $\text{div}(f) = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = \epsilon(1 - x_1^2) > 0$ on D. Therefore, there is no closed orbit in D.

3 Lyapunov Stability Theory

Consider the nonlinear system $\dot{x} = f(x)$, where $x(t) \in \mathbb{R}^n$ is the true state vector, $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz.

Definition: 3.1: Equilibrium

$x^* \in \mathbb{R}^n$ is an equilibrium if $f(x^*) = 0$.
Note: if we have a solution $x(t)$ with $x(0) = x^*$, then $x(t) = x^*, \forall t \geq 0$.

Definition: 3.2: Stability

Consider $\dot{x} = f(x)$ with equilibrium $x^* = 0$. We say $x^* = 0$ is stable if $\forall \epsilon > 0$, $\exists \delta > 0$ s.t. if $\|x(0)\| < \delta$, then $\|x(t)\| < \epsilon$ for all $t > 0$. If not, then x^* is unstable.
In Logic notation: $(\forall \epsilon > 0)(\exists \delta > 0)\ \|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \forall t \geq 0$.

Remark 3. Instability does not imply unboundedness.

Example: Van der Pol oscillator. The equilibrium $x = 0$ is unstable, but the solutions are attracted to the limit cycle, thus bounded.

Definition: 3.3: Asymptotic Stability

Consider $\dot{x} = f(x)$ with $f(0) = 0$, $x^* = 0$ is asymptotically stable if

1. It is stable
2. It is attractive: $\exists \delta_0 > 0$ s.t. if $\|x(0)\| < \delta_0$, then $x(t) \to 0$ as $t \to \infty$.

Remark 4.

1. Stability does not imply attractivity. *e.g.* $\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 \end{cases}$. 0 is stable, but not attractive.
2. Attractivity does not imply stability.

Definition: 3.4: Exponential Stability

Consider $\dot{x} = f(x)$ with $f(0) = 0$. We say $x^* = 0$ is exponentially stable if there exists $c, \alpha, \delta > 0$ s.t. $\forall x(0) \in B_\delta(0)$, $\|x(t)\| \leq c\,\|x(0)\|\, e^{-\alpha t}$, $\forall t \geq 0$.

Remark 5. If asymptotic stability or exponential stability hold for any $x(0)$, then we say globally asymptotic stability (GAS) or globally exponential stability (GES).

Consider $\dot{x} = f(x)$ with $f(0) = 0$ and $f : \mathbb{R}^n \to \mathbb{R}^n$ be Lipschitz. A domain $D \subset \mathbb{R}^n$ is an open connected set. Assume $0 \in D$. Let $V : D \to \mathbb{R}$ be continuously differentiable (C^1) on D. Recall the notion of Lie derivative or derivative of V along $\dot{x} = f(x)$ or along solutions of $\dot{x} = f(x)$:

$$\begin{aligned} \dot{V}(x) &= \frac{\partial V}{\partial x}\dot{x}(t) = \frac{\partial V}{\partial x} f(x) \\ &= \left[\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}\right] \begin{bmatrix} f_1(x) \\ \vdots \\ f_n(x) \end{bmatrix} \\ &= \sum_{i=1}^{n} \frac{\partial V}{\partial x_i} f_i(x) = L_f V(x) \end{aligned}$$

Notice: if $\phi(t,x_0)$ is a solution of $\dot{x}=f(x)$, then

$$\begin{aligned}\dot{V}(x_0) &= \frac{d}{dt}V(\phi(t,x_0))\bigg|_{t=0} \\ &= \frac{\partial V}{\partial x}(\phi(t,x_0))\,\frac{d\phi(t,x_0)}{dt}\bigg|_{t=0} = \frac{\partial V}{\partial x}(x_0)f(x_0)\end{aligned}$$

If $\dot{V}(x)<0$, then V will decrease along solutions of $\dot{x}=f(x)$. *i.e.* $\forall x_0\in\mathbb{R}^n$, $V(\phi(t,x_0))$ is a decreasing function of time.

***Theorem:* 3.1: Lyapunov's First Theorem**

Consider $\dot{x}=f(x)$ with $f(0)=0$, and f is locally Lipschitz. Let $D\subset\mathbb{R}^n$ be a domain containing 0. Let $V:D\to\mathbb{R}$ be C^1, satisfying:

1. V is positive definite at 0. *i.e.* $V(0)=0$ and $V(x)>0\ \forall x\in D\setminus\{0\}$.
2. $\dot{V}$ is negative semi-definite, *i.e.* $\dot{V}(x)\le 0$, $\forall x\in D$.

Then $x^*=0$ is stable.
Moreover, if $\dot{V}$ is negative definite, *i.e.* $\dot{V}(x)<0$, $\forall x\in D\setminus\{0\}$, then $x^*=0$ is asymptotically stable.

Proof. Suppose 1, 2 hold.

Step 1: Find a sublevel set of V inside D.
Let $\epsilon>0$, reduce ϵ as necessary s.t. $B_\epsilon(0)\subset D$. Let $c_{\min}=\min_{\|x\|=\epsilon}V(x)$. $c_{\min}$ exists because V is C^1 and $\{x:\|x\|=\epsilon\}$ is compact. Also $c_{\min}>0$ by 1.
Choose $c\in(0,c_{\min})$, and define the sublevel set of V, $\Omega_c=\left\{x\in\overline{B_\epsilon(0)}:V(x)\le c\right\}$.
Claim: $\Omega_c\subset B_\epsilon(0)$, the interior of $\overline{B_\epsilon(0)}$.
Suppose not. Suppose $\exists p\in\Omega_c$ s.t. $\|p\|=\epsilon$. Then $V(p)\ge c_{\min}>c$. Contradiction.

Step 2: Establish that Ω_c is positively invariant.
$\dot{V}(x)=L_fV(x)\le 0$, $\forall x\in\partial\Omega_c$ by 2. This then follows Theorem 2.4

Step 3: $\exists\delta>0$ s.t. $B_\delta(0)\subset\Omega_c$.
Since V is continuous and $V(0)=0$ by 1. $\exists\delta>0$, $\|x\|<\delta\Rightarrow V(x)\le c$. *i.e.* $B_\delta\subset\Omega_c\subset\Omega_\epsilon$.

Since this construction works for any $\epsilon>0$, we have proved $\|x(0)\|<\delta\Rightarrow\|\phi(t,x_0)\|<\epsilon,\forall t\ge 0$, *i.e.* $x^*=0$ is stable. □

Example: $\dot{x}_1=-x_2,\dot{x}_2=-x_1-x_2$, $\dot{x}=Ax$ where $A=\begin{bmatrix}0 & -1\\ -1 & -1\end{bmatrix}$.

Proof. Solve a Lyapunov equation $A^TP+PA=-Q$, where $Q>0$, *i.e.* $x^TQx>0,\forall x\ne 0,Q=Q^T$, for unknown $P=P^T$, $P>0$.

Choose $V(x)=x^TPx$,

$$\dot{V}=2x^TP\dot{x}=2x^TPAx=x^TA^TPx+x^TPAx=-x^TQx<0,\forall x\ne 0$$

Therefore, $x^*=0$ is asymptotically stable. □

Example: Pendulum with friction $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}x_2\\ -\frac{g}{l}\sin(x_1)-\frac{k}{ml^2}x_2\end{bmatrix}$.

Proof. $V(x)=\frac{1}{2}ml^2x_2^2+mgl(1-\cos x_1)$, $\dot{V}=-kx_2^2\le 0$. Only negative semi-definite. □

Question: How to Lyapunov's theorem to characterize globally asymptotic stability?
Note: The previous Theorem 3.1 cannot be used to characterize GAS in the following example:

$$\dot{x}_1 = \frac{-6x_1}{(1+x_1^2)^2} + 2x_2$$
$$\dot{x}_2 = \frac{-2(x_1+x_2)}{(1+x_1^2)^2}$$

Choose the Lyapunov function $V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$. $V(x)$ is p.d. at $x=0$ and n.s.d. at $x=0$.
However, not all level sets of V are bounded, $\exists c^* > 0$ s.t. Ω_{c^*} is not compact, and we cannot characterize the GAS.

There is a simple fix to ensure every sublevel set of V is compact: $V(x) \to \infty$ as $\|x\| \to \infty$. That is $V(x)$ is radially unbounded.

***Theorem:* 3.2: Barbashin-Krasovskii**

Consider $\dot{x} = f(x)$ with $f : \mathbb{R}^n \to \mathbb{R}^n$ locally Lipschitz, and $f(0) = 0$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function s.t.

1. V is p.d. at 0: $V(0) = 0$, and $V(x) > 0, \forall x \neq 0$.
2. $\dot{V}$ is n.d. at 0: $\dot{V}(0) = 0$, and $\dot{V}(x) < 0, \forall x \neq 0$.
3. V is radially unbounded: $V(x) \to \infty$ as $\|x\| \to \infty$.

Then $x^* = 0$ is globally asymptotically stable (GAS).

Proof. The stability part does not change. It remains to show that $\forall x(0) \in \mathbb{R}^n, x(t) \to 0$.
To the end, consider any $x(0) \in \mathbb{R}^n$ and let $c = V(x(0))$. $3 \Rightarrow \forall c > 0, \exists r > 0$ s.t. $V(x) > c$ if $\|x\| > r$.
Therefore, $\Omega_c \subset \overline{B}_r(0)$, the closed ball of radius r centered at 0. Ω_c is bounded.
We can reapply the attracting argument for all $x(0) \in \mathbb{R}^n$, $V(x(t)) \to 0$.
Since $V(x(t))$ is decreasing and converges, $V(x(t)) \to \epsilon > 0$, $\dot{V} < -\gamma$, then

$$V(x(t)) = V(x(0)) + \int_0^t \dot{V}(x(\tau))d\tau \leq V(x(0)) - \gamma t$$

Therefore, by continuity, $x(t) \to 0$. □

3.1 Stability of LTI Systems

$$\dot{x} = Ax, x(t) \in \mathbb{R}^n$$

***Theorem:* 3.3:**

$x^* = 0$ is asymptotically stable $\Leftrightarrow \sigma(A) \subset \mathbb{C}^-$ (spectrum of A lies in the Re < 0 half plane). *i.e.* A is Hurwitz.

We seek a Lyapunov characterization. Consider a quadratic Lyapunov function $V(x) = x^T P x$, where $P = P^T$ and p.d. *i.e.* $x^T P x > 0, x \neq 0$ and $x^T P x = 0$ for $x = 0$.

$$\begin{aligned}\dot{V} &= \dot{x}^T P x + x^T P \dot{x} = 2x^T P \dot{x} \\ &= 2x^T P A x = x^T (A^T P + PA) x \\ &= -x^T Q x, \text{ for some } Q = Q^T\end{aligned}$$

We need to solve $A^T P + PA = -Q$, the Lyapunov equation.

Theorem: 3.4:

A is Hurwitz ($\sigma(A) \subset \mathbb{C}^-$) if and only if for any $Q = Q^T$ p.d., there exists a unique $P = P^T$ p.d. s.t. $A^T P + PA = -Q$.

Example: $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 - x_2 \end{cases}$, $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Proof. Parametrize $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$. $A^T P + PA = -Q$ gives three equations, and solving the equations gives $P = \begin{bmatrix} \frac{3}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix}$, which is p.d. □

Indirect methods we can use the linear system approach to analyze a nonlinear system

1. Linearize about an equilibrium x^*
2. Use the linear theorems
3. Deduction about stability of x^* for the nonlinear system.

3.2 Exponential Stability

Recall Definition 3.4. What is the Lyapunov characterization?

Theorem: 3.5: Exponential Stability

Consider $\dot{x} = f(x)$, f locally Lipschitz, $f(0) = 0$, and let D be a domain containing 0 and $V : D \to \mathbb{R}$ is a C^1 function. Suppose $\exists \gamma_1, \gamma_2 > 0, \beta > 0, k > 0$ s.t. $\forall x \in D$

1. $\gamma_1 \|x\|^k \leq V(x) \leq \gamma_2 \|x\|^k$
2. $\dot{V}(x) = L_f V(x) \leq -\beta \|x\|^k$

Then $x^* = 0$ is exponentially stable. Moreover, if $D = \mathbb{R}^n$, then $x^* = 0$ is globally exponentially stable.

Proof. Let $\epsilon > 0$ be s.t. $B_\epsilon(0) \subset D$. Let $c_0 > 0$ be s.t. $\Omega_{c_0} = \left\{x \in \overline{B}_\epsilon(0) : V(x) \leq c_0\right\} \subset B_\epsilon(0)$. This is always doable by 1. Now consider any $x \in D$ by 1 and 2,

$$L_f V(x) \leq -\beta \|x\|^k \leq -\frac{\beta}{\gamma_2} V(x)$$

Therefore, for all $x_0 \in \Omega_{c_0}$,

$$\frac{d}{dt} V(\phi(t, x_0)) = L_f V(\phi(t, x_0)) \leq -\frac{\beta}{\gamma_2} V(\phi(t, x_0)), \forall t \geq 0$$

Integrate both sides,

$$V(\phi(t, x_0)) \leq V(x_0) - \frac{\beta}{\gamma_2} \int_0^t V(\phi(\tau, x_0)) d\tau$$

By Lemma 1.1, $V(\phi(t, x_0)) \leq V(x_0) \exp\left(-\frac{\beta}{\gamma_2} t\right)$. Now use 1,

$$\gamma_1 \|\phi(t, x_0)\|^k \leq V(\phi(t, x_0)) \leq V(x_0) \exp\left(-\frac{\beta}{\gamma_2} t\right) \leq \gamma_2 \|x_0\|^k \exp\left(-\frac{\beta}{\gamma_2} t\right)$$

Therefore, $\|\phi(t, x_0)\| \leq \left(\frac{\gamma_2}{\gamma_1}\right)^{1/k} \|x_0\| \exp\left(-\frac{\beta}{\gamma_2} t\right)$. Hence $x^* = 0$ is exponentially stable. □

3.3 Converse Theorem; LaSalle Invariance Principle; Barbalat's Lemma

Theorem: 3.6: Massera (Converse Theorem)

Let x^* be an asymptotically stable equilibrium of $\dot{x} = f(x)$ where f is locally Lipschitz. Then there exists a ball $B_r(x^*)$ and a C^1 function $V : B_r(x^*) \to \mathbb{R}$ s.t. V is p.d. at x^* and $L_f V(x)$ is n.d. at x^*. If x^* is GAS, then additionally $V : \mathbb{R}^n \to \mathbb{R}$ is p.d. at x^*, $L_f V(x)$ is n.d. at x^*.

Example: Consider a pendulum with friction $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a \sin x_1 - b x_2, b > 0 \end{cases}$. Find a Lyapunov function based on the total energy $V(x) = a(1 - \cos x_1) + \frac{1}{2} x_2^2$.
$V(x)$ is p.d. at $x^* = 0$ over $-\frac{\pi}{2} < x_1 < \frac{\pi}{2}$. $L_f V(x) = -b x_2^2 \leq 0$ n.s.d.

Question: Do we need to find another Lyapunov function?
No. Notice that $L_f V(x) < 0$ except at $x_2 = 0$, where $L_f V(x) = 0$. Solutions move along decreasing level sets of V. Then the solutions remain trapped, approaching points where $L_f V(x) = 0$. $L_f V(x) = 0$ gives $x_2 = 0$, $\dot{x}_2 = 0$, and $\dot{x}_1 = 0$. Solutions can maintain $L_f V(x) = 0$ only at $x^* = 0$. Thus as $V(x) \to 0$, $x \to 0$, $x^* = 0$ is asymptotically stable.

These observations can be formalized in Lasalle Invariance Principle.

Theorem: 3.7: LaSalle Invariance Principle

Consider the nonlinear system $\dot{x} = f(x)$ where $f(0) = 0$ and f is locally Lipschitz. Let $D \subset \mathbb{R}$ be a domain containing 0 and let Ω be a compact, positively invariant set under the system. Let $V : D \to \mathbb{R}$ be a C^1 function s.t. $\forall x \in \Omega$, $L_f V(x) \leq 0$. Define $E = \{x \in \Omega : L_f V(x) = 0\}$. Let m be the largest positively invariant set in E. Then $\forall x_0 \in \Omega$, $\phi(t, x_0) \to m$ as $t \to \infty$.

Proof. Let $x_0 \in \Omega$. We claim $\exists c_0 \in \mathbb{R}$ s.t. $\lim_{t \to \infty} V(\phi(t, x_0)) = c_0$.
We know $L_f V(x) \leq 0, \forall x \in \Omega$, so $V(\phi(t, x_0))$ is non-increasing. Also, V is continuous, and Ω is compact, so V achieves its minimum on Ω. Since $V(\phi(t, x_0))$ is a non-increasing function bounded from below, it has a limit c_0 as $t \to \infty$.

Claim: the positive limit set $L^+(x_0) \neq \emptyset$ and $\forall x \in L^+(x_0), V(x) = c_0$.
This is because $\phi(t, x_0)$ is bounded. Since $x_0 \in \Omega$ and Ω is compact and positively invariant. $\phi(t, x_0) \in \Omega, \forall t \geq 0$.
Apply Theorem 2.5, $L^+(x_0) \neq 0$, it is compact and invariant. Let $p \in L^+(x_0)$. This means $\exists \{t_k\}_k$ with $t_k \to \infty$ s.t. $\phi(t_k, x_0) \to p$.
By continuity of V, $V(\phi(t_k, x_0)) \to V(p)$ as $t_k \to \infty$. But we also know that $V(\phi(t, x_0)) \to c_0$, so $V(p) = c_0$.

Claim: $L^+(x_0) \subset E = \{x \in \Omega : L_f V(x) = 0\} \subset \Omega$.
$\forall p \in L^+(x_0)$ and $\forall t \geq 0$, $\phi(t, p) \in L^+(x_0)$, because $L^+(x_0)$ is invariant.
Then we have that $V(\phi(t, p)) = c_0, \forall t \geq 0$, V is constant, $\frac{d}{dt} V(\phi(t, p)) = 0, \forall t \geq 0$.
In particular, $\frac{d}{dt} V(\phi(t, p))\big|_{t=0} = L_f V(p) = 0$.
Also, $x_0 \in \Omega$ and Ω is compact, $L^+(x_0 \in \Omega)$. Taken together, these statements imply $p \in E$.

Claim: $L^+(x_0) \subset m$.
$L^+(x_0) \subset E$ by previous step. $L^+(x_0)$ is positively invariant. By Theorem 2.5, $L^+(x_0) \subset m$, which is the largest positively invariant set in E.

Claim: $\phi(t, x_0) \to m$ as $t \to \infty$.
By Theorem 2.5, $\phi(t, x_0) \to L^+(x_0)$ as $t \to \infty$, but $L^+(x_0) \subset m$. □

Remark 6. In practice, we want m to be a single point which is the equilibrium.

Back to the example: $V(x) = a(1-\cos x_1) + \frac{1}{2}x_2^2$. $\Omega = \left\{x \in \mathbb{R}^2 : V(x) \leq c\right\}$ is compact, positively invariant for sufficiently small c. $L_f V(x) = -bx_2^2$, $E = \{x : x_2 = 0\}$, $m = \{x : x_1 = x_2 = 0\} = \{0\}$. From Theorem 3.7, x^* is asymptotically stable.

Fact: Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function.

1. $\dot{f}(t) \to 0 \not\Rightarrow f$ converges to a constant. *e.g.* $f(t) = \sin(\log(t))$, $\dot{f}(t) = \cos(\log(t))\frac{1}{t} \to 0$ as $t \to \infty$, but $f(t)$ does not converge.
2. $f(t)$ converges as $t \to \infty \not\Rightarrow \dot{f}(t) \to 0$. *e.g.* $f(t) = e^{-t}\sin e^{2t}$, $\dot{f}(t)$ is unbounded.
3. If f is bounded from below. *i.e.* $\exists c \in \mathbb{R}$ s.t. $f(t) \geq c$ and f is non-increasing. *i.e.* $f(t) \leq 0$, then f converges as $t \to \infty$, *i.e.* $\lim_{t\to\infty} f(t) = c'$ for some $c' \geq c$.

***Definition:* 3.5: Uniform Continuous (Formal)**

A function $g : \mathbb{R} \to \mathbb{R}$ is uniformly continuous if $\forall \epsilon > 0$, $\forall t, t' \geq 0$, $|t - t'| < \delta \Rightarrow |g(t) - g(t')| < \epsilon$.

Remark 7. A sufficient condition for g to be uniformly continous is that its derivative is bounded.

***Lemma:* 3.1: Barbalat's Lemma**

If the differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a finite limit as $t \to \infty$ and if $\dot{f}(t)$ is uniformly continuous, then $\dot{f}(t) \to 0$ as $t \to \infty$.

Corollary 1. *If the differentiable function $f : \mathbb{R} \to \mathbb{R}$ has a finite limit as $t \to \infty$ and if $\ddot{f}(t)$ exists and is bounded, then $\dot{f}(t) \to 0$ as $t \to \infty$.*

***Theorem:* 3.8: Lyapunov-Like**

Consider a C^1 function $V : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ satisfying

1. V is lower bounded
2. $\dot{V}(x,t)$ is negative semi-definite at 0
3. $\dot{V}(x,t)$ is uniformly continuous in t

Then $\dot{V}(x,t) \to 0$ as $t \to \infty$.

Adaptive Control Consider a LTI system $\begin{cases} \dot{x} = Ax + Bu \\ y = Cx \end{cases}$, where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}$ is the input and $y \in \mathbb{R}$ is the output. We assume x, u, y are available for measurement. Given $r(t)$ a reference signal, find a controller s.t. $y(t) \to r(t)$, assuming A, B, C are unknown.
Define the error $e = r - y$, assume $r(t)$ is generated by a linear exogenous system $\dot{w} = Sw, r = Ew$.
Define the error model: $e = Ew - Cx$,

$$\begin{aligned}
\dot{e} &= E\dot{w} - C\dot{x} \\
&= Esw - C(Ax + Bu) \\
&= -CBu + ESw - CAx \\
&= -\beta u + \beta\psi\phi,
\end{aligned}$$

where $\beta = CB \in \mathbb{R}$ and $\mathrm{sgn}(\beta)$ is known, $\psi = -\frac{1}{\beta}(CA, CB)$ is a row vector of unknown parameters, $\phi = \begin{bmatrix} x \\ w \end{bmatrix}$ is called the regressor, and is known.

Now we have a scalar error model $\dot{e} = -\beta u + \beta\psi\phi$. We want $e(t) \to 0$. Choose a controller $u = ke + \hat{\psi}\phi$, where ke is for the closed-loop stability, $\hat{\psi}\phi$ is to achieve tracking (internal model principle), $\hat{\psi}$ is an estimation of the unknown ψ.

Define the parameter estimation error $\tilde{\psi} = \psi - \hat{\psi}$, then the closed-loop error model is $\dot{e} = -\beta ke - \beta\tilde{\psi}\phi$. Consider the Lyapunov function $V = \frac{1}{2}e^2 + \frac{1}{2}|\beta|\tilde{\psi}\tilde{\psi}^T$. V is p.d. at $(e^*, \tilde{\psi}^*) = 0$.

$$
\begin{aligned}
\dot{V} = L_f V &= e\dot{e} + |\beta|\tilde{\psi}\dot{\tilde{\psi}}^T \\
&= e[-\beta ke - \beta\tilde{\psi}\phi] + |\beta|\tilde{\psi}\dot{\tilde{\psi}}^T \\
&= -\beta ke^2 - \beta e\tilde{\psi}\phi + |\beta|\tilde{\psi}\dot{\tilde{\psi}}^T.
\end{aligned}
$$

Since $\beta k > 0$, $-\beta ke^2$ is n.d. Choose $\dot{\tilde{\psi}}^T = \text{sgn}(\beta)e\phi$ to cancel out the remaining terms.
Since $\tilde{\psi} = \psi - \hat{\psi}$, $\dot{\tilde{\psi}} = \dot{\psi} - \dot{\hat{\psi}} = -\dot{\hat{\psi}}$, we get $\dot{\hat{\psi}} = -\text{sgn}(\beta)e\phi^T$.

We now have $\dot{V} = -\beta ke^2 \leq 0$. At this point, we have V p.d. at $(0, 0)$ and $\dot{V} \leq 0$. So from Theorem 3.1, we can conclude the equilibrium $(e^*, \tilde{\psi}^*) = (0, 0)$ is stable.
Notice V is radially unbounded, so $e(t)$ and $\tilde{\psi}(t)$ are bounded. We can also assume that $w(t)$ is bounded. Then we know $\phi(t)$ is bounded. Then $\dot{e} = -\beta ke - \beta\tilde{\psi}\phi$ is bounded. Then $\ddot{V} = -2\beta ke\dot{e}$ is bounded. Therefore $\dot{V}$ is uniformly continous. By Lemma 3.1, $\dot{V}(t) \to 0$ along solution. But $\dot{V} = -\beta ke^2$, we conclude $e(t) \to 0$ as $t \to \infty$.

Remark 8. Notice we don't conclude $\tilde{\psi}(t) \to \infty$ from this method. Parameter convergence requires an extra condition called persistency of excitation.

3.4 Stability of Perturbed Systems

Consider the system

$$\dot{x} = f(x) + g(t, x),$$

where $f : \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz, $g : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ is continuous in t and locally Lipschitz in x. We regard $g(t, x)$ as a perturbation term.

First suppose $x = 0$ is GES of unperturbed system

$$\dot{x} = f(x)$$

Assume $g(t, 0) = 0$ for all t. Using converse Lyapunov theorems (Khalil Theorem 4.14), there exists $V : \mathbb{R}^n \to \mathbb{R}$ for $\dot{x} = f(x)$ satisfying

1. $c_1 \|x\|^2 \leq V(x) \leq c_2 \|x\|^2$
2. $\dot{V}(x) \leq -c_3 \|x\|^2$
3. $\left\|\frac{\partial V}{\partial x}\right\| \leq c_4 \|x\|$ for some $c_1, c_2, c_3, c_4 > 0$.

Suppose the perturbation satisfies a linear growth bound $\|g(t, x)\| \leq \gamma \|x\|$ for $t \geq 0$, $x \in \mathbb{R}^n$ and $\gamma \geq 0$ a constant.

Note any function g with $g(t, 0) = 0$ and g locally Lipschitz, uniformly in t, in a bounded neighborhood of 0 will satisfy $\|g(t, x)\| \leq \gamma \|x\|$ on that neighborhood.

Now consider $\dot{V}$ along solutions of $\dot{x} = f(x) + g(t,x)$,

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x}\dot{x} = \frac{\partial V}{\partial x}(f(x) + g(t,x)) \\
&\leq -c_3 \|x\|^2 + \frac{\partial V}{\partial x} g(t,x) \\
&\leq -c_3 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t,x)\| \\
&\leq -c_3 \|x\|^2 + c_4 \|x\| \gamma \|x\| \\
&= -(c_3 - \gamma c_4) \|x\|^2
\end{aligned}$$

If $\gamma < \frac{c_3}{c_4}$, then $c_3 - \gamma c_4 > 0$, $\dot{V} \leq -(c_3 - \gamma c_4)\|x\|^2 \leq 0$ n.d.

Lemma: 3.2: Khalil 9.1

Consider the system $\dot{x} = f(x) + g(t,x)$ and $x = 0$ is GES for $\dot{x} = f(x)$. Let $V(x)$ be a Lyapunov function for $\dot{x} = f(x)$ satisfying Khalil 4.14, and $g(t,x)$ satisfies $\|g(t,x)\| \leq \gamma \|x\|$ with $\gamma < \frac{c_3}{c_4}$. Then $x = 0$ is exponentially stable for $\dot{x} = f(x) + g(t,x)$.

Remark 9. In practice, we often do not know c_is. Then we write for $\gamma > 0$ sufficiently small.

The Lemma is conceptually important, because it highlights that ES is robust to perturbations.

Example: Consider $\dot{x} = Ax + g(t,x)$, where $A \in \mathbb{R}^{n\times n}$ is Hurwitz and $\|g(t,x)\| \leq \gamma \|x\|$, $\forall t \geq 0, x \in \mathbb{R}^n$. There exists $P = P^T$ p.d. solving the Lyapunov equation $A^T P + PA = -Q$ with $Q = Q^T$ p.d. For the nominal system $\dot{x} = Ax$, we choose the Lyapunov function $V = x^T Q x$. This Lyapunov function satisfies

1. $\lambda_{\min}(P)\|x\|^2 \leq V(x) \leq \lambda_{\max}(P)\|x\|^2$
2. $\dot{V} = -x^T Q x \leq -\lambda_{\min}(Q)\|x\|^2$
3. $\left\|\frac{\partial V}{\partial x}\right\| = \left\|2x^T P\right\| \leq 2\|P\|\|x\| \leq 2\lambda_{\max}(P)\|x\|^2$

Now consider $\dot{V}$ for $\dot{x} = Ax + g(t,x)$. Compute

$$\begin{aligned}
\dot{V} &= 2x^T P\dot{x} = 2x^T P(Ax + g(t,x)) \\
&= x^T(A^T P + PA)x + 2x^T P g(t,x) = -x^T Q x + 2x^T P g(t,x) \\
&\leq -\lambda_{\min}(Q)\|x\|^2 + 2\left\|x^T P\right\| \|g(t,x)\| \\
&\leq -\lambda_{\min}(Q)\|x\|^2 + 2\|P\|\|x\|\gamma\|x\| \\
&\leq -\lambda_{\min}(Q)\|x\|^2 + 2\lambda_{\max}(P)\gamma\|x\|^2 \\
&= -(\lambda_{\min}(Q) - 2\gamma\lambda_{\max}(P))\|x\|^2 .
\end{aligned}$$

We want $\gamma < \frac{\lambda_{\min}(Q)}{2\lambda_{\max}(P)}$.

Note P depends on the choice of Q. The best of Q for least restrictive bound is $Q = I$.

Example: $\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -4x_1 - 2x_2 + \beta x_2^3 \end{cases}$, where $\beta > 0$ is unknown.

Rewrite as a perturbed system $\dot{x} = f(x) + g(t,x)$, where $f(x) = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} x$, $g(x) = \begin{bmatrix} 0 \\ \beta x_2^3 \end{bmatrix}$. The solution of the Lyapunov equation $A^T P + PA = -I$ is $P = \begin{bmatrix} 3/2 & 1/8 \\ 1/8 & 5/16 \end{bmatrix}$. The conditions hold with $c_3 = 1$, $c_4 = 2\lambda_{\max}(P) = 3.026$.

Now look at $\|g(x)\| = \beta \|x_2\|^3 = \beta \|x_2\|^2 \|x_2\| \leq \beta \|x_2\|^2 \|x\| \leq \beta k_2^2 \|x\|$, which holds for all $\|x_2\| \leq k_2$. We don't know if such a bound on x_2 holds. Consider $\dot{x} = f(x) + g(x)$,

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x}(f(x) + g(x))(f(x) + g(x)) \\
&\leq -\|x\|^2 + c_4 \|x\| \beta k_2^2 \|x\| \\
&\leq -\|x\|^2 + 3.026\beta k_2^2 \|x\|^2
\end{aligned}$$

$\dot{V} \leq 0$ if $\beta < \frac{1}{3.026 k_2^2}$ Denote $\Omega_c = \left\{x \in \mathbb{R}^2 : V(x) \leq c\right\}$ closed and bounded. The boundary is $\partial\Omega_c = \left\{x : V(x) = \frac{3}{2}x_1^2 + \frac{1}{4}x_1 x_2 + \frac{5}{16}x_2^2\right\}$. We need the largest x_2 on $\partial\Omega$. Take $V(x) = c$, derivative w.r.t. x_1, set to 0, solve for x_2. This gives $x_1 = -\frac{3}{4}x_2$. $x_2^2 = \frac{96c}{29}$. $\forall x \in \Omega_c$, $|x_2| < k_2$, $k_2^2 < \frac{96c}{29}$, $\beta \leq \frac{0.1}{c}$. This gives a region of attraction.

***Lemma:* 3.3: Non-Vanishing Perturbation**

Let $x = 0$ be an equilibrium of $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a Lyapunov function for the system satisfying Khalil 4.14 holding $B_r(0)$, Consider $\dot{x} = f(x) + g(t, x)$. Suppose $\|g(t, x)\| \leq \delta < \frac{c_3}{c_4}\sqrt{\frac{c_1}{c_2}}\theta r$, where $0 < \theta < 1$. If $\|x(0)\| < \sqrt{\frac{c_2}{c_1}}r$, then $\|x(t)\| \leq \frac{c_4}{c_3}\sqrt{\frac{c_2}{c_1}}\frac{\delta}{\theta}$.

Proof.

$$\begin{aligned}
\dot{V} &= \frac{\partial V}{\partial x}(f(x) + g(x)) \leq -c_3 \|x\|^2 + \left\|\frac{\partial V}{\partial x}\right\| \|g(x)\| \\
&\leq -c_3 \|x\|^2 + c_4 \|x\| \delta = -c_3 \|x\|^2 + \frac{c_4}{c_3}\delta + \theta c_3 \|x\|^2 - \theta c_3 \|x\|^2 \\
&= -(1-\theta)c_3 \|x\|^2 - \theta c_3 \|x\|^2 + c_4 \delta \|x\| \\
&= -(1-\theta)c_3 \|x\|^2 - (\theta c_3 \|x\| - c_4 \delta) \|x\|
\end{aligned}$$

If $\|x\| \geq \frac{\delta c_4}{\theta c_3}$, then $\dot{V} \leq -(1-\theta)c_3 \|x\|^2$. □

Most Common Tricks/Techniques:

1. Cauchy-Schwarz: $u^T v \leq \|u\| \|v\|$
2. Matrix norm: $\|Ax\| \leq \|A\| \|x\|$
3. Young's Inequality: $2\|a\| \|b\| \leq \|a\|^2 + \|b\|^2$; $2\|a\|^2 \leq \|a\|^2 + 1$
4. $ab \leq \frac{a^2}{2\epsilon} + \frac{\epsilon b^2}{2}$
5. Comparison Lemma: Consider $\dot{y} = f(t, y)$, $y(t_0) = y_0$. Suppose $\dot{V} \leq f(t, V)$, and $V(t_0) \leq y_0$. Then $V(t) \leq y(t)$, $\forall t \geq t_0$, $\dot{V} \leq -\gamma V$.

Theorem: 3.9:

Consider the perturbed system

$$\dot{x} = f(t,x) + g(t),$$

where f is locally Lipschitz in x and piecewise-continuousin t. Suppose $x = 0$ is GES for $\dot{x} = f(t,x)$. Suppose $g(t) \to 0$ exponentially. Then $x = 0$ is stable for the perturbed system.

Proof. By Theorem 4.14 of Khalil, $\exists V_1 : \mathbb{R}^n \to \mathbb{R}$ s.t.

1. $c_1 \|x\|^2 \leq V_1(t,x) \leq c_2 \|x\|^2$
2. $\frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x} f(t,x) \leq -c_3 \|x\|^2$
3. $\left\| \frac{\partial V_1}{\partial x} \right\| \leq c_4 \|x\|$

for some $c_1, c_2, c_3, c_4 > 0$.
Since $g(t) \to 0$ exponentially, $\exists(\overline{c_2}, A_2)$ s.t. $\dot{\nu} = A_2\nu$ and $\|g(t)\| \leq \|\overline{c_2}\nu\|$.
Notice that A_2 is Hurwitz, so $\exists P_2 = P_2^T$ s.t. $A_2^T P_2 + P_2 A_2 = I$.

Let $V(t,x,\nu) = V_1(t,x) + c_5 \nu^T A_2 \nu$ where $c_5 > 0$ TBD. Then

$$\begin{aligned}
\dot{V} &= \dot{V}_1 + c_5 2\nu^T P_2 \dot{\nu} = \frac{\partial V_1}{\partial t} + \frac{\partial V_1}{\partial x}(f(t,x) + g(t)) + c_5 2\nu^T P_2 A_2 \nu \\
&\leq -c_3 \|x\|^2 + \frac{\partial V_1}{\partial x} g(t) + c_5 \nu^T (A_2^T P_2 + P_2 A_2)\nu \\
&= -c_3 \|x\|^2 + \frac{\partial V_1}{\partial x} g(t) - c_5 \|\nu\|^2 \\
&\leq -c_1 \|x\|^2 + \left\| \frac{\partial V}{\partial x} \right\| \|g(t)\| - c_5 \|\nu\|^2 \\
&\leq -c_3 \|x\|^2 + c_4 \|x\| \overline{c_2} \|\nu\| - c_5 \|\nu\|^2
\end{aligned}$$

Also $\|x\| \|\nu\| \leq \frac{\|x\|^2}{2\epsilon} + \frac{\epsilon\|\nu\|^2}{2}$ for $\epsilon > 0$ TBD. Then

$$\begin{aligned}
&\leq -c_3 \|x\|^2 + \overline{c_2} c_4 \left[\frac{\|x\|^2}{2\epsilon} + \frac{\epsilon \|\nu\|^2}{2} \right] - c_5 \|\nu\|^2 \\
&\leq -\left[c_3 - \frac{\overline{c_2} c_4}{2\epsilon} \right] \|x\|^2 - \left[c_5 - \frac{\epsilon \overline{c_2} c_4}{2} \right] \|\nu\|^2
\end{aligned}$$

Choose $\epsilon > 0$ s.t. $c_3 > \frac{\overline{c_2} c_4}{2\epsilon}$ and $c_5 > \frac{\epsilon \overline{c_2} c_4}{2}$.
We have $\dot{V} \leq -\gamma_1 \|x\|^2 - \gamma_2 \|\nu\|^2 \leq -\gamma V$.

Now we can apply Lemma 1.2 to get $V(t) \leq \exp(-\gamma t) V(0)$, $(x(t), \nu(t)) \to 0$ exponentially. Therefore, $x = 0$ is GES. □

3.5 Input-to-State Stability

Consider the control system

$$\dot{x} = f(x,u),$$

where $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ is locally Lipschitz in x and u. Suppose $u(t) \in \mathbb{R}^n$ is a piecewise continuous bounded function of t. Suppose we know the unforced system $\dot{x} = f(x,0)$ has an equilibrium at $x = 0$ that is GAS.

Example: Consider the linear system $\dot{x} = Ax + Bu$, $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^n$. Suppose $A \in \mathbb{R}^{n\times n}$ is Hurwitz. We know the solution is

$$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)}Bu(\tau)d\tau.$$

Since A is Hurwitz, $\exists(C, \lambda)$, $\lambda > 0$ s.t. $\left\|e^{At}\right\| \leq Ce^{-\lambda t}$, for all $t \geq 0$.

$$\begin{aligned}
\|x(t)\| &\leq \left\|e^{At}\right\| \|x(0)\| + \int_0^t \left\|e^{A(t-\tau)}\right\| \|B\| \|u(\tau)\| \, d\tau \\
&\leq Ce^{-\lambda t} \|x(0)\| + \int_0^t Ce^{-\lambda(t-\tau)} \|B\| \|u(\tau)\| \, d\tau \\
&\leq Ce^{-\lambda t} \|x(0)\| + Ce^{-\lambda t} \int_0^t e^{\lambda\tau} \|B\| \, d\tau \sup_{\tau\in[0,t]} \|u(\tau)\| \\
&\leq Ce^{-\lambda t} \|x(0)\| + \frac{C\|B\|}{\lambda} \sup_{\tau\in[0,t]} \|u(\tau)\|
\end{aligned}$$

This shows bounded input implies bounded state. We want to generalize this property to nonlinear systems.

***Definition:* 3.6: Class κ Functions**

A function $\alpha : [0, T] \to [0, \infty)$ belongs to class κ if it is strictly increasing and $\alpha(0) = 0$. It belongs to κ_∞ if $T = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

***Definition:* 3.7: Class κL Functions**

A continuous function $\beta : [0, T] \times [0, \infty] \to [0, \infty)$ belongs to class κL if for each fixed s, the mapping $\beta(\cdot, s)$ belongs to class κ and for each fixed r, $\beta(r, \cdot)$ is decreasing and $\beta(r, s) \to 0$ as $s \to \infty$.

***Definition:* 3.8: Input-to-State Stability**

A system $\dot{x} = f(x, u)$ is Input-to-State Stable (ISS) if there exists a class κL function β and class κ function γ s.t. $\forall x(0)$ and any piecewise continuous bounded input $u(t)$, $x(t)$ exists and

$$\|x(t)\| \leq \beta(\|x(0)\|, t) + \gamma\left(\sup_{\tau\in[0,t]} \|u(\tau)\|\right)$$

Remark 10. For ISS, bounded input implies bounded states. ISS implies $x = 0$ is GAS for $\dot{x} = f(x, 0)$.

***Theorem:* 3.10:**

Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1 function satisfying $\alpha(\|x\|) \leq V(x) \leq \alpha(\|x\|)$ with α_1, α_2 in class κ_∞. $\dot{V}(x) = \frac{\partial V}{\partial x} f(x, u) \leq -w(x)$, for all $\|x\| \geq \rho(\|x\|) > 0$ where ρ is class κ and $w(x)$ is a continuous positive definite (at $x = 0$) function. Then $\dot{x} = f(x, u)$ is ISS.

Example: $\dot{x} = -x^3 + u$. Note $x = 0$ is GAS for $\dot{x} = -x^3$.
Try Lyapunov function $V = \frac{1}{2}x^2$.

$$\begin{aligned}
\dot{V} &= x\dot{x} = x(-x^3 + u) = -x^4 + xu \\
&= -(1-\theta)x^4 - \theta x^4 + xu \text{ for } 0 < \theta < 1 \\
&\leq -(1-\theta)x^4
\end{aligned}$$

for all $|x| \geq \left(\frac{|u|}{\theta}\right)^{1/3}$. The system is ISS.

***Lemma:* 3.4:**

Consider $\dot{x} = f(x,u)$ and $f(x,u)$ is C^1 and globally Lipschitz in x and u. If $x = 0$ is GES for $\dot{x} = f(x,0)$, then $\dot{x} = f(x,u)$ is ISS.

***Theorem:* 3.11: Cascade System**

Consider the cascade system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_2)$$

Suppose $x_2 = 0$ is GAS for $\dot{x}_2 = f_2(x_2)$, $\dot{x}_1 = f_1(x_1, x_2)$ is ISS with input x_2, then $x = (0,0)$ is GAS.