Introduction To Functional Analysis

This is mainly from MIT 18.102 Introduction To Functional Analysis (`https://ocw.mit.edu/courses/18-102-introduction-to-functional-analysis-spring-2021/`)

1 Normed and Banach Spaces

1.1 Basic Banach Spaces

Definition: 1.1: Vector Space

V is a vector space over $\mathbb{R}$ or $\mathbb{C}$ or a field $\mathbb{K}$ if V has two operations:

- $+ : V \times V \to V$, $(v_1, v_2) \to v_1 + v_2$
- $\cdot : \mathbb{K} \times V \to V$, $(\alpha, v) \to \alpha v$

Along with some axioms: commutativity, associativity, identity and inverse of addition. Identity of multiplication and distributivity.

Example: $\mathbb{R}^n$, $\mathbb{C}^n$, $C([0,1]) = \{f : [0,1] \to \mathbb{C} : f \text{ continuous}\}$ are vector spaces.

Definition: 1.2: Dimension of Vector Spaces

A vector space V is finite dimensional if every linearly independent set is finite. *i.e.* $\forall E \subset V$ s.t. $\forall v_1, ..., v_N \in E$, $\sum_{i=1}^{N} a_i v_i = 0 \Rightarrow a_1 = \cdots = a_N = 0$, then E is finite. V is infinite dimensional if V is not finite dimensional.

Example: $C([0,1])$ is infinite dimensional. $E = \{f_n(x) = x^n : n \in \mathbb{N} \cup \{0\}\}$ is a linearly independent infinite set.

Definition: 1.3: Norm

A norm on a vector space V is a function $\|\cdot\| : V \to [0, \infty)$ with the following properties:

1. Definiteness: $\|v\| = 0 \Leftrightarrow v = 0$
2. Homogeneity: $\|\lambda v\| = |\lambda| \|v\|$ for all $v \in V$ and $\lambda \in \mathbb{K}$
3. Triangle Inequality: $\|v_1 + v_2\| \leq \|v_1\| + \|v_2\|$.

A *semi-norm* is a function $\|\cdot\| : V \to [0, \infty)$ that satisfies 2 and 3, but not necessarily 1.
A vector space V with a norm $\|\cdot\|$ is a *normed space.*

Definition: 1.4: Metric

Let X be a set. $d : X \times X \to [0, \infty)$ is a metric if

1. $d(x, y) = 0 \Leftrightarrow x = y$
2. $\forall x, y \in X$, $d(x, y) = d(y, x)$
3. $\forall x, y, z \in X$, $d(x, y) \leq d(x, z) + d(z, y)$

Theorem: 1.1: Metric Induced by Norm

Let $\|\cdot\|$ be a norm on a vector space V. Then $d(v, w) = \|v - w\|$ defines a metric on V called the metric induced by the norm.

Proof. 1 in Definition 1.3 $\Rightarrow$ 1 in Definition 1.4.
2: $\|v - w\|\|(-1)(w - v)\| = |-1|\|w - v\| = \|w - v\|$
3 in Definition 1.3 $\Rightarrow$ 3 in Definition 1.4. $\square$

Example: The Euclidean norm of $\mathbb{R}^n$ and $\mathbb{C}^n$ is given by $\|x\|_2 = \left(\sum_{i=1}^{n} |x_i|^2\right)^{1/2}$. We can also have $\|x\|_\infty = \max_i |x_i|$. In general, for $p \geq 1$, $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$.

Example: Let X be a metric space. Define $C_\infty(X) = \{f : X \to \mathbb{C} : f \text{ continuous and bounded.}\}$. $C_\infty(X)$ is a vector space. $\|u\|_\infty = \sup_{x \in X} |u(x)|$ is a norm on $C_\infty(X)$.

Proof. 1, 2 are easily satisfied.
For 3, let $u, v \in C_\infty(X)$, then $\forall x \in X$, $|u(x) + v(x)| \leq |u(x)| + |v(x)| \leq \|u\|_\infty + \|v\|_\infty$.
$\|u\|_\infty + \|v\|_\infty$ is an upper bound for $|u(x) + v(x)|$. Thus $\|u + v\|_\infty = \sup |u(x) + v(x)| \leq \|u\|_\infty + \|v\|_\infty$. $\square$

Note that $u_n \to u$ in $C_\infty(X) \Leftrightarrow \|u_n - u\| \to 0$ as $n \to \infty \Leftrightarrow \forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $\forall x \in X$, $|u_n(x) - u(x)| < \epsilon \Leftrightarrow u_n \to u$ uniformly on X. *i.e.* Convergence of functions in a continuous and bounded space of functions $C_\infty(X)$ is equivalent to uniform convergence of sequence of functions in X.

Definition: 1.5: l^p Spaces

The l^p space is the space of sequences $l^p = \{\{a_j\}_{j=1}^\infty : \|a\|_p < \infty\}$, where l^p-norm is defined by

$$\|a\|_p = \begin{cases} \left(\sum_{j=1}^\infty |a_j|^p\right)^{1/p}, 1 \leq p < \infty \\ \sup_{1 \leq j < \infty} |a_j|, p = \infty \end{cases}$$

Example: $\{\frac{1}{j}\}_{j=1}^\infty \in l^p$ for all $p > 1$, but not in l^1 (By p-series test).

Definition: 1.6: Banach Space

A normed space is a Banach space if it is complete w.r.t. the metric induced by the norm. *i.e.* All Cauchy sequences converges.

Example: $\mathbb{R}^n$ and $\mathbb{C}^n$ are complete, thus Banach w.r.t. any of l^p norms.

Theorem: 1.2:

Let X be a metric space, then $C_\infty(X)$ is Banach space.

Proof. We want to show that $C_\infty(X)$ is complete, *i.e.* every Cauchy sequence $\{u_n\}$ in $C_\infty(X)$ converges in $C_\infty(X)$.

Firstly, we show that $u_n \to u$ exists and is bounded.
Let $\{u_n\}$ be a Cauchy sequence in $C_\infty(X)$. Then $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$, $\|u_n - u_m\|_\infty < 1$ by definition of Cauchy sequences and choosing $\epsilon = 1$.
Also $\forall n \geq N_0$, $\|u_n\|_\infty = \|u_n - u_{N_0} + u_{N_0}\|_\infty \leq \|u_n - u_{N_0}\|_\infty + \|u_{N_0}\|_\infty < 1 + \|u_{N_0}\|_\infty$.
Let $B = \|u_1\|_\infty + \cdots + \|u_{N_0}\|_\infty + 1$. Then $\|u_n\|_\infty \leq B$ for all n. $\|u_n\|_\infty$ is bounded.

Since $\forall x \in X$, $|u_n(x) - u_m(x)| \leq \sup |u_n(x) - u_m(x)| = \|u_n - u_m\|_\infty$, then $\forall x \in X$, $\{u_n(x)\}_{n=1}^\infty$ is Cauchy in $\mathbb{C}$.
By Completeness of $\mathbb{C}$, $\forall x \in X$, $\{u_n(x)\}_{n=1}^\infty$ converges in $\mathbb{C}$. Define $u : X \to \mathbb{C}$ s.t. $u(x) = \lim\limits_{n\to\infty} u_n(x)$.
Then $\forall x \in X$, $|u(x)| = \lim\limits_{n\to\infty} |u_n(x)| \leq B$, Thus $\sup\limits_{x\in X} |u(x)| \leq B$, u is bounded.

Now we show $\|u - u_n\| \to 0$ as $n \to \infty$.
Let $\epsilon > 0$. Since $\{u_n\}$ is Cauchy in $C_\infty(X)$, $\exists N \in \mathbb{N}$ s.t. $\forall n, m \geq N$, $\|u_n - u_m\|_\infty < \frac{\epsilon}{2}$.
Let $x \in X$, $|u_n(x) - u_m(x)| \leq \|u_n - u_m\|_\infty < \frac{\epsilon}{2}$. Let $m \to \infty$, then $\forall n \geq N$, $|u_n(x) - u(x)| < \frac{\epsilon}{2}$. Therefore, $\|u_n - u\|_\infty < \frac{\epsilon}{2} < \epsilon$. Thus $\|u_n - u\|_\infty \to 0$. $u_n \to u$ uniformly on X. u is continuous.

Thus $u \in C_\infty(X)$. $C_\infty(X)$ is complete and therefore a Banach space. □

Example: $\forall p \geq 1$, l^p is a Banach space.

Example: $C_0 = \{a \in l^\infty : \lim\limits_{j\to\infty} a_j = 0\}$ is a Banach space with $\|a\|_\infty = \sup\limits_j |a_j|$.

***Definition:* 1.7: Summable Sequence**

Let $\{v_n\} \subset V$ be a sequence in V. The series $\sum_n v_n$ is summable if $\left\{\sum_{n=1}^m v_n\right\}_{m=1}^\infty$ converges and $\sum_n v_n$ is absolutely summable if $\sum_n \|v_n\|$ converges.

***Theorem:* 1.3:**

If $\sum_n v_n$ is absolutely summable, then $\left\{\sum_{n=1}^m v_n\right\}_{m=1}^\infty$ is Cauchy in V.

Proof. Same as in $\mathbb{R}$. □

***Theorem:* 1.4:**

V is a Banach space $\Leftrightarrow$ every absolutely summable series is summable.

Proof. (⇒) Suppose V is a Banach space. Let v_n be an absolute summable series.
By Theorem 1.3, $\left\{\sum_{n=1}^{m} v_n\right\}_{m=1}^{\infty}$ is Cauchy in V. By Definition 1.6, $\left\{\sum_{n=1}^{m} v_n\right\}_{m=1}^{\infty}$ converges, thus it is summable.

(⇐) Suppose every absolutely summable series is summable. Let $\{v_n\}_n$ be a Cauchy sequence in V.
We want to show that $\{v_n\}$ converges in V.
$\{v_n\}$ is Cauchy $\Rightarrow \forall k \in \mathbb{N}, \exists N_k \in \mathbb{N}$ s.t. $\forall n, m \geq N_k$, $\|v_n - v_m\| < 2^{-k}$.
Define $n_k = N_1 + \cdots + N_k$. Then $N_k \leq n_1 < n_2 < \cdots$.
Thus $\forall k \in \mathbb{N}$, $\|v_{n_{k+1}} - v_{n_k}\| < 2^{-k}$, $\sum_k (v_{n_{k+1}} - v_{n_k})$ is absolutely summable and thus $\sum(v_{n_{k+1}} - v_{n_k})$ is summable.
$\Rightarrow \left\{\sum_{k=1}^{m}(v_{n_{k+1}} - v_{n_k})\right\}_{m=1}^{\infty}$ converges in V. Thus $\left\{v_m = \sum_{k=1}^{m-1}(v_{n_{k+1}} - v_{n_k}) + v_{n_1}\right\}_{m=1}^{\infty}$ converges in V.
The subsequence $\{v_{n_m}\}$ converges in V. Thus $\{v_n\}_n$ converges in V by metric space theory. □

***Theorem:* 1.5: Holder's Inequality**

Let $n \in \mathbb{N}$, $a_k, b_k \in \mathbb{R}$, $1 \leq k \leq n$, if $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\sum_{k=1}^{n} |a_k b_k| \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^q\right)^{1/q}$$

***Theorem:* 1.6: Minkowski's Inequality**

Let $n \in \mathbb{N}$, $a_k, b_k \in \mathbb{R}$, $1 \leq k \leq n$, if $1 \leq p < \infty$, then

$$\left(\sum_{k=1}^{n} |a_k + b_k|^p\right)^{1/p} \leq \left(\sum_{k=1}^{n} |a_k|^p\right)^{1/p} \left(\sum_{k=1}^{n} |b_k|^p\right)^{1/p}$$

1.2 Operators and Functionals

***Definition:* 1.8: Linear Operators**

Let V, W be vector spaces, we say a map $T : V \to W$ is linear if $\forall \lambda_1, \lambda_2 \in \mathbb{K}$, $\forall v_1, v_2 \in V$, $T(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 T v_1 + \lambda_2 T v_2$. T is often called a linear operator.

Example: Let $K : [0,1] \times [0,1] \to \mathbb{C}$ be continuous functions. For $f \in C([0,1])$, define $Tf(x) = \int_0^1 K(x,y) f(y) dy$. Then $Tf \in C([0,1])$ and $\forall \lambda_1, \lambda_2 \in C$, $f_1, f_2 \in C([0,1])$, $T(\lambda_1 f_1 + \lambda_2 f_2) = \lambda_1 T f_1 + \lambda_2 T f_2$. T is a linear operator. It is the inverse of differential operator.

***Definition:* 1.9: Continuous Operators**

$T : V \to W$ is continuous on V if $\forall v \in V$, $\forall \{v_n\}$ with $v_n \to v \Rightarrow T v_n \to T v$, or equivalently, for all open sets $U \subset W$, $T^{-1}(U) = \{v \in V : Tv \in U\}$ is open in V.

Theorem: 1.7: Bounded Linear Operator

A linear operator $T : V \to W$ is continuous if and only if $\exists C > 0$ s.t. $\forall v \in V$, $\|Tv\|_W \leq C\|v\|_V$. We say T is a bounded linear operator.

Note: The image of T is not bounded unless T is the zero map, but bounded subsets of V are always mapped to bounded subsets of W.

Proof. ($\Leftarrow$) Suppose $\|Tv\| \leq C\|v\|$.
Let $v \in V$ and suppose $v_n \to v$. Then $\|Tv_n - Tv\| = \|T(v_n - v)\| \leq C\|v_n - v\| \to 0$ as $n \to \infty$.
By squeeze theorem, $\|Tv_n - Tv\| \to 0$, T is continuous by the Definition 1.9 (1).

($\Rightarrow$) Suppose T is continuous.
Let $B_W(0, 1)$ be the ball centered at 0 in W with radius 1. Then $T^{-1}(B_W(0, 1)) = \{v \in V : Tv \in B_W(0, 1)\}$ is an open set in V by Definition 1.9 (2).
$0 \in T^{-1}(B_W(0, 1))$ since T is a linear map $T0 = 0$. Therefore, $\exists r > 0$ s.t. $B_V(0, r) \subset T^{-1}(B_W(0, 1))$.
Let $v \in V \setminus \{0\}$. Then $\left\|\frac{r}{2\|v\|}v\right\| = \frac{r}{2} < r$, $\frac{r}{2\|v\|}v \in B_V(0, r)$ and $T\left(\frac{r}{2\|v\|}v\right) \in B_W(0, 1)$.
$\left\|T\left(\frac{r}{2\|v\|}v\right)\right\| < 1 \Rightarrow \|Tv\| < \frac{2}{r}\|v\|$, so we can choose $C = \frac{2}{r}$, s.t. $\forall v \in V$, $\|Tv\|_W \leq C\|v\|_V$. □

Example: $T : C([0, 1]) \to C([0, 1])$ given by $Tf(x) = \displaystyle\int_0^1 K(x, y)f(y)dy$, where $K(x, y) \in C([0, 1] \times [0, 1])$ is a bounded linear operator.

Proof. Let $f \in C([0, 1])$ and $\|f\|_\infty = \sup\limits_{x\in[0,1]} |f(x)|$.
Then for all $x \in [0, 1]$,

$$\begin{aligned}
|Tf(x)| &= \left|\int_0^1 K(x, y)f(y)dy\right| \\
&\leq \int_0^1 |K(x, y)||f(y)|dy \\
&\leq \int_0^1 \|K\|_\infty\|f\|_\infty dy = \|K\|_\infty\|f\|_\infty
\end{aligned}$$

Thus $\|Tf\|_\infty \leq \|K\|_\infty\|f\|_\infty$ □

Definition: 1.10: Operator Norm

Let V, W be normed spaces. Define $B(V, W)$ to be the space of all bounded linear operators. $B(V, W)$ is a vector space. Define the operator norm

$$\|T\| = \sup_{\|v\|=1} \|Tv\|$$

Note: $T \in B(V, W) \Rightarrow \exists C > 0$ s.t. $\forall v \in V$, $\|Tv\| \leq C\|v\|$.

Theorem: 1.8:

The operator norm is a norm, so $B(V, W)$ is a normed space.

Proof. Definiteness: Suppose $Tv = 0\ \forall \|v\| = 1$. Then $\forall v \in V \setminus \{0\}$, $0 = T\left(\frac{v}{\|v\|}\right) = \frac{1}{\|v\|}Tv$. Then $Tv = 0$ for all $v \in V$. T is the zero operator.

Homogeneity: $\|\lambda T\| = \sup\limits_{\|v\|=1} \|\lambda T v\| = |\lambda| \sup\limits_{\|v\|=1} \|Tv\| = |\lambda| \|T\|$.

Triangle inequality: If $S, T \in B(V, W)$, $v \in V$, $\|v\| = 1$.
$\|(S+T)(v)\| \overset{\text{By Linearity}}{=} \|Sv+Tv\| \overset{\text{Triangle inequality of norm}}{\leq} \|Sv\|+\|Tv\| \overset{\text{Definition of Operator Norm}}{\leq} \|S\|+\|T\|$ □

Remark 1. If $v \neq 0$, then $\left\|T\left(\frac{v}{\|v\|}\right)\right\| \leq \|T\| \Rightarrow \|Tv\| \leq \|T\|\|v\|$.

Example: For $Tf(x) = \int_0^1 K(x,y)f(y)dy$, $\|T\| \leq \|K\|_\infty$.

***Theorem:* 1.9:**

If W is a Banach space, then $B(V, W)$ is a Banach space.

Proof. Suppose $\{T_n\}_n \subset B(V, W)$ s.t. $C = \sum_n \|T_n\| < \infty$. We want to show that $\sum_n T_n$ is summable.
Let $v \in V$, $m \in \mathbb{N}$. $\sum\limits_{n=1}^{m} \|T_n v\| \leq \sum\limits_{n=1}^{m} \|T_n\|\|v\| \leq \|v\| \sum \|T_n\| = C\|v\|$
Thus $\left\{\sum\limits_{n=1}^{m} \|T_n v\|\right\}_{m=1}^{\infty}$ is bounded, $\sum_n \|T_n v\|$ converges.
Thus $\sum_n T_n v$ is absolutely summable in W. Since W is a Banach space, by Theorem 1.4, $\sum_n T_n v$ is summable in W.

Define $T : V \to W$ s.t. $Tv = \lim\limits_{m\to\infty} \sum\limits_{n=1}^{m} T_n v$. We want to show that $T \in B(V, W)$.

Linearity: $\forall \lambda_1, \lambda_2 \in \mathbb{K}$, $v_1, v_2 \in V$,

$$T(\lambda_1 v_1 + \lambda_2 v_2) = \lim_{m\to\infty} \sum_{n=1}^{m} T_n(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \lim_{m\to\infty} \sum_{n=1}^{m} T_n v_1 + \lambda_2 \lim_{m\to\infty} \sum_{n=1}^{m} T_n v_2 = \lambda_1 T v_1 + \lambda_2 T v_2$$

$T \in B(V, W)$ (Bounded): Let $v \in V$, $\|v\| = 1$.

$$\|Tv\| = \left\|\lim_{m\to\infty} \sum_{n=1}^{m} T_n v\right\| = \lim_{m\to\infty} \left\|\sum_{n=1}^{m} T_n v\right\| \leq \lim_{m\to\infty} \sum_{n=1}^{m} \|T_n v\| \leq \lim_{m\to\infty} \sum_{n=1}^{m} \|T_n\|\|v\| = \sum_n \|T_n\| = C$$

Thus $\|Tv\| \leq C$ for all $v \in V$, $\|v\| = 1$. $\|Tv\| \leq C\|v\|\ \forall v \in V$. Therefore, $T \in B(V, W)$.

Now we show that $\sum\limits_{n=1}^{m} T_n \to T$.
Let $v \in V$, $\|v\| = 1$.

$$\left\|Tv - \sum_{n=1}^{m} T_n v\right\| = \left\|\lim_{m'\to\infty} \sum_{n=1}^{m'} T_n v - \sum_{n=1}^{m} T_n v\right\| = \left\|\lim_{m'\to\infty} \sum_{n=m+1}^{m'} T_n v\right\|$$
$$\leq \lim_{m'\to\infty} \left\|\sum_{n=m+1}^{m'} T_n v\right\| \leq \lim_{m'\to\infty} \sum_{n=m+1}^{m'} \|T_n v\| \leq \lim_{m'\to\infty} \sum_{n=m+1}^{m'} \|T_n\| = \sum_{n=m+1}^{\infty} \|T_n\|$$

Thus, $\left\| T - \sum_{n=1}^{m} T_n \right\| \leq \sum_{n=m+1}^{\infty} \|T_n\| \to 0$ as $m \to \infty$. By squeeze theorem, $\left\| T - \sum_{n=1}^{m} T_n \right\| \to 0$.

Thus $\sum_{n=1}^{m} T_n \to T$, and $B(V, W)$ is a Banach space. □

Definition: 1.11: Dual Space and Functionals

If V is a normed space, $V' = B(V, \mathbb{K})$ is the dual space of V. Since $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ is complete, V' is a Banach space. An element in V' is called a functional.

Example: $\forall 1 \leq p < \infty$, The dual of l^p space is $(l^p)' = l^{p'}$ where $\frac{1}{p'} + \frac{1}{p} = 1$. $(l^1)' = l^\infty$, $(l^2)' = l^2$, but $(l^\infty)' \neq l^1$.

1.3 Quotient Spaces

Definition: 1.12: Subspace

$W \subset V$ is a subspace if $\forall \lambda_1, \lambda_2 \in \mathbb{K}$, $w_1, w_2 \in W$, $\lambda_1 w_1 + \lambda_2 w_2 \in W$

Theorem: 1.10: Banach Subspace

A subspace W of a Banach space V is a Banach space if and only if $W \subset V$ is closed.

Definition: 1.13: Quotient Space

Let $W \subset V$ be a subspace. Define equivalence relation on V by $v \sim v' \Leftrightarrow v - v' \in W$. Define $[v] = \{v' \in V : v' \sim v\}$ to be the equivalence class of v. Usually, we write $[v]$ as $v + W$.
The quotient space is $V/W = \{[v] : v \in V\}$ the collection of equivalence classes. V/W is a vector space with $(v_1 + W) + (v_2 + W) = (v_1 + v_2) + W$ and $\lambda(v + W) = \lambda v + W$.
Note: $W = 0 + W = w + W$ for all $w + W$.

Theorem: 1.11:

Let $\|\cdot\|$ be a semi-norm on V. Then $E = \{v \in V : \|v\| = 0\}$ is a subspace of V. Let $\|v + E\|_{V/E} = \|v\|$ $\forall v + E \in V/E$. Then $\|v + E\|_{V/E}$ defines a norm on V/E.

Proof. $\forall v_1, v_2 \in E$ and $\lambda_1, \lambda_2 \in \mathbb{K}$, $\|\lambda_1 v_1 + \lambda_2 v_2\| \leq |\lambda_1|\|v_1\| + |\lambda_2|\|v_2\|$ by homogeneity and triangle inequality of semi-norm.
Since $\|v_1\| = \|v_2\| = 0$, $\|\lambda_1 v_1 + \lambda_2 v_2\| = 0$, E is a subspace.

We now check that $\|v + E\|_{V/E} = \|v\|$ is well defined. Suppose $v + E = v' + E$, *i.e.* $\exists e \in E$ s.t. $v = v' + e$. Then $\|v\| = \|v' + e\| \leq \|v'\| + \|e\| = \|v'\|$. Similarly, $\|v'\| \leq \|v\|$. Thus $\|v\| = \|v'\|$.

Norm: Homogeneity and triangle inequality comes from the semi-norm. Definiteness comes that everything evaluates to 0 is in the same equivalence class in the quotient space. □

Theorem: 1.12: Baire Category Theorem

If M is a complete metric space and $\{C_n\}$ is a collection of closed subsets of M s.t. $M = \bigcap_{n\in\mathbb{N}} C_n$, then at least one C_n containts an open ball $B(x,r) = \{y \in M : d(x,y) < r\}$.

Proof. Assume that $\exists$ a collection of closed subsets $\{C_n\}$ s.t. $M = \bigcap_{n\in\mathbb{N}} C_n$ and none of C_n contains an open ball.

Since M contains an open ball, but C_1 does not, then $M \neq C_1$, $\exists p_1 \in M \setminus C_1$.
Since C_1 is closed, $M \setminus C_1$ is open. $\exists \epsilon_1 > 0$ s.t. $B(p_1, \epsilon_1) \cap C_1 = \emptyset$.
Now, since C_2 does not contain an open balls, $B(p_1, \frac{\epsilon}{3}) \not\subset C_2$, $\exists p_2 \in B(p_1, \frac{\epsilon}{3})$ s.t. $p_2 \notin C_2$.
Since C_2 is closed, $\exists 0 < \epsilon_2 < \frac{\epsilon_1}{3}$ s.t. $B(p_2, \epsilon_2) \cap C_2 = \emptyset$.

By induction, we can find a sequence of points $\{p_k\}_k$ in M and $\epsilon_k \in (0, \epsilon_1)$ s.t. $\forall k$, $p_k \in B(p_{k-1}, \frac{\epsilon_{k-1}}{3})$, $B(p_k, \epsilon_k) \cap C_k = \emptyset$.

Now, we show that $\{p_k\}$ is Cauchy.
$\forall k \in \mathbb{N}$, $\forall l \in \mathbb{N}$,

$$\begin{aligned}
d(p_k, p_{k+l}) &\leq d(p_k, p_{k+1}) + \cdots + d(p_{k+l-1}, p_{k+1}) \\
&< \frac{\epsilon_k}{3} + \cdots + \frac{\epsilon_{k+l-1}}{3} \\
&< \frac{\epsilon_1}{3^k} + \cdots + \frac{\epsilon_1}{3^{k+l}} < \epsilon_1 \sum_{m=0}^{\infty} 3^{-m} \\
&= \frac{\epsilon}{2} 3^{-k+1}
\end{aligned}$$

Thus, $\{p_k\}$ is Cauchy.

Since M is complete, $\exists p \in M$ s.t. $p_k \to p$.
Now $\forall k \in \mathbb{N}$,

$$d(p_{k+1}, p_{k+1+l}) < \epsilon_{k+1}\left(\frac{1}{3} + \cdots + \frac{1}{3^k}\right) < \epsilon_{k+1} \sum_{m=0}^{\infty} 3^{-m} = \epsilon_{k+1} \frac{3}{2}$$

Take limit as $l \to \infty$, $d(p_{k+1}, p) \leq \frac{3}{2}\epsilon_{k+1} < \frac{1}{2}\epsilon_k$ Thus $d(p_k, p) \leq d(p_k, p_{k+1}) + d(p_{k+1}, p) < \frac{1}{3}\epsilon_k + \frac{1}{2}\epsilon_k < \epsilon_k$.
Thus $p \in B(p_k, \epsilon_k)$, $p \notin C_k$ for any k. $p \notin \bigcup_n C_n = M$. Contradiction. □

Theorem: 1.13: Uniform Boundedness Theorem

Let B be a Banach space. $\{T_n\}$ be a sequence in $B(B, V)$ (a sequence of bounded linear operators). If $\forall b \in B$, $\sup_n \|T_n b\| < \infty$, then $\sup_n \|T_n\| < \infty$.

Proof. Define $C_k = \{b \in B : \|b\| \leq 1 \text{ and } \sup_n \|T_n b\| \leq k\}$ for $k \in \mathbb{N}$.
If $\{b_n\} \subset C_k$ and $b_n \to b$, then $\|b\| = \lim_{n\to\infty} \|b_n\| \leq 1$ and $\forall m \in \mathbb{N}$, $\|T_m b\| = \lim_{n\to\infty} \|T_m b_n\| \leq k$. Thus $b \in C_k$, C_k is closed.

Since $\forall b \in B$, $\sup_n \|T_n b\| < \infty$, we can always find some integer k to bound the sup, $\{b \in B : \|b\| \leq 1\} = \bigcup_k C_k$ is a complete metric space as union of closed sets.

By Theorem 1.12, there exists C_k that contains an open ball $B(b_0, \delta_0)$.
Let $b \in B(0, \delta_0)$, *i.e.* $\|b\| < \delta_0$. Then $b_0 + b \in B(b_0, \delta_0)$, $\sup_n \|T_n(b_0 + b)\| \leq K$.

$$\sup_n \|T_n b\| = \sup_n \|T_n(b_0 + b) - T_n b_0\| \leq \sup_n \|T_n b_0\| + \sup_n \|T_n(b_0 + b)\| \leq k + k = 2k$$

Let $n \in \mathbb{N}$, $\|b\| = 1$. Then $\left\|T_n\left(\frac{\delta_0}{2} b\right)\right\| \leq 2k$. $\|T_n b\| \leq \frac{4k}{\delta_0}$. Thus $\|T_n\| \leq \frac{4k}{\delta_0}$ and $\sup_n \|T_n\| \leq \dfrac{4k}{\delta_0} < \infty$. □

1.4 Open Mapping and Closed Graph Theorem

***Theorem:* 1.14: Open Mapping Theorem**

If B_1, B_2 are Banach spaces and $T \in B(B_1, B_2)$ is a surjective bounded linear operator, then T is an open map *i.e.* $\forall$ open subset $U \subset B_1$, $T(U)$ is open in B_2.

Proof. Firstly, we prove that if $B(0,1) = \{b \in B_1, \|b\| < 1\}$, then $T(B(0,1))$ contains an open ball in B_2 centered at 0.
Since T is surjective, $B_2 = \bigcup_{n\in\mathbb{N}} \overline{T(B(0,n))}$.
By Theorem 1.12, $\exists n_0 \in \mathbb{N}$ s.t. $\overline{T(B(0,n_0))}$ contains an open ball. By linearity, $\overline{n_0 T(B(0,1))}$ contains an open ball. Since n_0 is just a constant rescaling, $\overline{T(B(0,1))}$ contains an open ball.
i.e. $\exists v_0 \in B_2$ and $r > 0$ s.t. $B(v_0, 4r) \subset \overline{T(B(0,1))}$.
Then $\exists v_1 = Tu_1 \in T(B(0,1))$ for some $u_1 \in B(0,1)$ s.t. $\|v_0 - v_1\| < 2r$. Then $B(v_1, 2r) \subset B(v_0, 4r) \subset \overline{T(B(0,1))}$
Let $\|v\| < r$, then $\frac{1}{2}(2v + v_1) \in \frac{1}{2}\overline{T(B(0,1))} = \overline{T(B(0,\frac{1}{2}))}$. Then,

$$\begin{aligned} v &= \frac{1}{2}(2v + v_1) - \frac{1}{2}v_1 = \frac{1}{2}(2v + v_1) - \frac{1}{2}Tu_1 \\ &= -T\frac{u_1}{2} + \frac{1}{2}(2v + v_1) \in -T\frac{u_1}{2} + \overline{T\left(B\left(0, \frac{1}{2}\right)\right)} = \overline{T\left(-\frac{u_1}{2} + B\left(0, \frac{1}{2}\right)\right)} \subset \overline{T(B(0,1))} \end{aligned}$$

Thus $B(0,r) \subset \overline{T(B(0,1))}$. Rescale by 2^{-n}, $B(0, 2^{-n}r) = 2^{-n}B(0,r) \subset 2^{-n}\overline{T(B(0,1))} = \overline{T(B(0,2^{-n}))}$ for any $n \in \mathbb{N}$.

Now we show that $B(0, \frac{r}{2}) \subset T(B(0,1))$.
Let $\|v\| < \frac{r}{2}$. Then $v \in \overline{T(B(0,\frac{1}{2}))} \Rightarrow \exists b_1 \in B(0,\frac{1}{2})$ s.t. $\|v - Tb_1\| < \frac{r}{4}$. Thus $v - Tb_1 \in \overline{T(B(0,\frac{1}{4}))}$.
Then $\exists b_2 \in B(0, \frac{1}{4})$ s.t. $\|v - Tb_1 - Tb_2\| < \frac{r}{8}$. Continuing the iteration, we get a sequence $\{b_k\}$ in B_1 s.t. $\|b_k\| < 2^{-k}$, $\|v - \sum_{k=1}^{n} Tb_k\| < 2^{-n-1}r$. The series $\sum b_k$ is absolutely summable in B_1.
Since B_1 is a Banach space, by Theorem 1.4, $\sum b_k$ is summable, $\exists b \in B_1$ s.t. $b = \sum b_k$ and $\|b\| = \lim_{n\to\infty} \|\sum_{k=1}^{n} b_k\| \leq \lim_{n\to\infty} \sum_{k=1}^{n} \|b_k\| = \sum_{k=1}^{\infty} \|b_k\| < \sum_{k=1}^{\infty} 2^{-k} = 1$
Moreover, since T is continuous, $Tb = \lim_{n\to\infty} T\sum_{k=1}^{n} b_k = \lim_{n\to\infty} \sum_{k=1}^{n} Tb_k = v$. Since $\|b\| < 1$, we have $v = Tb \in T(B(0,1))$.
Thus $B(0, \frac{r}{2}) \subset T(B(0,1))$ (as by definition, $\|v\| < \frac{r}{2}$, $v \in B(0, \frac{r}{2})$). *i.e.* $T(B(0,1))$ contains an open ball in B_2 centered at 0.

We have shown the specific case. Now, suppose $U \subset B_1$ is open and $b_2 = Tb_1 \in T(U)$. Then $\exists \epsilon > 0$ s.t. $b_1 + B(0,\epsilon) = B(b_1, \epsilon) \subset U$. Let $\delta > 0$

$$B(b_2, \epsilon\delta) = b_2 + \epsilon B(0,\delta) \subset b_2 + \epsilon T(B(0,1)) = Tb_1 + \epsilon T(B(0,1)) = T(b_1 + B(0,\epsilon)) \subset T(U)$$

This shows the general case. $\square$

Corollary 1. *If B_1, B_2 are Banach spaces, $T \in B(B_1, B_2)$ is a bijective bounded linear operator, then $T^{-1} \in B(B_2, B_1)$.*

Proof. T^{-1} is continuous if and only if $\forall$ open $U \subset B$, $(T^{-1})^{-1}(U) = T(U)$ is open by Theorem 1.14. $\square$

***Theorem:* 1.15:**

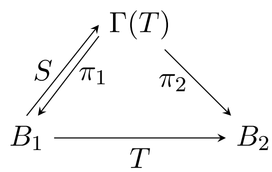
Let B_1, B_2 be Banach spaces, then $B_1 \times B_2$ with norm $\|(b_1, b_2)\| = \|b_1\| + \|b_2\|$ is a Banach space.

***Theorem:* 1.16: Closed Graph Theorem**

If B_1, B_2 are Banach spaces, $T : B_1 \to B_2$ is a linear operator, then $T \in B(B_1, B_2) \Leftrightarrow \Gamma(T) = \{(u, Tu) : u \in B_1\} \subset B_1 \times B_2$ is closed.

Proof. ($\Rightarrow$) Suppose $T \in B(B_1, B_2)$. Let $\{(u_n, Tu_n)\}$ be a sequence in $\Gamma(T)$ s.t. $u_n \to u$ and $Tu_n \to v$. Then by continuity, $v = \lim_{n\to\infty} Tu_n = T(\lim_{n\to\infty} u_n) = Tu$. Thus $(u, v) = (u, Tu) \in \Gamma(T)$, $\Gamma(T)$ is closed.

($\Leftarrow$) Define $\pi_1 : \Gamma(T) \to B_1$ s.t. $\pi_1(u, Tu) = u$, $\pi_2 : \Gamma(T) \to B_2$ s.t. $\pi_2(u, Tu) = Tu$.



Since $\Gamma(T) \subset B_1 \times B_2$ is a closed subspace of the Banach space $B_1 \times B_2$, then $\Gamma(T)$ is a Banach space.
Since $\|\pi_1(u, v)\| = \|u\| \le \|u\| + \|v\| = \|(u, v)\|$, $\pi_1 \in B(\Gamma(T), B_1)$, similarly, $\pi_2 \in B(\Gamma(T), B_2)$.
Also $\pi_1 : \Gamma(T) \to B_1$ is bijective, thus $S = \pi_1^{-1} : B_1 \to \Gamma(T)$ is a bounded linear operator.
Then $T = \pi_2 \circ S : B_1 \to B_2$ is a bounded linear operator as the composition of bounded linear operators. $\square$

Remark 2. Theorem 1.14 and Theorem 1.16 are logically equivalent.

1.5 Hahn-Banach Theorem

Given a general non-trivial normed space, the dual space $V' = B(V, \mathbb{K}) = \{0\}$ is not necessarily true. The Hahn-Banach Theorem tells us that the dual space contains many elements.

***Definition:* 1.14: Partial Order**

A partial order on a set E is a relation $\le$ on E s.t.
1. $\forall e \in E$, $e \le e$
2. $\forall e, f \in E$, $e \le f$ and $f \le e \Rightarrow e = f$
3. $\forall e, f, g \in E$, $e \le f$ and $f \le g \Rightarrow e \le g$

An *upper bound* of a set $D \subset E$ is an element $e \in E$ s.t. $\forall d \in D$, $d \le e$. A maximal element of E is an element $e \in E$ s.t. if $f \in E$ and $e \le f$, then $e = f$. Similar definition for minimal element.

***Definition:* 1.15: Chain**

If $(E, \le)$ is a partially ordered set, a chain in E is a set C s.t. $\forall e, f \in C$, either $e \le f$ or $f \le e$

Lemma: 1.1: Zorn's Lemma

If every chain in a non-empty partially ordered set E has an upper bound, then E has a maximal element

Definition: 1.16: Hamel Basis

A Hamel basis $H \subset V$ (V a vector space) is a linearly independent set s.t. every element of V is a finite linear combination of elements of H.

Example: $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$ is a Hamel basis for $\mathbb{R}^2$.

Theorem: 1.17:

If V is a vector space, then V has a Hamel basis.

Proof. Let $E = \{\text{linearly independent subsets of } V\}$.
Define a partial order $\leq$ on E by inclusion, *i.e.* for $e, e' \subset V$, $e \leq e' \Leftrightarrow e \subset e'$.
Let C be a chain in E. Define $c = \bigcup_{e \in C} e$. Then $\forall e \in C$, $e \leq c$, c is an upper bound for C.
Let $v_1, ..., v_N \in c$. $\exists e_1, ..., e_N \in C$ s.t. $\forall j, v_j \in e_j$.
Since C is a chain, $\exists J$ s.t. $\forall j = 1, ..., N$, $e_j \leq e_J$ (equivalently, $e_j \subset e_J$). Therefore, $v_1, ..., v_N \in e_J$. $v_1, ..., v_N$ are linearly independent, since $e_J \in E$. Thus $C \in E$.
By Lemma 1.1, E has a maximal element H.
Assume H does not span V, then $\exists v \in V$ s.t. v cannot be written as a finite linear combination of elements in H.
$H \cup \{v\}$ is a linearly independent subset of V. Then $H < H \cup \{v\}$, H is not maximal. Contradiction.
Thus H spans V and by definition, H is a Hamel basis. □

Lemma: 1.2:

Let V be a normed space, $M \subset V$ be a subspace and $u : M \to \mathbb{C}$ be linear s.t. $|u(t)| \leq C\|t\|, \forall t \in M$. Let $x \notin M$. Then $\exists u' : M' \to \mathbb{C}$ which is linear on $M' = M + \mathbb{C}x = \{t + ax : t \in M, a \in \mathbb{C}\}$ s.t. $u'|_M = u$ and $\forall t' \in M'$, $|u'(t')| \leq C\|t'\|$

Proof. If $t' \in M' = M + \mathbb{C}x$, then there exists unique $t \in M$ and $a \in \mathbb{C}$ s.t. $t' = t + ax$.
If $t + ax = \tilde{t} + \tilde{a}x$, then $(a - \tilde{a})x = \tilde{t} - t \in M$, $a = \tilde{a}$, $t = \tilde{t}$. Otherwise, $(a - \tilde{a})x \neq 0$, and $(a - \tilde{a})x \notin M$.
Once we choose $\lambda \in \mathbb{C}$, $u'(t + ax) = u(t) + a\lambda$ is well-defined on M' and $u' : M' \to \mathbb{C}$ is linear.

WLOG, assume $C = 1$. We want to choose $\lambda \in \mathbb{C}$ s.t. $\forall t \in M$, $a \in \mathbb{C}$, $|u(t) + a\lambda| \leq \|t + ax\|$, which always holds for $a = 0$.
Consider the case $a \neq 0$. Then we can divide both sides by $|a|$.
$|u(\frac{t}{-a}) - \lambda| \leq \|\frac{t}{-a} - x\|$ $\forall t \in M$, which is equivalent to $|u(t) - \lambda| \leq \|t - x\|$.

We firstly show that $\exists \alpha \in \mathbb{R}$ s.t. $|w(t) - \alpha| \leq \|t - x\|$, $\forall t \in M$ where $w(t) = \text{Re}(u(t)) = \frac{w(t) + \overline{w(t)}}{2}$
Note $\forall t \in M$, $|w(t)| = |\text{Re}(u(t))| \leq |u(t)| \leq \|t\|$
Then $\forall t_1, t_2 \in M$, $w(t_1) - w(t_2) = w(t_1 - t_2) \leq |w(t_1 - t_2)| \leq \|t_1 - t_2\| \leq \|t_1 - x\| + \|t_2 - x\|$
Thus, $w(t_1) - \|t_1 - x\| \leq w(t_2) + \|t_2 - x\|, \forall t_1, t_2 \in M$. Therefore, $\sup_{t \in M}[w(t) - \|t - x\|] \leq w(t_2) + \|t_2 - x\|$, $\forall t_2 \in M$.
Then $\sup_{t \in M}[w(t) - \|t - x\|] \leq \inf_{t \in M} w(t) + \|t - x\|$. We choose α between them.

Then $\forall t \in M$, $w(t) - \|t-x\| \leq \alpha \leq w(t) + \|t-x\| \Rightarrow -\|t-x\| \leq \alpha - w(t) \leq \|t-x\| \Rightarrow |w(t)-\alpha| \leq \|t-x\|$. We can repeat this for imagnary part by replacing x with ix. Then it defines u' on all $M + \mathbb{C}x$. □

Theorem: 1.18: Hahn-Banach Theorem

Let V be a normed space, $M \subset V$ a subspace, and $u : M \to \mathbb{C}$ a linear map s.t. $\forall t \in M$, $|u(t)| \leq C\|t\|$ for all $t \in M$ (bounded lienar functional), then there exists a continuous extension $U \in V' = B(V, \mathbb{C})$ s.t. $U|_M = u$ and $\|U(t)\| \leq C\|t\|$ for all $t \in V$.

Proof. Strategy: Firstly, apply Lemma 1.1 for all continuous extensions of u to get a maximal element U. Then use Lemma 1.2 to show that U is defined on all of V.

Let $E = \{(v, N)\}$ where N is a subspace of V and v is a continuous extension of u to N.
Define $\leq$ on E by $(v_1, N_1) \leq (v_2, N_2)$ if $N_1 \subset N_2$ and $v_2|_{N_1} = v_1$. Then $\leq$ is a partial order.
Let $C = \{(v_i, N_i), i \in I\}$ be a chain in E. Then $\forall i_1, i_2 \in I$, either $(v_{i_1}, N_{i_1}) \leq (v_{i_2}, N_{i_2})$ or vice versa.

Let $N = \bigcup_{i\in I} N_i$. We show that N is a subspace.
Let $v_1, v_2 \in N$ and $a_1, a_2 \in \mathbb{C}$, $\exists i_1, i_2 \in I$ s.t. $v_1 \in N_{i_1}$ and $v_2 \in N_{i_2}$.
Then since C is a chain, WLOG we assume $N_{i_1} \subset N_{i_2}$. Then $v_1, v_2 \in N_{i_2}$. $a_1v_1 + a_2v_2 \in N_{i_2} \subset N$, N is a subspace.
Define $v : N \to \mathbb{C}$, $v(t) = v_i(t)$ if $t \in N_i$.
Well-defined: suppose $t \in N_{i_1} \cap N_{i_2}$, WLOG assume $(v_{i_1}, N_{i_1}) \leq (v_{i_2}, N_{i_2})$
Since v_{i_2} extend v_{i_1}, $v_{i_2}|_{N_{i_1}} = v_{i_1}$, $v_{i_2}(t) = v_{i_1}(t)$, v is well defined.

Similarly, we can show that v is linear and is an extension of any v_i. Thus $\forall i \in I$, $(v_i, N_i) \leq (v, N)$, *i.e.* (v, N) is an upper bound of C.

By Lemma 1.1, E has a maximal element (U, N). We want to show that $N = V$.
Assume $N \neq V$. Let $x \notin N$, by Lemma 1.2, there exists a continuous extension of U to $N + \mathbb{C}x$ and $(v, N + \mathbb{C}x) \in E$.
Then $(U, N) < (v, N + \mathbb{C}x)$, (U, N) is not maximal. Contradiction. Thus $N = V$. □

Theorem: 1.19:

If V is a normed space, then $\forall v \in V \setminus \{0\}$, $\exists f \in V'$ s.t. $\|f\| = 1$ and $f(v) = \|v\|$.

Proof. Define $u : \mathbb{C}v \to \mathbb{C}$ by $u(\lambda v) = \lambda\|v\|$. Then $|u(t)| \leq \|t\|$, $\forall t \in \mathbb{C}v$ and $u(v) = \|v\|$.
By Theorem 1.18, $\exists f \in V'$ extending u s.t. $\forall t \in V$, $|f(t)| \leq \|t\|$. Then $f(v) = u(v) = \|v\|$.
Since $|f(t)| \leq \|t\|$, $\forall t \in V$, $\|f\| \leq 1$. But $1 = f\left(\frac{v}{\|v\|}\right) \leq \|f\|$. Thus $\|f\| = 1$. □

1.6 Double Dual

Definition: 1.17: Double Dual

The double dual of V is $V'' = (V')'$ (dual of the dual)

Example: Let $v \in V$. Define $T_v : V' \to \mathbb{C}$ by $T_v(v') = v'(v)$, where v' is a functional in V' and v is a fixed vector in V. Then $T_v \in V''$.

Proof. T_v is linear, since v is fixed and v' is a bounded linear functional.
T_v is bounded, since $|T_v(v')| = |v'(v)| \leq \|v'\|\|v\|$.
Thus $T_v \in (V')' = V''$ and $\|T_v\| \leq \|v\|$. ☐

***Definition:* 1.18: Isometry**

If V, W are normed space, then $T \in B(V, W)$ is isometric if $\forall v \in V$, $\|Tv\| = \|v\|$.

***Theorem:* 1.20:**

Let $v \in V$. Define $T_v : V' \to \mathbb{C}$ s.t. $T_v(v') = v'(v)$. Then the map $T : V' \to V''$ s.t. $T(v) = T_v$ is isometric.

Proof. We have shown that $T(v) = T_v$ is a bounded linear operator $T \in B(V, V'')$ and $\|T_v\| \leq \|v\|$ in the previous example.
Now, we show that $\forall v \in V$, $\|T_v\| = \|v\|$.
If $v = 0$, it is trivial that $\|T_0\| = \|0\|$.
If $v \in V \setminus \{0\}$, then by Theorem 1.19, $\exists f \in V'$ st. $\|f\| = 1$ and $f(v) = \|v\|$.
Then $\|v\| = |f(v)| = \|T_v(f)\| \leq \|T_v\|\|f\| = \|T_v\|$. Thus $\|T_v\| = \|v\|$. ☐

***Definition:* 1.19: Reflexive Banach Space**

A Banach space is reflexive if $V = V''$ in the sense that $v \mapsto T_v$ is onto.

Example: For $1 < p < \infty$, l^p is reflexive. $(l^1)'$ is not reflexive, since $(l^1)' = l^\infty$, but $(l^\infty)' \neq l^1$. c_0 the sequences converging to zero is not reflexive, $(c_0)' = l^1$, but $(l^1)' = l^\infty \neq c_0$.

2 Lebesgue Measure and Integrals

Why do we need Lebesgue measure and Lebesgue integrals? Compared with Riemann integrals, Lebesgue integration has more and better limiting theorems. Consider the space of Riemann integrable functions on $[0,1]$:

$$L^1_R([0,1]) = \{f : [0,1] \to \mathbb{C} : f \text{ is Riemann integrable on } [0,1]\}$$

We can define $\|f\|_1 = \int_0^1 |f(x)|dx$ for $f \in L^1_R([0,1])$ as a semi-norm. However, even if we quotient out the $\|f\|_1 = 0$ subspace to get a norm, $L^1_R([0,1])$ is still not Banach. The completion of $L^1_R([0,1])$ is the Lebesgue integrable functions.

***Definition:* 2.1: Indicator Function**

$$1_E(x) = \begin{cases} 1, x \in E \\ 0, x \notin E \end{cases}$$

How should we integrate $1_E(x)$? If $E = [a,b]$, then $\int 1_E(x)dx = l([a,b])$. For more general E, $\int 1_E(x)dx = m(E)$ where $m(E)$ is the measure (length) of E.

We want to define measure of subsets of $\mathbb{R}$ with the following properties:

1. $m(E)$ is well-defined $\forall E \subset \mathbb{R}$
2. If I is an interval, $m(I) = l(I)$, regardless of its topology (open/close intervals)
3. If $\{E_n\}$ is a countable collection of disjoint sets, then $m\left(\bigcup_n E_n\right) = \sum_n m(E_n)$
4. m is translation invariant: If $E \subset \mathbb{R}$, $x \in \mathbb{R}$, then $m(x+E) = m(\{x+y : y \in E\}) = m(E)$.

However, such a function $m : \mathcal{P}(\mathbb{R}) \to [0,\infty)$ does not exist. We drop the first assumption, and still satisfying 2, 3 and 4, which gives the set of Lebesgue measurable sets.

Notation: If $I \subset \mathbb{R}$ is an interval, then $l(I)$ denotes its length.

2.1 Measures

***Definition:* 2.2: Outer Measure**

For $A \subset \mathbb{R}$, define the outer measure of A as

$$m^*(A) = \inf\left\{\sum_n l(I_n) : \{I_n\} \text{ a countable collection of open intervals s.t. } A \subset \bigcup_n I_n\right\}$$

Example: $m^*(\{0\}) = 0$

Proof. Let $\epsilon > 0$. Then $\{0\} \subset (-\frac{\epsilon}{2}, \frac{\epsilon}{2})$. $m^*(\{0\}) \le l(-\frac{\epsilon}{2}, \frac{\epsilon}{2}) = \epsilon$. Thus $m^*(\{0\}) = 0$. □

***Theorem:* 2.1:**

If $A \subset \mathbb{R}$ is countable, then $m^*(A) = 0$.

Proof. If A is countable, then $A = \{a_n : n \in \mathbb{N}\}$ can be enumerated.
Let $\epsilon > 0$. We show that $m^*(A) \le \epsilon$.
For each $n \in \mathbb{N}$, let $I_n = (a_n - \frac{\epsilon}{2^{n+1}}, a_n + \frac{\epsilon}{2^{n+1}})$.
$a_n \in I_n$ for each n, thus $A \subset \cup_n I_n$.
Then $m^*(A) \le \sum_{n=1}^{\infty} l(I_n) = \sum_{n=1}^{\infty} \frac{\epsilon}{2^n} = \epsilon$. Thus $m^*(A) = 0$. □

Example: $m^*(\mathbb{Q}) = 0$

***Theorem:* 2.2:**

If $A \subset B$, then $m^*(A) \le m^*(B)$.

Proof. Any covering of B should also cover A. Infimum over covering of A should be smaller. □

***Theorem:* 2.3:**

Let $\{A_n\}$ be a countable collection of subsets of $\mathbb{R}$. Then $m^*\left(\bigcup_n A_n\right) \le \sum_n m^*(A_n)$.

Proof. If $\exists n$ s.t. $m*(A_n) = \infty$ or $\sum m*(A_n) = \infty$, then the inequality is true.
Suppose $\forall n$ $m^*(A_n) < \infty$ and $\sum m*(A_n) < \infty$.
Let $\epsilon > 0$. For each n, let $\{I_{nk}\}_{k\in\mathbb{N}}$ be a collection of open intervals s.t. $A_n \subset \bigcup_{k\in\mathbb{N}} I_{nk}$ and $\sum_{k=1}^{\infty} l(I_{nk}) < m^*(A_n) + \frac{\epsilon}{2^n}$.
Then $\bigcup_{n\in\mathbb{N}} A_n \subset \bigcup_{n,k\in\mathbb{N}} I_{nk}$.
Thus, by Theorem 2.2,

$$m^*\left(\bigcup_n A_n\right) \le \sum_{n,k} l(I_{nk}) = \sum_n \sum_k l(I_{nk}) < \sum_n m^*(A_n) + \sum_n \frac{\epsilon}{2^n} = \sum_n m^*(A_n) + \epsilon$$

Let $\epsilon \to 0$, we get $m^*\left(\bigcup_n A_n\right) \le \sum_n m^*(A_n)$. □

***Theorem:* 2.4:**

If $I \subset \mathbb{R}$ is an interval, then $m^*(I) = l(I)$.

Proof. Suppose $I = [a, b]$. Then $\forall \epsilon > 0$, $I \subset (a-\epsilon, b+\epsilon)$, $m^*(I) \le l(a-\epsilon, b+\epsilon) = b-a+2\epsilon$, $m^*(I) \le b-a$.
Now, we need to show that $b - a \le m^*(I)$. Let $\{I_n\}_n$ be a collection of open intervals s.t. $[a, b] \subset \cup I_n$.
Since $[a, b]$ is compact by Heine Borel Theorem, then $\exists \{J_k\}_{k=1}^N \subset \{I_n\}$ s.t. $[a, b] \subset \bigcup_{k=1}^N J_k$ (Any cover of compact sets have finite subcover).
Since $a \in \cup_{k=1}^N J_k$, $\exists k_1$ s.t. $a \in J_{k_1}$. By rearranging the intervals, we can assume $k_1 = 1$. *i.e.* $a \in J_1 = (a_1, b_1)$.
If $b_1 > b$, then we are done. Otherwise $b_1 \le b$, then $b_1 \in [a, b]$. $\exists k_2$ s.t. $b_1 \in J_{k_2}$. By rearranging, assume

$k_2 = 2$, $b_1 \in J_2 = (a_2, b_2)$.
We continue until $b_k > b$. Thus $\exists K$, $1 \le K \le N$ s.t. $\forall k = 1, ..., K-1$, $b_k \le b$ and $a_{k+1} < b_k < b_{k+1}$, and $b < b_K$. Then,

$$\begin{aligned}\sum_n l(I_n) \ge \sum_{k=1}^{N} l(J_k) \le \sum_{k=1}^{K} l(J_k) \\ &= (b_K - a_K) + (b_{K-1} - a_{K-1}) + \cdots + (b_1 - a_1) \\ &= b_K + (b_{K-1} - a_K) + (b_{K-2} - a_{K-1}) + \cdots + (b_1 - a_2) - a_1 \\ &\ge b_k - a_1 \ge b - a\end{aligned}$$

Thus $m^*(I) \ge b - a$. Therefore, $m^*(I) = b - a$.

If I is any finite interval, $[a, b], (a, b], [a, b), (a, b)$, then $\forall \epsilon > 0$, $[a + \epsilon, b - \epsilon] \subset I \subset [a - \epsilon, b + \epsilon]$.
$m * ([a + \epsilon, b - \epsilon]) \le m^*(I) \le m^*([a - \epsilon, b + \epsilon])$, so $b - a - 2\epsilon \le m^*(I) \le b - a + 2\epsilon$.
Let $\epsilon \to 0$, $b - a \le m^*(I) \le b - a$. Therefore, $m^*(I) = b - a$.
If $I = \mathbb{R}, (-\infty, a), (a, \infty), (-\infty, a], [a, \infty)$, then $m^*(I) = \infty$ □

Theorem: 2.5:

$\forall A \subset \mathbb{R}$ and $\epsilon > 0$, there exists an open set O s.t. $A \subset O$ and $m^*(A) \le m^*(O) \le m^*(A) + \epsilon$

Proof. It is clear if $m^*(A) = \infty$, so we suppose $m^*(A) < \infty$.
Let $\{I_n\}_n$ be a collection of open intervals s.t. $A \subset \bigcup_n I_n$ and $\sum_n l(I_n) \le m^*(A) + \epsilon$.
Take $O = \bigcup_n I_n$, O is open. $A \subset O$ and $m^*(O) = m^*\left(\bigcup_n I_n\right) \le \sum_n m^*(I_n) = \sum_n l(I_n) \le m^*(A) + \epsilon$. □

Definition: 2.3: Measurable Sets

A set $E \subset \mathbb{R}$ is Lebesgue measurable if $\forall A \subset \mathbb{R}$, $m^*(A) = m^*(A \cap E) + m^*(A \cap E^C)$.

Remark 3. Since $\forall A, E$, $A = (A \cap E) \cup (A \cap E^C)$, $m^*(A) \le m^*(A \cap E) + m^*(A \cap E^C)$ always hold by Theorem 2.3. Thus E is measurable if $\forall A \subset \mathbb{R}$, $m^*(A \cap E) + m^*(A \cap E^C) \le m^*(A)$.

Theorem: 2.6:

$\emptyset$, $\mathbb{R}$ are measurable. $E \subset \mathbb{R}$ is measurable $\Leftrightarrow$ $E^C \subset \mathbb{R}$ is measurable.

Theorem: 2.7:

If $m^*(E) = 0$ (E has zero outer measure), then E is measurable.

Proof. Let $A \subset \mathbb{R}$. Then $A \cap E \subset E$, $m^*(A \cap E) \le m^*(E) = 0 \Rightarrow m^*(A \cap E) = 0$. Thus $m^*(A \cap E) + m^*(A \cap E^C) = m^*(A \cap E^C) \le m^*(A)$. □

Theorem: 2.8:

If E_1, E_2 are measurable, then $E_1 \cup E_2$ is measurable.

Proof. Let $A \subset \mathbb{R}$. Since E_2 is measurable, then $m^*(A \cap E_1^C) = m^*(A \cap E_1^C \cap E_2) + m^*(A \cap E_1^C \cap E_2^C)$ by Definition 2.3, setting $A = A \cap E_1^C$, $E = E_2$.
Then $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^C)$ (because $A \cap E_1 \cap E_2$ is included in the first set).
Then $m^*(A \cap (E_1 \cup E_2)) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^C) \overset{E_1 \text{ is measurable}}{=} m^*(A) - m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^C) = m^*(A) - m^*(A \cap (E_1 \cup E_2)^C)$
Rearranging the terms, we get $m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^C) \leq m^*(A)$ □

***Theorem:* 2.9:**

If $E_1, ..., E_n$ are measurable, then $\bigcup_{k=1}^{n} E_k$ is measurable.

Proof. We prove by induction. $n = 1$ is trivial.
IH: Suppose $\bigcup_{k=1}^{n} E_k$ holds for $n = m$.
When $n = m + 1$. Let $E_1, ..., E_{m+1}$ be measurable. Then $\bigcup_{k=1}^{m+1} E_k = \bigcup_{k=1}^{m} E_k \cup E_{m+1}$ is measurable as the union of two measurable sets by Theorem 2.8. □

2.1.1 Sigma Algebra

***Definition:* 2.4: Sigma Alegebra**

A non-empty collection of sets $A \subset \mathcal{P}(\mathbb{R})$ is an *algebra* if:

1. $E \in A \Rightarrow E^C \in A$
2. $E_1, ..., E_n \in A \Rightarrow \bigcup_{k=1}^{n} E_k \in A$

An algebra A is a σ-algebra if also

3. if $\{E_n\}_{n=1}^{\infty}$ is a countable collection of elements of A, then $\bigcup_{n=1}^{\infty} E_n \in A$

Remark 4. By De Morgan's law, $E_1, ..., E_n \in A \Rightarrow \bigcap_{k=1}^{n} E_k = \left(\bigcup_{k=1}^{n} E_k^C\right)^C \in A$. Thus if $E \in A$, then $\emptyset = E \cap E^C \in A$, and $\mathbb{R} = \emptyset^C \in A$.

Similarly, if A is a σ-algebra, then $\{E_n\}_n \subset A \Rightarrow \bigcap_n E_n \in A$.

Example: $A = \{\emptyset, \mathbb{R}\}$, $A = \mathcal{P}(\mathbb{R})$, $A = \{E \subset \mathbb{R} : E \text{ or } E^C \text{ is countable}\}$ are σ-algebra

Proof. For the third one, E is countable, E^C is uncountable, but $(E^C)^C$ is then countable.
Suppose $\{E_n\} \subset A$. If $\forall n$, E_n is countable, then $\cup_n E_n$ is countable, $\cup_n E_n \in A$.
If $\exists N_0$ s.t. $E_{N_0}^C$ is countable, then $(\cup E_n)^C = \cap E_n^C \subset E_{N_0}^C$, $(\cup E_n)^C$ is countable. Thus $\cup E_n \in A$. □

Theorem: 2.10: Borel Measure

Let $\Sigma = \{A : A$ is a sigma algebra cotaining all subsets of $\mathbb{R}\}$. (*e.g.* $\mathcal{P}(\mathbb{R}) \in \Sigma$) Define $\mathcal{B} = \bigcap_{A\in\Sigma} A \subset \mathcal{P}(\mathbb{R})$. Then $\mathcal{B}$ is the smallest σ-algebra containing all subsets of $\mathbb{R}$. This is the Borel Measure.

Proof. Suppose $E \in \mathcal{B}$. Then $\forall A \in \Sigma$, $E \in A$, and thus $E^C \in A$, $E^C \in \bigcap_{A\in\Sigma} A = \mathcal{B}$. Therefore $\mathcal{B}$ is closed under complement.
Similarly, we can show that it is closed under countable union: those sets in the countable union must be in every $A \in \Sigma$, and then we can apply closure under countable union within each A. □

Lemma: 2.1:

Let A be an algebra, $\{E_n\}_n$ be a collection of elements of A. Then $\exists\{F_n\}_n$ a collection of elements of A that are disjoint s.t. $\bigcup_n E_n = \bigcup_n F_n$. (Thus we only need to check 3 for disjoint collections $\{E_n\}$ for 3 for σ-alg)

Proof. Let $G_n = \bigcup_{k=1}^{n} E_k$. Then $G_1 \subset G_2 \subset \cdots$, and $\bigcup_n E_n = \bigcup_n G_n$.
Take $F_1 = G_1$ and $F_{n+1} = G_{n+1} \setminus G_n$ for all $n \geq 1$. Then $\bigcup_{k=1}^{n} F_k = \bigcup_{k=1}^{n} G_n$. And $\bigcup_k E_k = \bigcup_k F_k$ for countable unions. □

Theorem: 2.11: Additivity of Lebesgue Measure

Let $A \subset \mathbb{R}$, $E_1, ..., E_n$ be disjoint measurable sets. Then $m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) = \sum_{k=1}^{n} m^*(A \cap E_k)$.

Proof. By induction, $n = 1$ is trivially true.
IH: Suppose $m^*\left(A \cap \left(\bigcup_{k=1}^{n} E_k\right)\right) = \sum_{k=1}^{n} m^*(A \cap E_k)$ is true for $n = m$.
IS: When $n = m+1$. Let $E_1, ..., E_{m+1}$ be measurable disjoint sets. Let $A \subset \mathbb{R}$. Since $E_k \cap E_{m+1} = \emptyset$ for all $k = 1, ..., m$. $A \cap \left(\bigcup_{k=1}^{m+1} E_k\right) \cap E_{m+1} = A \cap E_{m+1}$, and $A \cap \left(\bigcup_{k=1}^{m+1} E_k\right) \cap E_{m+1}^C = A \cap \left(\bigcup_{k=1}^{m+1} E_k\right)$.
Since E_{m+1} is measurable, by Definition 2.3,

$$\begin{aligned}
m^*(A \cap (\cup_{k=1}^{m+1} E_k)) &= m^*(A \cap (\cup_{k=1}^{m+1} E_k) \cap E_{m+1}) + m^*(A \cap (\cup_{k=1}^{m+1} E_k) \cap E_{m+1}^C) \\
&= m^*(A \cap E_{m+1}) + m^*(A \cap (\cup_{k=1}^{m} E_k)) \\
&= m^*(A \cap E_{m+1}) + \sum_{k=1}^{m} m^*(A \cap E_k) \text{By IH} \\
&= \sum_{k=1}^{m+1} m^*(A \cap E_k)
\end{aligned}$$

□

Theorem: 2.12:

The collection $\mathcal{M}$ of measurable sets is a σ-algebra.

Proof. We have shown that $\mathcal{M}$ is an algebra. By Lemma 2.1, we just need to show $\mathcal{M}$ is closed under countable disjoint unions. Let $\{E_n\}$ be a collection of disjoint measurable sets. Let $A \subset \mathbb{R}$, $E = \bigcup_{n=1}^{\infty} E_n$.
We want to show that $m^*(A \cap E^C) + m^*(A \cap E) \leq m^*(A)$.
Let $N \in \mathbb{N}$. Since $\mathcal{M}$ is an algebra, $\cup_{n=1}^{N} E_n \in \mathcal{M}$.

$$\begin{aligned} m^*(A) &= m^*(A \cap (\cup_{n=1}^{N} E_n)) + m^*(A \cap (\cup_{n=1}^{N} E_n)^C) \\ &\geq m^*(A \cap (\cup_{n=1}^{N} E_n)) + m^*(A \cap E^C) \\ &= \sum_{n=1}^{N} m^*(A \cap E_n) + m^*(A \cap E^C) \end{aligned}$$

Let $N \to \infty$, $m^*(A) \geq \sum_{n=1}^{\infty} m^*(A \cap E_n) + m^*(A \cap E^C) \geq m^*(A \cap \cup_n E_n) + m^*(A \cap E^C) = m^*(A \cap E) + m^*(A \cap E^C)$ □

Theorem: 2.13:

$\forall a \in \mathbb{R}$, (a, ∞) is measurable.

Proof. Let $A \subset \mathbb{R}$, $A_1 = A \cap (a, \infty)$, $A_2 = A \cap (-\infty, a]$. We want to show that $m^*(A_1) + m^*(A_2) \leq m^*(A)$.
If $m^*(A) = \infty$, then done. Suppose $m^*(A) < \infty$
Let $\epsilon > 0$, $\{I_n\}_n$ be a collection of open intervals s.t. $\sum_n l(I_n) \leq m^*(A) + \epsilon$.
Define $J_n = I_n \cap (a, \infty)$, $K_n = I_n \cap (-\infty, a]$. Then each J_n and K_n is either an interval or an empty set.
Then $A_1 \subset \cup_n J_n$, $A_2 \subset \cup_n K_n$ and $l(I_n) = l(J_n) + l(K_n)$,

$$m^*(A_1) + m^*(A_2) \leq \sum_n m^*(J_n) + \sum_n m^*(K_n) = \sum_n l(J_n) + l(K_n) = \sum_n (I_n) \leq m^*(A) + \epsilon$$

Let $\epsilon \to 0$, $m^*(A_1) + m^*(A_2) \leq m(A)$. □

Theorem: 2.14:

Every open set is measurable, and thus $\mathcal{B} \subset \mathcal{M}$

Proof. For all $b \in \mathbb{R}$, $(-\infty, b) = \bigcup_{n=1}^{\infty} \left(-\infty, b - \frac{1}{n}\right] = \bigcup_{n=1}^{\infty} \left(b - \frac{1}{n}, \infty\right)^C \in \mathcal{M}$, because $\left(b - \frac{1}{n}, \infty\right)$ is measurable. Complements are measurable by measurable by Definition 2.4 and countable unions of measurable sets are measurable.
Thus any $(a, b) = (-\infty, b) \cap (a, \infty)$ is measurable because σ-alg is closed under intersections.
Finally every open subset of $\mathbb{R}$ is a countable union of open intervals. Thus all open sets are measurable. □

2.1.2 Lebesgue Measure

Definition: 2.5: Lebesgue Measure

If $E \in \mathcal{M}$ is measurable, then the Lebesgue measure of E is $m(E) = m^*(E)$.

Theorem: 2.15:

If $A, B \in M$ and $A \subset B$, then $m(A) \leq m(B)$. Every interval is Lebesgue measurable and $m(l) = I(l)$.

Proof. These properties are inherited from outer measures Definition 2.2. For closed intervals, $[a, b] = (b, \infty)^C \cap (-\infty, a)^C$ and $(b, \infty)^C$ and $(-\infty, a)^C$ are measurable. □

Theorem: 2.16:

Suppose $\{E_n\}$ is a countable collection of disjoint measurable sets. Then $m\left(\bigcup_n E_n\right) = \sum_n m(E_n)$.

Proof. Since E_n are measurable, $\cup_n E_n \in \mathcal{M}$ by Theorem 2.12,

Thus $m\left(\bigcup_n E_n\right) \overset{\text{Definition 2.5}}{=} m^*\left(\bigcup_n E_n\right) \overset{\text{Theorem 2.3}}{\leq} \sum_n m^*(E_n) = \sum_n m(E_n)$.

We now show that $\sum_n m(E_n) \leq m\left(\bigcup_n E_n\right)$.

Let $N \in \mathbb{N}$, $m\left(\bigcup_{n=1}^N E_n\right) = m^*\left(\mathbb{R} \cap \left(\bigcup_{n=1}^N E_n\right)\right) = \sum_{n=1}^N m^*(\mathbb{R} \cap E_n) = \sum_{n=1}^N m(E_n)$.

Thus, $\sum_{n=1}^N m(E_n) = m\left(\bigcup_{n=1}^N E_n\right) \leq m(\cup_n E_n)$.

Let $N \to \infty$, $\sum_n m(E_n) \leq m(\cup_n E_n)$. Thus $m\left(\bigcup_n E_n\right) = \sum_n m(E_n)$. □

Theorem: 2.17: Translation Invariance

If $E \in \mathcal{M}$ and $x \in \mathbb{R}$, then $E + x = \{y + x : y \in E\}$ is measurable and $m(E) = m(E + x)$.

Theorem: 2.18: Continuity of Lebesgue Measure

Suppose $\{E_k\}_k$ is a collection of measurable sets s.t. $E_1 \subset E_2 \subset \cdots$. Then $m\left(\bigcup_{k=1}^\infty E_k\right) = \lim_{n\to\infty} m\left(\bigcup_{k=1}^N m(E_n)\right)$

Proof. Let $F_1 = E_1$, $F_{k+1} = E_{k+1} \setminus E_k$ for $k \geq 1$. Then $F_{k+1} = E_{k+1} \cap E_k^C \in \mathcal{M}$. Then $\{F_k\}$ is a disjoint collection of measurable sets.

Also, $\forall n \in \mathbb{N}$, $\bigcup_{k=1}^{n} F_k = E_n$ and $\bigcup_{k=1}^{\infty} F_k = \bigcup_{k=1}^{\infty} E_k$.

Then $m\left(\bigcup_{k=1}^{\infty} E_k\right) = m\left(\bigcup_{k=1}^{\infty} F_k\right) \overset{\text{Theorem 2.16}}{=} \sum_{k=1}^{\infty} m(F_k) = \lim_{n\to\infty} \sum_{k=1}^{n} m(F_k) \overset{\text{By construction}}{=} \lim_{n\to\infty} m(E_n)$ ☐

2.2 Measurable Functions

We want to define $\int_a^b f = \lim \sum_{i=1}^{n} y_{i-1} l(f^{-1}[y_{i-1}, y_i])$. If f is a general function, $f^{-1}[y_{i-1}, y_i]$ need not be an interval.

***Definition:* 2.6: Extended Real Numbers**

We define the extended real numbers $[-\infty, \infty] = \mathbb{R} \cup \{\pm\infty\}$ s.t. $x \pm \infty = \pm\infty$, $\forall x \in \mathbb{R}$ and $0(\pm\infty) = 0$, $x(\pm\infty) = \infty$, $\forall x \in \mathbb{R} \setminus \{0\}$.

***Definition:* 2.7: Measurable Functions**

Let $E \subset \mathbb{R}$ be measurable, $f : E \to [-\infty, \infty]$ is Lebesgue measurable if $\forall \alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty]) \in \mathcal{M}$ is measurable.

***Theorem:* 2.19:**

Let $E \subset \mathbb{R}$ be measurable, $f : E \to [-\infty, \infty]$. Then, the follwoing are equivalent:

1. $\forall \alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty]) \in \mathcal{M}$.
2. $\forall \alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty]) \in \mathcal{M}$.
3. $\forall \alpha \in \mathbb{R}$, $f^{-1}([-\infty, \alpha)) \in \mathcal{M}$.
4. $\forall \alpha \in \mathbb{R}$, $f^{-1}([-\infty, \alpha]) \in \mathcal{M}$.

Proof. ($1 \Rightarrow 2$) Suppose $\forall \alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty]) \in \mathcal{M}$. Then $\forall \alpha \in \mathbb{R}$, $[a, \infty] = \bigcap_n \left(\alpha - \frac{1}{n}, \infty\right]$. $f^{-1}([\alpha, \infty]) = \bigcap_n f^{-1}\left(\left(\alpha - \frac{1}{n}, \infty\right]\right)$ is measurable as countable intersection of measurable sets.

($2 \Rightarrow 1$) Suppose $\forall \alpha \in \mathbb{R}$, $f^{-1}([\alpha, \infty]) \in \mathcal{M}$. Then $\forall \alpha \in \mathbb{R}$, $(a, \infty] = \bigcup_n \left[\alpha + \frac{1}{n}, \infty\right]$, $f^{-1}((\alpha, \infty]) = \bigcup_n f^{-1}\left(\left[\alpha + \frac{1}{n}, \infty\right]\right)$ is measurable.

$2 \Leftrightarrow 3$, because $[-\infty, \alpha) = ([\alpha, \infty])^C$. $1 \Leftrightarrow 4$, because $[-\infty, \alpha] = ((\alpha, \infty])^C$. ☐

***Theorem:* 2.20:**

If $E \subset \mathbb{R}$ is measurable and $f : E \to \mathbb{R}$ is a measurable function, then $\forall F \in \mathcal{B}$ (Borel σ-alg), $f^{-1}(F)$ is measurable.

Proof. f is measurable, then $\forall a < b$, $f^{-1}((a, b)) = f^{-1}([-\infty, b) \cap (a, \infty]) = f^{-1}([-\infty, b)) \cap f^{-1}((a, \infty])$ is measurable. Thus $\forall a < b$, $f^{-1}(a, b)$ is measurable. $f^{-1}(U)$ is therefore measurable for all open $U \subset \mathbb{R}$ as countable union of open intervals. ☐

Theorem: 2.21:

If $f : E \to \mathbb{R}$ is measurable, then $f^{-1}(\{\infty\})$ and $f^{-1}(\{-\infty\})$ are measurable.

Proof. $f^{-1}(\{\infty\}) = \bigcap_n f^{-1}((n, \infty])$ is measurable. Similarly, $f^{-1}(\{-\infty\}) = \bigcap_n f^{-1}([-\infty, -n))$ is measurable. □

Theorem: 2.22:

If $f : \mathbb{R} \to \mathbb{R}$ is continuous, then f is measurable.

Proof. $\forall \alpha \in \mathbb{R}$, $f^{-1}((\alpha, \infty]) = f^{-1}((\alpha, \infty))$ is an open set as pre-image of an open set. Thus measurable. □

Theorem: 2.23:

Let $E \subset \mathbb{R}$, $F \subset \mathbb{R}$ be measurable. Define $\chi_F(x) = \begin{cases} 1, x \in F \\ 0, x \notin F \end{cases}$. Then $\chi_F : E \to \mathbb{R}$ is measurable.

Proof. Let $\alpha \in \mathbb{R}$, $\chi_F^{-1}((\alpha, \infty]) = \begin{cases} \emptyset, \alpha \geq 1 \\ E \cap F, 0 \leq \alpha < 1 \\ E, \alpha < 0 \end{cases}$ is measurable. □

Theorem: 2.24: Algebraic Operations Measurability

Suppose $E \subset \mathbb{R}$ is measurable, $f, g : E \to \mathbb{R}$ are measurable and $c \in \mathbb{R}$. Then $cf, f + g, fg : E \to \mathbb{R}$ are measurable.

Proof.

1. If $c = 0$, then $cf = 0$ is continuous, thus measurable. If $c > 0$, let $\alpha \in \mathbb{R}$, $cf(x) > \alpha \Leftrightarrow f(x) > \frac{\alpha}{c}$. $(cf)^{-1}((\alpha, \infty]) = f^{-1}((\alpha/c, \infty])$ is measurable, same for $c < 0$
2. Let $\alpha \in \mathbb{R}$. $f(x) + g(x) > \alpha \Leftrightarrow f(x) > \alpha - g(x) \Leftrightarrow \exists r \in \mathbb{Q}$ s..t $f(x) > r > \alpha - g(x)$ ($\mathbb{Q}$ is dense in $\mathbb{R}$). *i.e.* $f(x) > r$ and $g(x) > \alpha - r$. Thus $x \in f^{-1}((r, \infty]) \cap g^{-1}((\alpha - r, \infty])$. Then $(f + g)^{-1}((\alpha, \infty]) = \bigcup_{r \in \mathbb{Q}} \left(f^{-1}((r, \infty]) \cap g^{-1}((\alpha - r, \infty])\right)$ is measurable
3. We show that f^2 is measurable. Let $\alpha \in \mathbb{R}$. If $\alpha < 0$, then $(f^2)^{-1}((\alpha, \infty]) = E$ is measurable. If $\alpha \geq 0$, then $f^2(x) > \alpha \Leftrightarrow f(x) > \sqrt{\alpha}$ or $f(x) < -\sqrt{\alpha}$. $(f^2)^{-1}((\alpha, \infty]) = f^{-1}((\sqrt{\alpha}, \infty]) \cup f^{-1}([-\infty, -\sqrt{\alpha}))$ is measurable.
   Then $fg = \frac{1}{4}((f + g)^2 - (f - g)^2)$ is measurable.

□

Theorem: 2.25:

If $E \subset \mathbb{R}$ is measurable, $f_n : E \to [-\infty, \infty]$ is measurable for all n, then the following functions are measurable

1. $g_1(x) = \sup_n f_n(x)$
2. $g_2(x) = \inf_n f_n(x)$
3. $g_3(x) = \limsup_n f_n(x) = \lim_{n\to\infty} \sup_{k \geq n} f_n(x) = \inf_{n\to\infty} \sup_{k\geq n} f_n(x)$
4. $g_4(x) = \liminf_n f_n(x) = \lim_{n\to\infty} \inf_{k\geq n} f_n = \sup_{n\to\infty} \inf_{k\geq n} f_n$

Proof.
1. $x \in g_1^{-1}((\alpha, \infty]) \Leftrightarrow \sup_n f_n(x) > \alpha \Leftrightarrow$ there exists n s.t. $f_n(x) > \alpha$, *i.e.* $x \in f_n^{-1}((\alpha, \infty])$, $g_1^{-1}((\alpha, \infty]) = \bigcup_n f_n^{-1}((\alpha, \infty])$ is measurable
2. $g_2^{-1}([\alpha, \infty]) = \bigcap_n f_n^{-1}([\alpha, \infty])$ is measurable

g_3 is infimum of sequence of functions defined as supremum of f_n, thus measurable. Same for g_4. □

Theorem: 2.26:

If $E \subset \mathbb{R}$ is measurable, $f_n : E \to [-\infty, \infty]$ is measurable for all n, and $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in E$, then f is measurable.

Proof. If $\lim_{n\to\infty} f_n(x) = f(x)\ \forall x \in E$, then $f(x) = \limsup_{n\to\infty} f_n(x) = \liminf_{n\to\infty} f_n(x)$. By Theorem 2.25, both are measurable. □

Remark 5. If $f_n : [a, b] \to \mathbb{R}$ is Riemann integrable for all n, and $f_n \to f$, then f need not be Riemann integrable.

Example: Let $\mathbb{Q} \cap [0, 1] = \{r_1, r_2, ...\}$. $f_n(x) = \begin{cases} 1, x \in \{r_1, ..., r_n\} \\ 0,\ \text{else} \end{cases}$. $f_n(x)$ is Riemann integrable for all finite n. $\forall x \in [0, 1]$, $f_n(x) \to \chi_{\mathbb{Q}}(x)$, which is not Riemann integrable.

Definition: 2.8: Almost Everywhere

A statement $P(x)$ holds almost everywhere on E (a.e. on E) if $m(\{x \in E : P(x) \text{ does not hold}\}) = 0$

Theorem: 2.27:

If $f, g : E \to [-\infty, \infty]$, f is measurable and $f = g$ a.e. on E, then g is measurable.

Proof. Let $N = \{x \in E : f(x) \neq g(x)\}$. Then $N \in \mathcal{M}$ and $m(N) = 0$ by Definition 2.8.
Let $\alpha \in \mathbb{R}$, $N_\alpha = \{x \in N : g(x) > \alpha\} \subset N$, so $m^*(N_\alpha) = 0$, and $N_\alpha \in \mathcal{M}$.
Then $g^{-1}((\alpha, \infty]) = (f^{-1}((\alpha, \infty]) \cap N^C) \cup N_\alpha \in \mathcal{M}$. □

Definition: 2.9: Complex Measurable Functions

Let $E \subset \mathbb{R}$ be measurable, $f : E \to \mathbb{C}$ is measurable if $\text{Re}(f) : E \to \mathbb{R}$ and $\text{Im}(f) : E \to \mathbb{R}$ are measurable.

Theorem: 2.28: Properties of Complex Measurable Functions

If $f, g : E \to \mathbb{C}$ are measurable and $\alpha \in \mathbb{C}$, then αf, $f+g$, fg, $\bar{f}$, $|f|$ are measurable.

Theorem: 2.29:

If $f_n : E \to \mathbb{C}$ is measurable $\forall n$ and $\forall x \in E$, $\lim\limits_{n\to\infty} f_n(x) = f(x)$, then f is measurable.

2.2.1 Simple Functions

Definition: 2.10: Simple Functions

If $E \subset \mathbb{R}$ is measurable, a measurable function $\varphi : E \to \mathbb{C}$ is a simple function if $\varphi(E) = \{a_1, ..., a_n\}$ (range is finite).

Remark 6. If $\varphi : E \to \mathbb{C}$ is a simple function, $\varphi(E) = \{a_1, ..., a_n\}$, then $\forall i$, $A_i = \varphi^{-1}(\{a_i\})$ is measurable and $\forall i \neq j$, $A_i \cap A_j = \emptyset$, $\displaystyle\bigcup_{i=1}^{n} A_i = E$, $\forall x \in E$, $\varphi(x) = \displaystyle\sum_{i=1}^{n} a_i \chi_{A_i}(x)$.

Theorem: 2.30: Properties of Simple Functions

Scalar multiplications, linear combinations and products of simple functions are simple functions.

Theorem: 2.31:

If $f : E \to [0, \infty]$ is measurable, then $\exists$ sequence of simple functions $\{\varphi_n\}$ s.t.

1. $\forall x \in E$, $0 \leq \varphi_0(x) \leq \varphi_1(x) \leq \cdots \leq f(x)$
2. $\forall x \in E$, $\lim\limits_{n\to\infty} \varphi_n(x) = f(x)$
3. $\forall B \geq 0$, $\varphi_n \to f$ uniformly on $\{x \in E : f(x) \leq B\}$

Proof. For $n = 0, 1, 2, ...,$ $- \leq k \leq 2^{2n} - 1$, define $E_n^k = \{x \in E : k2^{-n} < f(x) \leq (k+1)2^{-n}\} = f^{-1}((k2^{-n}, (k+1)2^{-n}])$, and $F_n = f^{-1}((2^n, \infty])$, $\varphi_n = \displaystyle\sum_{k=0}^{2^{2n}-1} k2^{-n}\chi_{E_n^k} + 2^n \chi_{F_n}$.

e.g. $\varphi_1 = 0\chi_{f^{-1}((0,\frac{1}{2}])} + \frac{1}{2}\chi_{f^{-1}((\frac{1}{2},1])} + \chi_{f^{-1}((1,\frac{3}{2}])} + \frac{3}{2}\chi_{f^{-1}((\frac{3}{2},2])} + 2\chi_{f^{-1}((2,\infty])}$

By definition, $0 \leq \varphi_n(x) \leq f(x)$. If $x \in E_n^k$, then $k2^{-n} < f(x) \leq (k+1)2^{-n}$, $\varphi_n(x) = k2^{-n} < f(x)$. If $x \in F_n$, then $f(x) > 2^n = \varphi_n(x)$.

For 1, suppose $x \in E_n^k$. Then $k2^{-n} < f(x) \leq (k+1)2^n$, $(2k)2^{-n-1} < f(x) \leq (2k+2)2^{-n-1}$, so $x \in E_{n+1}^{2k} \cup E_{n+1}^{2k+1}$.

If $x \in E_{n+1}^{2k}$, then $\varphi_n(x) = k2^{-n} = (2k)2^{-n-1} = \varphi_{n+1}(x)$

If $x \in E_{n+1}^{2k+1}$, then $\varphi_n(x) = k2^{-n} = (2k)2^{-n-1} < (2k+1)2^{-n-1} = \varphi_{n+1}(x)$

Similarly, if $x \in F_n$, $\varphi_n(x) \leq \varphi_{n+1}(x)$.

Since $E = \left[\displaystyle\bigcup_{k=0}^{2^m-1} E_n^k\right] \cup F_n$, then $\forall x \in E$, $\varphi_n(x) \leq \varphi_{n+1}(x)$. Thus 1 is proved.

For 2 and 3, $\{y \in E : f(y) \leq 2^n\} = \displaystyle\bigcup_{k=0}^{2^{2n}-1} E_n^k$. Suppose $x \in E_n^k$, then $k2^{-n} < f(x) \leq (k+1)2^{-n}$.

Thus $0 \leq f(x) - \varphi_n(x) = f(x) - k2^{-n} \leq (k+1)2^{-n} - k2^{-n} = 2^{-n}$. Then 2 and 3 follow. ∎

Definition: 2.11:

If $f : E \to [-\infty, \infty]$, we define $f^+(x) = \max(f(x), 0)$ and $f^- = \max(-f(x), 0)$. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$.

Theorem: 2.32:

Let $E \subset \mathbb{R}$ be measurable, $f : E \to \mathbb{C}$ be measurable. Then there exists a sequence of functions $\{\phi_n\}$ s.t.

1. $\forall x \in E$, $0 \leq |\varphi_0(x)| \leq |\varphi_1(x)| \leq \cdots \leq |f(x)|$
2. $\forall x \in E$, $\lim_{n\to\infty} \varphi_n(x) = f(x)$
3. $\forall B \geq 0$, $\varphi_n \to f$ uniformly on $\{x \in E : f(x) \leq B\}$.

2.3 Lebesgue Integrals

2.3.1 Lebesgue Integral of a Non-negative Function

Definition: 2.12:

If $E \subset \mathbb{R}$ is measurable, define $L^+(E) = \{f : E \to [0, \infty] : f \text{ is measurable}\}$.

Definition: 2.13: Lebesgue Integral of Simple Functions

Let $\varphi \in L^+(E)$ be a simple function, $\varphi = \sum_{j=1}^{n} a_j \chi_{A_j}$, where $\forall j, A_j \subset E$, $\forall i \neq j$, $A_i \cap A_j = \emptyset$ and $\bigcup_{j=1}^{n} A_j = E$. The Lebesgue integral of φ is $\int_E \varphi = \sum_{j=1}^{m} a_j m(A_j) \in [0, \infty]$.

Theorem: 2.33: Properties of Lebesgue Integrals (Simple Functions)

Let $\varphi, \psi \in L^+(E)$ be simple functions. Then

1. If $c \geq 0$, then $\int_E c\varphi = c \int_E \varphi$
2. $\int_E (\varphi + \psi) = \int_E \varphi + \int_E \psi$
3. If $\varphi \leq \psi$, then $\int_E \varphi \leq \int_E \psi$
4. If $F \subset E$ is measurable, then $\int_F \varphi = \int_E \varphi \chi_F \leq \int_E \varphi$

Proof.

1. By Definition 2.10 and 2.13, $c\varphi = \sum_{j=1}^{n} (ca_j)\chi_{A_j}$.

   Then $\int_E c\varphi = \sum_{j=1}^{n} ca_j m(A_j) = c \sum_{j=1}^{n} a_j m(A_j) = c \int_E \varphi$

2. Write $\varphi = \sum_{j=1}^{n} a_j \chi_{A_j}$, $\psi = \sum_{k=1}^{m} b_k \chi_{B_k}$. Then $E = \bigcup_{j=1}^{n} A_j = \bigcup_{k=1}^{m} B_k$.

$\forall j,\ A_j = \bigcup_{k=1}^{m} A_j \cap B_k,\ \forall k,\ B_k = \bigcup_{j=1}^{n} B_k \cap A_j$, and these unions are disjoint. Then by Definition 2.5 (Additivity),

$$\int_E \varphi + \int_E \psi = \sum_{j=1}^{n} a_j m(A_j) + \sum_{k=1}^{m} b_k m(B_k) = \sum_{j,k} a_j m(A_j \cap B_k) + \sum_{k,j} b_k m(B_k \cap A_j) = \sum_{j,k} (a_j + b_k) m(A_j \cap B_k)$$

Since $\varphi + \psi = \sum_{j,k} (a_j + b_k)\chi_{A_j \cap B_k}$, then $\int_E (\varphi + \psi) = \sum_{j,k} (a_j + b_k) m(A_j \cap B_k) = \int_E \phi + \int_E \psi$

3. $\forall x \in E,\ \varphi(x) \leq \psi(x) \Leftrightarrow a_j \leq b_k$ whereever $A_j \cap B_k \neq \emptyset$. Thus

$$\int_E \varphi = \sum_{j=1}^{n} a_j m(A_j) = \sum_{j,k} a_j m(A_j \cap B_k) \leq \sum_{j,k} b_k m(A_j \cap B_k) = \sum_{k=1}^{m} b_k m(B_k) = \int_E \psi$$

□

Definition: 2.14: Lebesgue Integral of Non-negative Functions

If $f \in L^+(E)$, define

$$\int_E f = \sup\{\int_E \varphi : \varphi \in L^+(E) \text{ simple functions} \varphi \leq f\}$$

Theorem: 2.34:

If $E \subset \mathbb{R}$ s.t. $m(E) = 0$, then $\forall f \in L^+(E)$, $\int_E f = 0$. (Similar to Riemann integral over a single point)

Proof. Let $\varphi \in L^+(E)$ be simple. $\varphi = \sum_{j=1}^{n} a_j \chi_{A_j}$ with $\varphi \leq f$. Then $A_j \subset E, \forall j \Rightarrow m(A_j) = 0, \forall j \Rightarrow \int_E \varphi = \sum_{j=1}^{n} a_j m(A_j) = 0$. Thus, $\int_E \varphi = \sup\{0\} = 0$ □

Theorem: 2.35: Properties of Lebesgue Integrals (Non-negative Functions)

If $\varphi \in L^+(E)$ is simple, then the two definitions (2.13 and 2.14) agree.
If $f, g \in L^+(E)$, $c \in [0, \infty)$ and $f \leq g$ on E, then $\int_E cf = c\int_E f$, $\int_E f \leq \int_E g$
If $f \in L^+(E)$ and $F \subset E$ is measurable, then $\int_F f = \int_E f\chi_F \leq \int_E f$

Theorem: 2.36: Order Property of Lebesgue Integrals (Non-negative Functions)

If $f, g \in L^+(E)$ and $f \leq g$ a.e. on E, then $\int_E f \leq \int_E g$

Proof. Let $F = \{x \in E : f(x) \le g(x)\} = (g-f)^{-1}([0,\infty])$, F is measurable and $m(F^C) = 0$, since $f \le g$ a.e. Then

$$\int_E f = \int_{F\cup F^C} f = \int_F f + \int_{F^C} f = \int_F f \le \int_F g = \int_F g + \int_{F^C} f = \int_{F\cup F^C} f = \int_E g$$

□

Theorem: 2.37: Monotone Convergence Theorem

If $\{f_n\}$ is a sequence in $L^+(E)$ s.t. $f_1 \le f_2 \le \cdots$ pointwise on E and $f_n \to f$ pointwise on E. Then $\lim_{n\to\infty} \int_E f_n = \int_E f$. (Note: we don't require uniform convergence as in Riemann integration.)

Proof. $f_1 \le f_2 \le \cdots \overset{\text{By Theorem 2.36}}{\Rightarrow} \int_E f_1 \le \int_E f_2 \le \cdots$, so the integrals form a monotone sequence, $\lim_{n\to\infty} \int_E f_n$ exists in $[0,\infty]$.

Since $f_1 \le f_2 \le \cdots$ and $\lim_{n\to\infty} \int_E f_n = f(x)$ $\forall x$, then $f_1 \le f_2 \le \cdots \le f$. Thus $\forall n$, $\int_E f_n \le \int_E f_n$, $\lim_{n\to\infty} \int_E f_n \le \int_E f$

Now we show that $\int_E f \le \lim_{n\to\infty} \int_E f_n$.

Let $\varphi \in L^+(E)$ be simple, $\varphi = \sum_{j=1}^m a_j \chi_{A_j}$ with $\varphi \le f$. Let $\epsilon \in (0,1)$ and $E_n = \{x \in E : f_n(x) \ge (1-\epsilon)\varphi(x)\}$

Note $\forall x \in E$, $(1-\epsilon)\varphi(x) < f(x)$. Since $\forall x \in E$, $\lim_{n\to\infty} f_n(x) = f(x)$, $\bigcup_{n=1}^\infty E_n = E$. Since $f_1 \le f_2 \le \cdots$, then $E_1 \subset E_2 \subset \cdots$. Then we have

$$\int_E f_n \ge \int_{E_n} f_n \ge \int_{E_n} (1-\epsilon)\varphi(x) = (1-\epsilon)\int_E \varphi(x) = (1-\epsilon)\sum_{j=1}^m a_j m(A_j \cap E_n)$$

Taking the limit, we get $\lim_{n\to\infty} \int_E f_n \ge \lim_{n\to\infty} (1-\epsilon)\sum_{j=1}^m a_j m(A_j \cap E_n)$.

Since $E_1 \cap A_j \subset E_2 \cap A_j \subset \cdots$ and $\bigcup_{n=1}^\infty (E_n \cap A_j) = A_j$, by Theorem2.18, we get $\lim_{n\to\infty} m(A_j \cap E_n) = m\left(\bigcup_{n=1}^\infty E_n \cap A_j\right) = m(A_j)$.

Therefore, $\lim_{n\to\infty} \int_E f_n \ge \lim_{n\to\infty} (1-\epsilon)\sum_{j=1}^m a_j m(A_j \cap E_n) = (1-\epsilon)\sum_{j=1}^n a_j m(A_j) = (1-\epsilon)\int_E \varphi$.

Let $\epsilon \to 0$, $\int_E \varphi \le \lim_{n\to\infty} \int_E f_n$. Thus $\int_E f \le \lim_{n\to\infty} \int_E f_n$ □

Theorem: 2.38:

If $f \in L^+(E)$ and $\{\varphi_n\}$ is a sequence of simple functions s.t. $0 \le \varphi_1 \le \varphi_2 \le \cdots \le f$ and $\varphi_n \to f$ pointwise, then $\int_E f = \lim_{n\to\infty} \int_E \varphi_n$.

Theorem: 2.39: Additivity of Lebesgue Integral (Non-negative Functions)

If $f, g \in L^+(E)$, then $\displaystyle\int_E (f+g) = \int_E f + \int_E g$

Proof. Let $\{\varphi_n\}$ and $\{\psi_n\}$ be sequences of simple functions s.t. $0 \le \varphi_1 \le \varphi_2 \le \cdots \le f$, and $\varphi_n \to f$ pointwise, $0 \le \psi_1 \le \psi_2 \le \cdots \le g$, and $\psi_n \to g$ pointwise.
Then $0 \le \varphi_1 + \psi_1 \le \varphi_2 + \psi_2 \le \cdots \le f + g$ and $\varphi_n + \psi_n \to f + g$ pointwise. Then $\displaystyle\int_E (f+g) = \lim_{n\to\infty} \int_E \varphi_n + \psi_n = \lim_{n\to\infty} \left(\int_E \varphi_n + \int_E \psi_n \right) = \int_E f + \int_E g$ □

Theorem: 2.40:

If $\{f_n\}$ is a sequence in $L^+(E)$, then $\displaystyle\int_E \sum_n f_n = \sum_n \int_E f_n$.

Proof. By induction using Theorem 2.39, we have $\displaystyle\int_E \sum_{n=1}^N f_n = \sum_{n=1}^N \int_E f_n$.

Since $\displaystyle\sum_{n=1}^1 f_n \le \sum_{n=1}^2 f_n \le \cdots$ and $\displaystyle\sum_{n=1}^N f_n \to \sum_{n=1}^\infty f_n$ pointwise, then by Theorem 2.37,

$\displaystyle\int_E \sum_{n=1}^\infty f_n = \lim_{N\to\infty} \int_E \sum_{n=1}^N f_n = \lim_{N\to\infty} \sum_{n=1}^N \int_E f_n = \sum_{n=1}^\infty \int_E f_n$. □

Theorem: 2.41:

If $f \in L^+(E)$, then $\displaystyle\int_E f = 0 \Leftrightarrow f = 0$ a.e. on E.

Proof. ($\Leftarrow$) Since $f \le 0$ a.e., then $\displaystyle 0 \le \int_E f \le \int_E 0 = 0$

($\Rightarrow$) Let $F_n = \{x \in E : f(x) > \frac{1}{n}\}$, $F = \{x \in E : f(x) > 0\}$. Then $\displaystyle\bigcup_{n=1}^\infty F_n = F$, $F_1 \subset F_2 \subset \cdots$.

Then $\forall n$, $\displaystyle 0 \le \frac{1}{n} m(F_n) = \int_{F_n} \frac{1}{n} \le \int_{F_n} f \le \int_E f = 0$

Thus $\forall n, m(F_n) = 0$, $\displaystyle m(F) = m\left(\bigcup_{n=1}^\infty F_n\right) \overset{\text{Theorem 2.18}}{=} \lim_{n\to\infty} m(F_n) = 0$.

Thus $f = 0$ a.e. on E. □

Theorem: 2.42:

If $\{f_n\}$ is a sequence in $L^+(E)$ s.t. $f_1(x) \le f_2(x) \le \cdots$ for almost all $x \in E$ and $\displaystyle\lim_{n\to\infty} f_n(x) = f(x)$, then $\displaystyle\int_E f = \lim_{n\to\infty} \int_E f_n$.

Proof. Let $F = \{x \in E : \text{both conditions hold}\}$. Then $m(E \setminus F) = 0$, $f - \chi_F f = 0$ a.e. and $f_n - \chi_F f_n = 0$ a.e. for all n.

By Theorem 2.37 and 2.41, $\displaystyle\int_E f = \int_E f\chi_F = \int_F f = \lim_{n\to\infty} \int_F f_n = \lim_{n\to\infty} \int_E f_n$. ☐

Remark 7. Sets of measure zero don't affect Lebesgue integrals.

Lemma: 2.2: Fatou's Lemma

If $\{f_n\}$ is a sequence in $L^+(E)$, then $\displaystyle\int_E \liminf_{n\to\infty} f_n(x) \le \liminf_{n\to\infty} \int_E f_n(x)$.

Proof. Since $\displaystyle\liminf_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\inf_{k\ge n} f_k(x) \right)$ and $\inf_{k\ge1} f_k(x) \le \inf_{k\ge2} f_k(x) \le \cdots$, by Theorem 2.37, $\displaystyle\int_E \liminf_{n\to\infty} f_n = \lim_{n\to\infty} \int_E \inf_{k\ge n} f_k$.

$\forall j \ge n, x \in E$, $\displaystyle\inf_{k\ge n} f_k(x) \le f_j(x)$ by defintion, thus $\displaystyle\int_E \inf_{k\ge n} f_k(n) \le \int_E f_j(x)$ by Theorem 2.36.

Therefore, $\displaystyle\int_E \inf_{k\ge n} f_k(n) \le \inf j \ge n \int_E f_j(x)$

$\displaystyle\Rightarrow \int_E \liminf_{n\to\infty} f_n = \lim_{n\to\infty} \int_E \inf_{k\ge n} f_k \le \lim_{n\to\infty} \inf_{k\ge n} \int_E f_k = \liminf_{n\to\infty} \int_E f_n(x)$ ☐

Theorem: 2.43:

If $f \in L^+(E)$ and $\displaystyle\int_E f < \infty$, then $\{x \in E : f(x) = \infty\}$ is a set of measure zero.

Proof. Let $F = \{x \in E : f(x) = \infty\}$. Then $\forall n$, $n\chi_F \le f\chi_F$ (Definition of unbounded functions). By Theorem 2.36, $\forall n$, $\displaystyle nm(F) \le \int_E f\chi_F \le \int_E f < \infty$.

Then $\forall n$, $\displaystyle m(F) \le \frac{1}{n} \int_E f \to 0$ since $\displaystyle\int_E f < \infty$. Therefore, $m(F) = 0$. ☐

2.3.2 Lebesgue Integrable Functions

Definition: 2.15: Lebesgue Integrable Functions

Let $E \subset \mathbb{R}$ be measurable, a measurable function $f : E \to \mathbb{R}$ is Lebesgue integrable over E if $\displaystyle\int_E |f| < \infty$.

Note: $\displaystyle\int_E |f| = \int_E f^+ + \int_E f^-$. Thus f is integrable $\Leftrightarrow$ f^+ and f^- are both integrable.

Definition: 2.16: Lebesgue Integral

f $f : E \to \mathbb{R}$ is Lebesgue integrable, then the Lebesgue integral of f is $\displaystyle\int_E f = \int_E f^+ - \int_E f^-$.

Theorem: 2.44: Properties of Lebesgue Integrals

Suppose $f, g : E \to \mathbb{R}$ are integrable, then

1. $\forall c \in \mathbb{R}$, cf is integrable and $\displaystyle\int_E cf = c\int_E f$
2. $f + g$ is integrable and $\displaystyle\int_E (f+g) = \int_E f + \int_E g$
3. If A, B are disjoint measurable sets, then $\displaystyle\int_{A\cup B} f = \int_A f + \int_B f$

Proof.

1. scaling by $c \neq 0$ either swaps f^+ with f^- or doesn't change anything and follows from Theorem 2.35.
2. $|f+g| \leq |f| + |g|$, thus by Theorem 2.35, $\displaystyle\int_E |f+g| \leq \int_E |f| + \int_E |g| < \infty$, thus $f+g$ is integrable. $f+g = (f+g)^+ - (f+g)^- = (f^+ + g^+) - (f^- + g^-)$. Then $\displaystyle\int_E (f+g) = \int_E (f+g)^+ - \int_E (f+g)^- = \int_E (f^+ + g^+) - \int_E (f^- + g^-) = \int_E f^+ + \int_E g^+ - \int_E f^- - \int_E g^- = \int_E f + \int_E g$
3. $f\chi_{A\cup B} = f\chi_A + f\chi_B$ and follows 2

□

Theorem: 2.45: Order Properties of Lebesgue Integrals

Suppose $f, g : E \to \mathbb{R}$ are measurable, then:

1. If f is integrable, then $\displaystyle\left|\int_E f\right| \leq \int_E |f|$
2. If g is integrable and $f = g$ a.e., then f is integrable and $\displaystyle\int_E f = \int_E g$
3. If f and g are integrable and $f \leq g$ a.e., then $\displaystyle\int_E f \leq \int_E g$

Proof.

1. $\displaystyle\left|\int_E f\right| = \left|\int_E f^+ - \int_E f^-\right| \overset{\text{Triangle Inequality and Non-negativity}}{\leq} \int_E f^+ + \int_E f^- = \int_E f^+ + f^- = \int_E |f|$
2. If $f = g$ a.e., then $|f| = |g|$ a.e., $\displaystyle\int_E |f| = \int_E |g| < \infty$, thus f is integrable.

   Moreover, $|f-g| = 0$ a.e. $\displaystyle\left|\int_E f - \int_E g\right| = \left|\int_E (f-g)\right| \leq \int_E |f-g| = 0$
3. Define $h(x) = \begin{cases} g(x) - f(x), g(x) \geq f(x) \\ 0, \text{else} \end{cases}$. Then $h \in L^+(E)$, $h = g - f$ a.e. $\displaystyle\int_E |h| < \infty$. Thus

$$0 \leq \int_E h^+ = \int_E h = \int_E g - f = \int_E g - \int_E f$$

   Therefore, $\displaystyle\int_E f \leq \int_E g$.

□

Theorem: 2.46: Dominated Convergence Theorem

Let $g : E \to [0, \infty)$ be integrable, $\{f_n\}_n$ be a sequence of real-valued measurable functions s.t. $\forall n$, $|f_n| \le g$. Then $\exists f : E \to \mathbb{R}$ s.t. $f_n \to f$ pointwise a.e. Then $\lim\limits_{n\to\infty} \int_E f_n = \int_E f$.

Proof. Since $\forall n$, $|f_n| \le g$ a.e., then f_n is integrable. Moreover, $f_n \to f$ a.e., so f is measurable and $|f| \le g$ a.e. Thus, f is integrable, by Theorem 2.29.
Since changing f and f_n for all n on a set of measure zero does not affect the integrals, we can assume that $\forall n$, $|f_n| \le g$, and $\exists f : E \to \mathbb{R}$ s.t. $f_n \to f$
Note $\forall n$, $\left|\int_E f_n\right| \overset{\text{By Theorem 2.45}}{\le} \int_E |f_n| \le \int_E g$, therefore, $\left\{\int_E f_n\right\}_n$ is a bounded sequence in $\mathbb{R}$.
Since $g + f_n \ge 0$, by Lemma 2.2, $\int_E g - f = \int_E \liminf\limits_{n\to\infty}(g - f_n) \le \liminf\limits_{n\to\infty} \int_E g - f_n = \int_E g - \limsup\limits_{n\to\infty} \int_E f_n$
Similarly, $\int_E g + f \le \int_E g + \liminf\limits_{n\to\infty} \int_E f_n$. Then,

$$\limsup_{n\to\infty} \int_E f_n \le \int_E g - \int_E (g - f) = \int_E f = \int_E g + f - \int_E g \le \liminf_{n\to\infty} \int_E f_n$$

But $\limsup\limits_{n\to\infty} \int_E f_n \ge \liminf\limits_{n\to\infty} \int_E f_n$ by definition. Thus $\liminf\limits_{n\to\infty} \int_E f_n = \limsup\limits_{n\to\infty} \int_E f_n = \lim\limits_{n\to\infty} \int_E f_n = \int_E f$. $\square$

Theorem: 2.47: Agreement of Riemann and Lebesgue Integrals

Suppose $a < b$, $f \in C([a, b])$. Then $\int_{[a,b]} f = \int_a^b f(x)dx$. Lebesgue and Riemann integrals agree on $C([a, b])$.

Proof. If $f \in C([a, b])$, then $|f| \in C([a, b])$, *i.e.* $|f|$ is bounded, $\exists B \ge 0$ s.t. $|f| \le B$ on $[a, b]$
Then $\int_{[a,b]} |f| \le \int_{[a,b]} B = Bm([a, b]) = B(b - a) < \infty$. Thus f is Lebesgue integrable.
By considering $f^+ = \frac{f+|f|}{2}$ and $f^- = \frac{|f|-f}{2}$ separately and showing $\int_{[a,b]} f^\pm = \int_a^b f^\pm(x)dx$ and using linearity, we may assume that $f \ge 0$.
Let $\underline{x}^n = \left\{x_0^n = a, x_1^n, ..., x_{m_n}^n = b\right\}$ be a sequence of partitions of $[a, b]$ s.t. $\|\underline{x}^n\| = \max\limits_{1\le j\le m_n} \left|x_j^n - x_{j-1}^n\right| \to 0$.
Let $\xi_j^n = \left[x_{j-1}^n, x_j^n\right]$ s.t. $\inf\limits_{x\in\left[x_{j-1}^n, x_j^n\right]} f(x) = f(\xi_j^n)$.
By Riemann integration theory, $\lim\limits_{n\to\infty} \sum_{j=1}^{m_n} f(\xi_j^n)\left(x_j^n - x_{j-1}^n\right) = \int_a^b f(x)dx$
Let $N = \bigcup_{n=1}^{\infty} \underline{x}^n$. Then N is countable, $m(N) = 0$.
Let $f_n = \sum_{j=1}^{m_n} f(\xi_j^n)\chi_{[x_{j-1}^n, x_j^n)} + 0\chi_{\{x_j^n\}}$, f_n is a non-negative simple function.
Note, $\forall n$, $\int_{[a,b]} f_n = \sum_{j=1}^{m_n} f(\xi_j^n)m([x_{j-1}^n, x_j^n)) = \sum_{j=1}^{m_n} f(\xi_j^n)(x_j^n - x_{j-1}^n)$
Also, $\forall x \in [a, b] \setminus N$, $0 \le f_n(x) \le f(x)$.

We show that if $x \in [a,b] \setminus N$, then $f_n \to f$ pointwise a.e.
Let $x \in [a,b] \setminus N$, $\epsilon > 0$. Since f is continuous at x, $\exists \delta > 0$ s.t. if $|x - y| < \delta$, then $|f(x) - f(y)| < \epsilon$.
Since $\|\underline{x}^n\| = \max_{1 \le j \le m_n} \left|x_j^n - x_{j-1}^n\right| \to 0$, $\exists M \in \mathbb{N}$ s.t. $\forall n \ge M$, $\max_{1 \le j \le m_n} (x_j^n - x_{j-1}^n) < \delta$.

Let $n \ge M$. Then $f_n(x) = \sum_{j=1}^{m_n} f(\xi_j^n)\chi_{[x_{j-1}^n, x_j^n)}(x) = f(\xi_k^n)$ for some unique k s.t. $x \in [x_{k-1}^n, x_k^n)$.
Then since $\xi_k^n \in [x_{k-1}^n, x_k^n]$ and $x_k^n - x_{k-1}^n < \delta$, then $|x - \xi_k^n| < \delta$ and $|f(x) - f_n(x)| = |f(x) - f(\xi_k^n)| < \epsilon$.
Thus $\lim_{n\to\infty} f_n(x) = f(x)$, $\forall x \in [a,b] \setminus N$.

By Theorem 2.46, $\int_{[a,b]} f = \lim_{n\to\infty} \int_{[a,b]} f_n = \lim_{n\to\infty} \sum_{j=1}^{m_n} f(\xi_j^n)(x_j^n - x_{j-1}^n) = \int_a^b f(x)dx$. □

Definition: 2.17: Complex Lebesgue Integrals

We can use the previous theorems to construct the corresponding statements for complex-valued integrable functions. $f : E \to \mathbb{C}$ is Lebesgue integrable if $\int_E |f| < \infty$ with $\int_E f = \int_E \mathrm{Re}(f) + i\int_E \mathrm{Im}(f)$.

Theorem: 2.48: Order Property (Complex Valued)

If $f : E \to \mathbb{C}$ is integrable, then $\left|\int_E f\right| \le \int_E |f|$.

Proof. Clear if $\int_E f = 0$. Suppose $\int_E f \ne 0$.
Let $\alpha = \frac{\overline{\int_E f}}{|\int_E f|}$. Then $|\alpha| = 1$ and

$$\begin{aligned} \left|\int_E f\right| &= \alpha \int_E f = \int_E \alpha f \overset{\alpha f \text{ is real}}{=} \mathrm{Re} \int_E \alpha f = \int_E \mathrm{Re}\,(\alpha f) \\ &\le \int_E |\mathrm{Re}(\alpha f)| \le \int_E |\alpha f| = \int_E |f| \end{aligned}$$

□

2.4 Lp space

Definition: 2.18: L^P Norm

If $f : E \to \mathbb{C}$ is measurable and $1 \le p < \infty$, then we define $\|f\|_{L^P(E)} = \left(\int_E |f|^p\right)^{1/p}$
And $\|f\|_{L^\infty(E)} = \inf\{M > 0 : m(\{x \in E : |f(x)| > M\}) = 0\} = \text{ess sup}_{x\in E}|f(x)|$ is the infinity norm or the essential supremum.

Theorem: 2.49:

If $f : E \to \mathbb{C}$ is measurable, then $|f(x)| \le \|f\|_{L^\infty(E)}$ a.e. on E. If $E = [a,b]$ and $f \in C([a,b])$, then $\|f\|_{L^\infty([a,b])} = \|f\|_\infty = \sup_{x\in[a,b]} |f(x)|$.

Remark 8. We denote $\|\cdot\|_{L^p(E)}$ by $\|\cdot\|_p$.

Theorem: 2.50: Holder Inequality

If $1 \le p \le \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$, and $f, g : E \to \mathbb{C}$ are measurable, then $\displaystyle\int_E |fg| \le \|f\|_p \|g\|_q$.

Theorem: 2.51: Minkowski Inequality

If $1 \le p \le \infty$ and $f, g : E \to \mathbb{C}$ are measurable, then $\|f + g\|_p \le \|f\|_p + \|g\|_p$.

Definition: 2.19: L^p space

For $1 \le p \le \infty$, define $L^p(E) = \left\{ f : E \to \mathbb{C} : f \text{ is measurable and } \|f\|_p < \infty \right\}$. We consider two elements $f, g \in L^p(E)$ to be the same element if $f = g$ a.e.

Remark 9. Strictly speaking, this means an element of $L^p(E)$ is an equivalence class: $[f] = \left\{ g : E \to \mathbb{C} : \|g\|_p < \infty \text{ and } g = f \text{ a.e.} \right\}$. We still refer to functions $f \in L^p(E)$ rather than $[f] \in L^p(E)$.

Theorem: 2.52:

$L^p(E)$ with pointwise addition and scalar multiplication is a vector space. Moreover, $\|\cdot\|_p$ is a norm on $L^p(E)$.

Proof. Note that by Theorem 2.45, if $f = g$ a.e., then $\displaystyle\int_E |f|^p = \int_E |g|^p$. Thus if $[f] = [g]$, then $\displaystyle\|[f]\|_p = \int_E |f|^p = \int_E |g|^p = \|[g]\|_p$. $\|\cdot\|_p$ is well-defined.

Definiteness: by Theorem 2.41, $\displaystyle\int_E |f|^p = 0 \Leftrightarrow f = 0$ a.e., $[f] = [0]$

Homogeneity and triangle inequality then follow from the definition and Theorem 2.51. □

Theorem: 2.53:

Let $E \subset \mathbb{R}$ be measurable. Then $\displaystyle f \in L^p(E) \Leftrightarrow \lim_{n\to\infty} \int_{[-n,n]\cap E} |f|^p < \infty$.

Proof. If $f \in L^p(E)$, then $\displaystyle\int_E |f|^p < \infty$. Note $\displaystyle\int_{[-n,n]\cap E} |f|^p = \int_E \chi_{[-n,n]} |f|^p$.

Since $\chi_{[-1,1]}|f|^p \le \chi_{[-2,2]}|f|^p \le \cdots$ on E and $\forall x \in E$, $\displaystyle\lim_{n\to\infty} \chi_{[-n,n]}(x)|f(x)|^p = |f(x)|^p$.

By Theorem 2.37, $\displaystyle\int_E |f|^p = \lim_{n\to\infty} \int_{[-n,n]\cap E} |f|^p$. □

Example: If $f : \mathbb{R} \to \mathbb{C}$ is measurable and $\exists C \ge 0$ and $q > 1$ s.t. for almost every $x \in \mathbb{R}$, $|f(x)| \le C(1 + |x|)^{-q}$, then $f \in L^p(\mathbb{R})$ $\forall p \ge 1$.

Proof. $\int_{[-n,n]} |f|^p \le \int_{[-n,n]} C^p(1+|x|)^{-pq} = \int_{-n}^{n} C^p(1+|x|)^{-pq}dx \le CB(p)$, where $B(p)$ is a constant depending on p. □

Theorem: 2.54: Density of L^p

Let $a < b$, $1 \le p < \infty$, $f \in L^p([a,b])$ and $\epsilon > 0$. Then $\exists g \in C([a,b])$ s.t. $g(a) = g(b) = 0$ and $\|f-g\|_p < \epsilon$. *i.e.* $C([a,b])$ is dense and a proper subset in $L^p([a,b])$.

Theorem: 2.55: Riesz-Fischer

For all $1 \le p \le \infty$, $L^p(E)$ is a Banach space.

Proof. For $1 \le p < \infty$, we show that every absolutely summable series is summable.

Let $\{f_k\}$ be a sequence in $L^p(E)$ s.t. $\sum_k \|f_k\|_p < \infty$. We want to show that $\exists f \in L^p(E)$ s.t. $\sum_{k=1}^n f_k \to f$, *i.e.* $\lim_{n\to\infty} \left\| \sum_{k=1}^n f_k - f \right\|_p = 0$

Define $g_n : E \to [0,\infty)$ by $g_n = \sum_{k=1}^n |f_k(x)|$. g_n is measurable. Then $\|g_n\|_p = \left\| \sum_{k=1}^n |f_k| \right\|_p \overset{\text{Triangle Inequality}}{\le} \sum_{k=1}^n \|f_k\|_p \le M < \infty$.

By Lemma 2.2, $\int_E \left(\sum_k |f_k| \right)^p = \int_E \liminf_{n\to\infty} |g_n|^p \le \liminf_{n\to\infty} \int_E |g_n|^p \le M^p$

Thus $\sum_k \|f_k(x)\| < \infty$ for almost every $x \in E$.

Define $f(x) = \begin{cases} \sum_k f_k(x), \sum_k |f_k(x)| < \infty \\ 0, \text{ else} \end{cases}$, $g(x) = \begin{cases} \sum_k |f_k(x)|, \sum_k |f_k(x)| < \infty \\ 0, \text{ else} \end{cases}$.

Then $\lim_{n\to\infty} \sum_{k=1}^n f_k(x) - f(x)$ a.e. on E and $\left| \sum_{k=1}^n f_k(x) - f(x) \right|^p \le \|g(x)\|^p$ a.e. on E.

Since $\left\| \sum_k |f_k| \right\|_p \le M$, then $\|g\|_p \le M$, $\int_E |g|^p < \infty$. Moreover, $\|f\|_p \le \|g\|_p \le M$. *i.e.* $f \in L^p(E)$.

Apply Theorem 2.46, $\lim_{n\to\infty} \int_E \left| \sum_{k=1}^n f_k - f \right|^p = 0$, *i.e.* $\left\| \sum_{k=1}^n f_k - f \right\|_p^p \to 0$. □

3 Hilbert Spaces

Definition: 3.1: Pre-Hilbert Space

A pre-Hilbert space H is a vector space over $\mathbb{C}$ with a Hermitian inner product $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{C}$ with the following properties

1. $\forall \lambda_1, \lambda_2 \in \mathbb{C}$, $v_1, v_2, w \in H$, $\langle \lambda_1 v_1 + \lambda_2 v_2, w \rangle = \lambda_1 \langle v_1, w \rangle + \lambda_2 \langle v_2, w \rangle$
2. $\forall v, w \in H$, $\langle v, w \rangle = \overline{\langle w, v \rangle}$
3. $\forall v \in H$, $\langle v, v \rangle \geq 0$ and $\langle v, v \rangle = 0 \Leftrightarrow v = 0$.

Also,

1. If $v \in H$ and $\langle v, w \rangle = 0$ for all $w \in H$, then $v = 0$
2. $\langle v, \lambda w \rangle = \overline{\langle \lambda w, v \rangle} = \overline{\lambda \langle w, v \rangle} = \bar{\lambda} \langle v, w \rangle$

Definition: 3.2: Norm on Pre-Hilbert Space

If H is a pre-Hilbert space, we define $\|v\| = \langle v, v \rangle^{1/2}$

Theorem: 3.1: Cauchy-Schwarz Inequality

$\forall u, v \in H$, $|\langle u, v \rangle| \leq \|u\| \|v\|$.

Proof. Let $f(t) = \|u + tv\|^2 = \langle u + tv, u + tv \rangle \geq 0$.
Then $f(t) = \langle u, u \rangle + t^2 \langle v, v \rangle + t \langle u, v \rangle + t \langle v, u \rangle = \|u\|^2 + t^2 \|v\|^2 + 2t\text{Re} \langle u, v \rangle$
The minimum of f is non-negative and $f'(t_{\min}) = 0$, so $t_{\min} = -\frac{\text{Re}\langle u,v \rangle}{\|v\|^2}$.
Then $0 \leq f(t_{\min}) = \|u\|^2 - \frac{(\text{Re}\langle u,v \rangle)^2}{\|v\|^2}$, $|\text{Re} \langle u, v \rangle| \leq \|u\| \|v\|$.
If $\langle u, v \rangle = 0$, then done. Otherwise, let $\lambda = \frac{\overline{\langle u,v \rangle}}{|\langle u,v \rangle|}$. Then $|\lambda| = 1$ and $|\langle u, v \rangle| = \lambda \langle u, v \rangle = \langle \lambda u, v \rangle = \text{Re} \langle \lambda u, v \rangle \leq \|\lambda u\| \|v\|$.
Since $|\lambda| = 1$ and $\langle \lambda u, \lambda u \rangle = \lambda \bar{\lambda} \langle u, u \rangle = \langle u, u \rangle$, we get $\|\lambda u\| \|v\| = \|u\| \|v\|$. □

Theorem: 3.2:

If H is a pre-Hilbert space, then $\|\cdot\|$ is a norm on H.

Proof. Definiteness: $\|v\| = 0 \Leftrightarrow \langle v, v \rangle = 0 \Leftrightarrow v = 0$
Homogeneity: If $\lambda \in \mathbb{C}$, $v \in H$, $\langle \lambda v, \lambda v \rangle = \lambda \bar{\lambda} \langle v, v \rangle = |\lambda|^2 \langle v, v \rangle$. Thus $\|\lambda v\| = |\lambda| \|v\|$.
Triangle inequality: Let $u, v \in H$. Then

$$\begin{aligned} \|u + v\|^2 &= \langle u + v, u + v \rangle = \|u\|^2 + \|v\|^2 + 2\text{Re} \langle u, v \rangle \\ &\leq \|u\|^2 + \|v\|^2 + 2 |\text{Re} \langle u, v \rangle| \\ &\leq \|u\|^2 + \|v\|^2 + 2 |\langle u, v \rangle| \text{ (Norm of Complex Numbers)} \\ &\leq \|u\|^2 + \|v\|^2 + 2 \|u\| \|v\| \text{ (By Theorem 3.1)} \\ &= (\|u\| + \|v\|)^2 \end{aligned}$$

□

Theorem: 3.3: Continuity of Hermitian Inner Product

If $u_n \to u$ and $v_n \to v$ in a pre-Hilbert space with norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$, then $\langle u_n, v_n \rangle \to \langle u, v \rangle$

Proof. If $u_n \to u$ and $v_n \to v$, *i.e.* $\|u_n - u\| \to 0$ and $\|v_n - v\| \to 0$ as $n \to \infty$, then

$$\begin{aligned} |\langle u_n, v_n \rangle - \langle u, v \rangle| &= |\langle u_n - u, v_n \rangle + \langle u, v_n - v \rangle| \\ &\leq |\langle u_n - u, v_n \rangle| + |\langle u, v_n - v \rangle| \ \text{(Triangle inequality)} \\ &\leq \|u_n - u\| \|v_n\| + \|u\| \|v_n - v\| \ \text{(By Theorem 3.1)} \\ &\leq \|u_n - u\| \sup_k \|v_k\| + \|u\| \|v_n - v\| \to 0 \text{ as } n \to \infty \end{aligned}$$

By squeeze theorem, $\langle u_n, v_n \rangle \to \langle u, v \rangle$ □

3.1 Basic Theory

Definition: 3.3: Hilbert Space

A Hilbert space H is a pre-Hilbert space which is complete w.r.t. the norm $\|\cdot\| = \langle \cdot, \cdot \rangle^{1/2}$.

Example: $\mathbb{C}^n = \{z = (z_1, ..., z_n) : z_j \in \mathbb{C}\}$ where $\langle z, w \rangle = \sum_{j=1}^{n} z_j \overline{w_j}$ is a Hilbert space.

Example: $l^2 = \left\{ a = \{a_k\}_k : a_k \in \mathbb{C} \text{ and } \sum_{k=1}^{\infty} |a_k|^2 < \infty \right\}$ where $\langle a, b \rangle = \sum_{k=1}^{\infty} a_k \overline{b_k}$ is a Hilbert space. Note $\langle a, a \rangle^{1/2} = \left(\sum_{k=1}^{\infty} |a_k|^2 \right)^{1/2} = \|a\|_{l^2}$ is the l^2 norm.

Example: If $E \subset \mathbb{R}$ is measurable, then $L^2(E) = \left\{ f : E \to \mathbb{C} : \int_E |f|^2 < \infty \right\}$ where $\langle f, g \rangle = \int_E f \overline{g}$ is a Hilbert space.

Theorem: 3.4: Parallelogram Law

If H is a pre-Hilbert space, then $\forall u, v \in H$, $\|u + v\|^2 + \|u - v\|^2 = 2 \left(\|u\|^2 + \|v\|^2 \right)$. Moreover, if H is a normed space satisfying the equation, then H is a pre-Hilbert space.

This implies that except for $p = 2$, other l^p or L^p spaces are not Hilbert space.

Definition: 3.4: Orthonormal Subsets

If H is a pre-Hilbert space, $u, v \in H$ are orthogonal if $\langle u, v \rangle = 0$. We write $u \perp v$. A subset $\{e_\lambda\}_{\lambda \in \Lambda} \subset H$ is orthonormal if $\forall \lambda \in \Lambda$, $\|e_\lambda\| = 1$ and $\lambda_1 \neq \lambda_2 \Rightarrow e_{\lambda_1} \perp e_{\lambda_2}$.

Remark 10. We are mainly interested in finite/countable orthonormal subsets, $\{e_1, ..., e_N\} = \{e_n\}_{n=1}^{N}$ and $\{e_n\}_{n=1}^{\infty}$.

Example: $\left\{\begin{pmatrix}1\\0\end{pmatrix},\begin{pmatrix}0\\1\end{pmatrix}\right\}$ is an orthonormal subset of $\mathbb{C}^2$, $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\}$ is an orthonormal subset of $\mathbb{C}^3$.

Example: Let $e_n = \left\{0, ..., \overset{n\text{th entry}}{1}, 0, ...\right\} \in l^2$, $\{e_n\}_{n=1}^{\infty}$ is orthonormal in l^2.

Example: $\frac{1}{\sqrt{2\pi}}e^{inx} \in L^2([-\pi,\pi])$. $\left\{\frac{1}{\sqrt{2\pi}}e^{inx}\right\}_{n\in\mathbb{Z}}$ is orthonormal in $L^2([-\pi,\pi])$

Proof. When $m \neq n$, $\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}}e^{imx}\overline{\frac{1}{\sqrt{2\pi}}e^{inx}} = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x} = 0$. (Consider $e^ix = \cos x + i \sin x$) □

***Theorem:* 3.5: Bessel**

If $\{e_n\}_n$ is a countable orthonormal subset of a pre-Hilbert space H, then $\forall u \in H$, $\sum_n |\langle u, e_n\rangle|^2 \leq \|u\|^2$.

Proof. (Finite case) Suppose $\{e_n\}_{n=1}^N$ is an orthonormal subset of H. Then

$$\begin{aligned}\left\|\sum_{n=1}^N \langle u,e_n\rangle e_n\right\|^2 &= \left\langle \sum_n \langle u,e_n\rangle e_n, \sum_m \langle u,e_m\rangle e_m\right\rangle \\ &= \sum_{n,m} \langle u,e_n\rangle \overline{\langle u,e_m\rangle} \langle e_n,e_m\rangle \\ &= \sum_n |\langle u,e_n\rangle|^2\end{aligned}$$

And $\left\langle u, \sum n = 1^N \langle u,e_n\rangle e_n\right\rangle = \sum_{n=1}^N \overline{\langle u,e_n\rangle}\langle u,e_n\rangle = \sum_{n=1}^N |\langle u,e_n\rangle|^2$

$$\begin{aligned}\text{Thus, } 0 &\leq \left\|u - \sum_{n=1}^N \langle u,e_n\rangle e_n\right\|^2 \\ &\leq \|u\|^2 + \left\|\sum_{n=1}^N \langle u,e_n\rangle e_n\right\|^2 - 2\text{Re}\left\langle u, \sum_{n=1}^N \langle u,e_n\rangle e_n\right\rangle \\ &= \|u\|^2 - \sum_{n=1}^N |\langle u,e_n\rangle|^2\end{aligned}$$

(Infinite case) Suppose $\{e_n\}_{n=1}^{\infty}$ is an orthonormal subset of H. Then $\forall N \in \mathbb{N}$, $\sum_{n=1}^N |\langle u,e_n\rangle|^2 \leq \|u\|^2$ and take $N \to \infty$ gives the desired result. □

3.1.1 Gram-Schmidt

Definition: 3.5: Maximal Orthonormal Subset

An orthonormal subset $\{e_\lambda\}_{\lambda\in\Lambda}$ of a pre-Hilbert space H is maximal if $u \in H$ and $\langle u, e_\lambda\rangle = 0\ \forall\lambda \in \Lambda$ $\Rightarrow u = 0$.

Example: $\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}0\\1\end{pmatrix}\right\}$ is maximal in $\mathbb{C}^2$, but $\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\1\\0\end{pmatrix}\right\}$ is not maximal in $\mathbb{C}^3$, since $\begin{pmatrix}0\\0\\1\end{pmatrix}$ is orthogonal to this set.

Example: $\{e_n\}_{n=1}^\infty$ is maximal subset of l^2.

Theorem: 3.6:

Every non-trivial pre-Hilbert space has a maximal orthonormal subset.

Theorem: 3.7:

Every non-trivial separable (having a countable dense subset) pre-Hilbert space has a countable maximal orthonormal subset.

Proof. Let $\{v_j\}_{j=1}^\infty$ be a countable dense subset of H s.t. $\|v_1\| \neq 0$
Claim: $\forall n \in \mathbb{N}$, $\exists m(n) \leq n$ and an orthonormal subset $\left\{e_1, ..., e_{m(n)}\right\}$ s.t.

1. $\operatorname{span}\left\{e_1, ..., e_{m(n)}\right\} = \operatorname{span}\{v_1, ..., v_n\}$
2. $\left\{e_1, ..., e_{m(n)}\right\} = \left\{e_1, ..., e_{m(n-1)}\right\} \cup \begin{cases}\emptyset, v_n \in \operatorname{span}\{v_1, ..., v_{n-1}\}\\ e_{m(n)}, \text{else}\end{cases}$

We prove the claim by induction:
Base case: $n = 1$, $e_1 = \frac{v_1}{\|v_1\|}$
Induction: Suppose the claim holds for $n = k$.
When $n = k + 1$:
If $v_{k+1} \in \operatorname{span}\{v_1, ..., v_k\}$, then $\operatorname{span}\left\{e_1, ..., e_{m(k)}\right\} = \operatorname{span}\{v_1, ..., v_k\} = \operatorname{span}\{v_1, ..., v_k, v_{k+1}\}$.

Suppose $v_{k+1} \notin \operatorname{span}\{v_1, ..., v_k\}$. Define $w_{k+1} = v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, e_j\rangle e_j \neq 0$, since $v_{k+1} \notin \operatorname{span}\{v_1, ..., v_k\}$
We can define a unit vector $e_{k+1} = \frac{w_{k+1}}{\|w_{k+1}\|}$, $\|e_{k+1}\| = 1$.

For any $j \leq k$, $\left\langle e_{m(k+1)}, e_k\right\rangle = \frac{1}{\|w_{k+1}\|}\left\langle v_{k+1} - \sum_{j=1}^{m(k)} \langle v_{k+1}, e_j\rangle e_j, e_1\right\rangle = \frac{1}{\|w_{k+1}\|}\left(\langle v_{k+1}, e_l\rangle - \langle v_{k+1}, e_l\rangle\right) =$ 0

Let $S - \bigcup_{n=1}^\infty \left\{e_1, ..., e_{m(n)}\right\}$ (may be finite or infinite). Then S is an orthonormal subset of H. We now show that H is maximal.
Suppose $u \in H$, $\forall l$, $\langle u, e_l\rangle = 0$. Since $\{v_j\}_j$ is dense in H, there exists a sequence $\left\{v_{j(k)}\right\}_{k=1}^\infty$ s.t. $v_{j(k)} \to u$ as $k \to \infty$.

By the first part of the claim, $v_{j(k)} \in \text{span}\left\{e_1, ..., e_{m(j(k))}\right\}$. Thus

$$\left\|v_{j(k)}\right\|^2 = \sum_{l=1}^{m(j(k))} \left|\left\langle v_{j(k)}, e_l\right\rangle\right|^2 \overset{\langle u,e_l\rangle=0}{=} \sum_{l=1}^{m(j(k))} \left|\left\langle v_{j(k)} - u, e_l\right\rangle\right|^2$$

$$\overset{\text{By Theorem 3.5}}{\leq} \left\|v_{j(k)} - u\right\|^2 \to 0 \text{ as } k \to \infty$$

By squeeze theorem, $\left\|v_{j(k)}\right\| = 0$ and thus $u = 0$. □

Definition: 3.6: Orthonormal Basis

Let H be a Hilbert space. An orthonormal basis of H is a countable maximal orthonormaal subset $\{e_n\}_{n\in\mathbb{N}}$.

Theorem: 3.8: Fourier-Bessel Series

If $\{e_n\}_n$ is an orthonormal basis in a Hilbert space H, then $\forall u \in H$, $\lim_{m\to\infty} \sum_{n=1}^{m} \langle u, e_n\rangle e_n = u$. *i.e.* $\sum_{n=1}^{\infty} \langle u, e_n\rangle e_n = u$

Proof. We show that $\left\{\sum_{n=1}^{m} \langle u, e_n\rangle e_n\right\}_m$ is Cauchy.

Let $\epsilon > 0$. By Theorem 3.5, $\sum_{n=1}^{\infty} |\langle u, e_n\rangle|^2 \leq \|u\|^2 < \infty$.

Thus, $\exists M \in \mathbb{N}$ s.t. $\forall N \geq M$, $\sum_{n=N+1}^{\infty} |\langle u, e_n\rangle|^2 < \epsilon^2$.

Then $\forall m > l \geq N$,

$$\left\|\sum_{n=1}^{m} \langle u, e_n\rangle e_n - \sum_{n=1}^{l} \langle u, e_n\rangle e_n\right\|^2 = \sum_{n=l+1}^{m} |\langle u, e_n\rangle|^2 \leq \sum_{n=l+1}^{\infty} |\langle u, e_n\rangle|^2 < \epsilon^2$$

Since H is complete, $\exists \bar{u} \in H$ s.t. $\bar{u} = \lim_{m\to\infty} \sum_{n=1}^{m} \langle u, e_n\rangle e_n$ in H.

By Theorem 3.3, $\forall l \in \mathbb{N}$, $\langle u - \bar{u}, e_l\rangle = \lim_{m\to\infty} \langle u - \sum_{n=1}^{m} \langle u, e_n\rangle e_n, e_l\rangle = (\langle u, e_l\rangle - \langle u, e_l\rangle) = 0$
Since $\{e_l\}_l$ is maximal, then $u - \bar{u} = 0$. □

Theorem: 3.9:

If H has an orthonormal basis, then H is separable.

Proof. Suppose $\{e_n\}$ is an orthonormal basis for H. Then $S = \bigcup_{m\in\mathbb{N}} \left\{\sum_{n=1}^{m} q_n e_n : q_1, ..., q_m \in \mathbb{Q} + i\mathbb{Q}\right\}$ is countable.
By Theorem 3.8, S is dense in H. □

Remark 11. If H is a Hilbert space, then H is separable $\Leftrightarrow$ H has an orthonormal basis.

Theorem: 3.10: Parseval's Identity

If H is a Hilbert space and $\{e_n\}_n$ is a countable orthonormal basis, then $\forall u \in H$, $\sum_n |\langle u, e_n\rangle|^2 = \|u\|^2$.

Proof. We have $u = \sum_n \langle u, e_n\rangle e_n$. Then

$$\begin{aligned}
\|u\|^2 &= \lim_{m\to\infty} \left\langle \sum_{n=1}^{m} \langle u, e_n\rangle e_n, \sum_{l=1}^{m} \langle u, e_l\rangle e_l \right\rangle \\
&= \lim_{m\to\infty} \sum_{n,l=1}^{m} \langle u, e_n\rangle \overline{\langle u, e_l\rangle} \langle e_n, e_l\rangle \\
&= \lim_{m\to\infty} \sum_{n=1}^{m} \langle u, e_n\rangle \overline{\langle u, e_n\rangle} \\
&= \sum_n |\langle u, e_n\rangle|^2
\end{aligned}$$

□

Theorem: 3.11:

If H is an infinite dimensional separable Hilbert space, then H is isometrically isomorphic to l^2. *i.e.* $\exists$ a bijective (bounded) linear operator $T : H \to l^2$ s.t. $\forall u, v \in H$, $\|Tu\|_{l^2} = \|u\|_H$ and $\langle Tu, Tv\rangle_{l^2} = \langle u, v\rangle_H$.

Proof. Since H is a separable Hilbert space, by Theorem 3.9, it has an orthonormal basis $\{e_n\}_{n\in\mathbb{N}}$ and $\forall u \in H$, $u = \sum_{n=1}^{\infty} \langle u, e_n\rangle e_n$.

Then $\|u\| = \left(\sum_{n=1}^{\infty} |\langle u, e_n\rangle|^2\right)^{1/2}$. Define $Tu = \{\langle u, e_n\rangle\}_{n=1}^{\infty} \in l^2$. T does the job. □

3.2 Fourier Series

Theorem: 3.12:

The subset $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ is an orthonormal subset of $L^2([-\pi, \pi])$.

Proof.

$$\left\langle e^{inx}, e^{imx}\right\rangle = \int_{-\pi}^{\pi} e^{inx}\overline{e^{imx}}dx = \int_{-\pi}^{\pi} e^{i(n-m)x}dx = \begin{cases} 2\pi, m = n \\ 0, \text{ else} \end{cases}$$

□

Definition: 3.7: Fourier Series

Let $f \in L^2([-\pi,\pi])$. The nth *Fourier coefficient* of f is $\hat{f}(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)e^{-int}dt$. The Nth *partial Fourier sum* of f is $S_N f(x) = \sum_{|n|\leq N} \hat{f}(n)e^{inx} = \sum_{|n|\leq N} \left\langle f, \frac{e^{int}}{\sqrt{2\pi}} \right\rangle \frac{e^{inx}}{\sqrt{2\pi}}$

The Fourier series of f is the formal series $\sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$.

Question: Do we have for all $f \in L^2([-\pi,\pi])$, $f(x) = \sum_{n\in\mathbb{Z}} \hat{f}(n)e^{inx}$?

i.e. $\|f - S_N f\|_2 = \left(\int_{-\pi}^{\pi} |f(x) - S_N f(x)|^2\, dx\right)^{1/2} \to 0$ as $N \to \infty$?

Equivalently, is $\left\{\frac{e^{inx}}{\sqrt{2\pi}}\right\}_{n\in\mathbb{Z}}$ maximal in $L^2([-\pi,\pi])$? *i.e.* if $\hat{f}(n) = 0\ \forall n$, then $f = 0$.

The answer to this question is yes.

Theorem: 3.13: Dirichlet Kernel

$\forall f \in L^2([-\pi,\pi])$, $N \in \mathbb{N}\cup\{0\}$. $S_N f(x) = \int_{-\pi}^{\pi} D_N(x-t)f(t)dt$,

where $D_N(x) = \begin{cases} \frac{2N+1}{2\pi}, x = 0 \\ \frac{\sin((N+\frac{1}{2})x)}{2\pi\sin(\frac{x}{2})}, x \neq 0 \end{cases}$. $D_N(x)$ is called the Dirichlet kernel.

Proof. If $f \in L^2([-\pi,\pi])$, $S_N f(x) = \sum_{|n|\leq N}\left(\int_{-\pi}^{\pi} f(t)e^{-int}dt\right)e^{inx} = \int_{-\pi}^{\pi} f(t)\left(\frac{1}{2\pi}\sum_{|n|\leq N} e^{in(x-t)}\right)dt$

$$\begin{aligned} D_N(x) &= \frac{1}{2\pi}\sum_{|n|\leq N} e^{inx} = \frac{1}{2\pi}e^{-iNx}\sum_{n=0}^{2N}(e^{ix})^n \\ &= \frac{1}{2\pi}e^{-iNx}\frac{1-e^{i(2N+1)}x}{1-e^{ix}} \\ &= \frac{1}{2\pi}\frac{e^{i\left(N+\frac{1}{2}\right)x} - e^{-i\left(N+\frac{1}{2}\right)x}}{e^{i\frac{x}{2}} - e^{-i\frac{x}{2}}} = \frac{1}{2\pi}\frac{\sin\left(N+\frac{1}{2}\right)x}{\sin\frac{x}{2}} \end{aligned}$$

□

Definition: 3.8: Cesaro-Fourier Mean

If $f \in L^2([-\pi,\pi])$. Define the Nth Cesaro-Fourier mean of f by $\sigma_N f(x) = \frac{1}{N+1}\sum_{k=0}^{N} S_k f(x)$.

Note: if the series converges, the Cesaro mean converges. Also if we have a sequence $\{1,-1,1,-1\}$ which does not converge, but the Cesaro mean converges.

Theorem: 3.14: Fejer Kernel

$\forall f \in L^2([-\pi,\pi])$, $\sigma_N f(x) = \displaystyle\int_{-\pi}^{\pi} K_N(x-t)f(t)dt$, where $K_N(x) = \begin{cases} \frac{N+1}{2\pi}, x=0 \\ \frac{1}{2\pi(N+1)}\left(\frac{\sin(\frac{N+1}{2}x)}{\sin\frac{x}{2}}\right)^2 \end{cases}$ is the Fejer kernel. Moreover,

1. $K_N(x) \geq 0$, $K_N(x) = K_N(-x)$, K_N is 2π-periodic
2. $\displaystyle\int_{-\pi}^{\pi} K_N(t)dt = 1$
3. $\forall \delta \in (0,\pi]$, then $\forall \delta \leq |x| \leq \pi$, $K_N(x) \leq \frac{1}{2\pi(N+1)\sin^2\frac{\delta}{2}}$

Proof. From Theorem 3.13, $\sigma_N f(x) = \dfrac{1}{N+1}\displaystyle\sum_{k=0}^{N} S_k f(x) = \int_{-\pi}^{\pi} \frac{1}{N+1}\sum_{k=0}^{N} D_k(x-t)f(t)dt$. Then

$$\begin{aligned} K_N(x) &= \frac{1}{N+1}\sum_{k=0}^{N} D_k(x) = \frac{1}{2\pi(N+1)}\frac{1}{2\left(\sin\frac{x}{2}\right)^2}\sum_{k=0}^{N} 2\sin\frac{x}{2}\sin\left(\left(k+\frac{1}{2}\right)x\right) \\ &= \frac{1}{2\pi(N+1)}\frac{1}{2\left(\sin\frac{x}{2}\right)^2}\sum_{k=0}^{N}[\cos kx - \cos(k+1)x] \\ &= \frac{1}{2\pi(N+1)}\frac{1}{2\left(\sin\frac{x}{2}\right)^2}(1-\cos((N+1)x)) \\ &= \frac{1}{2\pi(N+1)}\frac{1}{\left(\sin\frac{x}{2}\right)^2}\sin^2\left(()\frac{N+1}{2}x\right) \end{aligned}$$

1 follows since $\sin^2$ are positive and $K_N(x) = K_N(-x)$.

For 2, $\displaystyle\int_{-\pi}^{\pi} D_k(t)dt = \int_{-\pi}^{\pi}\sum_{n=-k}^{k} e^{int}dt = 1$. Then $\displaystyle\int_{-\pi}^{\pi} K_N(t)dt = \frac{1}{N+1}\sum_{k=0}^{N}\int_{-\pi}^{\pi} D_k(t)dt = \frac{N+1}{N+1} = 1$.

For 3, let $\delta \in (0,\pi]$. Then $\sin^2\frac{x}{2}$ is even and increasing on $[0,\pi]$. $\forall \delta \leq |x| \leq \pi$, $\sin^2\frac{x}{2} \leq \sin^2\frac{\delta}{2}$

Thus $K_N(x) \leq \dfrac{1}{2\pi(N+1)\sin^2\frac{\delta}{2}}\sin^2\left(\dfrac{N+1}{2}x\right) \leq \dfrac{1}{2\pi(N+1)\sin^2\frac{\delta}{2}}$ □

Theorem: 3.15: Fejer's Theorem

If $f \in C([-\pi,\pi])$ is 2π periodic, $f(\pi) = f(-\pi)$, then $\sigma_N f \to f$ uniformly on $[-\pi,\pi]$.

Proof. Firstly, we extend f by periodicity $f(x+2\pi) = f(x)$ to all of $\mathbb{R}$. Then $f \in C(\mathbb{R})$ is 2π-periodic. Thus f is uniformly continuous and bounded. *i.e.* $\|f\|_\infty = \sup_{x\in\mathbb{R}}|f(x)| = \sup_{x\in[-\pi,\pi]}|f(x)| < \infty$.

Let $\epsilon > 0$. Since f is uniformly continuous, $\exists \delta > 0$ s.t. if $|y-z| < \delta$, $|f(y) - f(z)| < \frac{\epsilon}{2}$.
Choose $M \in \mathbb{N}$ s.t. $\forall N \geq M$, $\frac{2\|f\|_2}{(N+1)\sin^2\frac{\delta}{2}} < \frac{\epsilon}{2}$ because LHS$\to 0$.

Since f and K_N are 2π-periodic, $\sigma_N f(x) = \displaystyle\int_{-\pi}^{\pi} K_N(x-t)f(t)dt \overset{\tau=x-t}{=} \int_{x-\pi}^{x+\pi} K_N(\tau)f(x-\tau)d\tau \overset{\text{periodic}}{=} \int_{-\pi}^{\pi} K_N(t)f(x-t)dt$.

Then $\forall N \geq M$, $\forall x \in [-\pi, \pi]$,

$$\begin{aligned}
&|\sigma_N f(x) - f(x)| = \left|\int_{-\pi}^{\pi} K_N(t) f(x-t)dt - \int_{-\pi}^{\pi} K_N(t) f(x) dt\right| \text{ (since } \int_{-\pi}^{\pi} K_N(t)dt) \\
&= \left|\int_{-\pi}^{\pi} K_N(t)(f(x-t) - f(x))dx\right| \\
&\leq \int_{-\pi}^{\pi} |K_N(t)(f(x-t) - f(x))|\, dx \text{ (By Theorem 2.45)} \\
&= \int_{|t|\leq\delta} K_N(t)\, |f(x-t) - f(x)|\, dt + \int_{\delta\leq|x|\leq\pi} K_N(t)|f(x-t) - f(t)|dt \\
&< \frac{\epsilon}{2}\int_{|t|<\delta} K_N(t)dt + 2\,\|f\|_\infty \int_{\delta\leq|t|\leq\pi} \frac{K_N(t)}{2\pi(N+1)\sin\frac{\delta}{2}}dt \text{ (By uniform continuity, choice of M)} \\
&\leq \frac{\epsilon}{2} + \frac{2\,\|f\|_\infty}{(N+1)\sin^2\frac{\delta}{2}} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.
\end{aligned}$$

Thus $\sigma_N f \to f$ uniformly. □

Remark 12. Same proof can be modified if instead of $K_N(x) \geq 0$, we have $\sup_N \int_{-\pi}^{\pi} |K_N(x)|dx < \infty$. Also, $\int_{-\pi}^{\pi} |D_N(x)|dx \sim \log N$.

Theorem: 3.16: Bounding Cesaro-Fourier Mean

$\forall f \in L^2([-\pi, \pi])$, $\|\sigma_N f\|_2 \leq \|f\|_{L^2}$

Proof. Suppose $f \in C([-\pi, \pi])$ 2π-periodic. Then $\sigma_N f(x) = \int_{-\pi}^{\pi} f(x-t)K_N(t)dt$,

$$\begin{aligned}
\int_{-\pi}^{\pi} |\sigma_N f(x)|^2\, dx &= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x-s)\overline{f(x-t)}K_N(s)K_N(t)dsdtdx \\
&= \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} K_N(s)K_N(t)\left[\int_{-\pi}^{\pi} f(x-s)\overline{f(x-t)}dx\right] dsdt \text{ (By Fubini)} \\
&\leq \int_{-\pi}^{\pi} K_N(s)K_N(t)\,\|f(\cdot - s)\|_2\,\|f(\cdot - t)\|_2\, dsdt \text{ (By Theorem 3.1)} \\
&= \|f\|_2^2 \int_{-\pi}^{\pi} K_N(s)ds \int_{-\pi}^{\pi} K_N(t)dt = \|f\|_2^2
\end{aligned}$$

Thus $\|\sigma_N f\| \leq \|f\|_2$ for $f \in C([-\pi, \pi])$

Let $f \in L^2([-\pi, \pi])$, $\exists\, \{f_n\}_n$ of 2π-periodic continuous functions s.t. $\|f_n - f\|_2 \to 0$.
Then $\|\sigma_N f_n - \sigma_N f\|_2 \to 0$
Thus $\|\sigma_N f\|_2 = \lim_{n\to\infty} \|\sigma_N f_n\|_2 \leq \lim_{n\to\infty} \|f_n\|_2 = \|f\|_2$. □

Theorem: 3.17: Convergence of Cesaro-Fourier Mean

$\forall f \in L^2([-\pi, \pi])$, $\|\sigma_N f - f\|_2 \to 0$ as $N \to \infty$. In particular, if $\hat{f}(n) = 0$, $\forall n$, then $f = 0$.

Proof. Let $f \in L^2([-\pi,\pi])$, $\epsilon > 0$. There exists $g \in C([-\pi,\pi])$ and 2π-periodic s.t. $\|f-g\|_2 < \frac{\epsilon}{3}$.
Since $\sigma_N g \to g$ uniformly on $[-\pi,\pi]$ by Theorem 3.15,
$\exists M \in \mathbb{N}$ s.t. $\forall N \geq M$, $\forall x \in [-\pi,\pi]$, $|\sigma_N g(x) - g(x)| < \frac{\epsilon}{3\sqrt{2\pi}}$
Then $\forall N \geq M$,

$$
\begin{aligned}
\|\sigma_N f - f\|_2 &\leq \|\sigma_N(f-g)\|_2 + \|\sigma_N g - g\|_2 + \|g-f\|_2 \text{ (By Triangle inequality)} \\
&\leq 2\|f-g\|_2 + \left(\int_{-\pi}^{\pi} |\sigma_N g - g|^2 \, dx\right)^{1/2} \text{ (By Theorem 3.16 and Definition 2.18)} \\
&< \frac{2\epsilon}{3} + \frac{\epsilon}{3}\left(\int_{-\pi}^{\pi} \frac{1}{\sqrt{2\pi}} dx\right)^{1/2} = \epsilon.
\end{aligned}
$$

Thus $\|\sigma_N f - f\|_2 \to 0$ as $N \to \infty$. □

Now we have shown that $\forall f \in L^2$, $\|S_N f - f\|_2 \to 0$. Carleson shows that $\forall f \in L^2$, $S_N f(x) \to f(x)$ a.e. Also $\forall 1 < p < \infty$, $\|S_N f - f\|_p \to 0$, but this doesn't hold for $p = 1$ or ∞.

3.3 Riesz Representation

Theorem: 3.18: Length Minimizer

Suppose $C \subset H$ is a subset of a Hilbert space H s.t. $C \neq \emptyset$, C is closed and C is convex, *i.e.* if $v_1, v_2 \in C$ and $t \in [0,1]$, then $tv_1 + (1-t)v_2 \in C$. Then there exists a unique $v \in C$ with $\|v\| = \inf_{u \in C} \|u\|$.

Proof. $a = \inf S \leftrightarrow a$ is a lower bound for S and $\exists \{s_n\} \in S$ s.t. $s_n \to a$.
Let $d = \inf_{u\in C} \|u\|$. Then $\exists \{u_n\}_n \in C$ s.t. $\|u_n\| \to d$. We want to show that $\{u_n\}$ is Cauchy.
Let $\epsilon > 0$, since $\|u_n\| \to d$, $\exists N \in \mathbb{N}$ s.t. $\forall n \geq N$, $2\|u_n\|^2 < 2d^2 + \frac{\epsilon^2}{2}$. Then $\forall n, m \geq N$,

$$
\begin{aligned}
\|u_n - u_m\|^2 &\leq 2\|u_n\|^2 + 2\|u_m\|^2 - 4\left\|\frac{u_n + u_m}{4}\right\|^2 \text{ (By Theorem 3.4)} \\
&\leq 2\|u_n\|^2 + 2\|u_m\|^2 - 4d^2 \text{ (By Definition of d as infimum)} \\
&< 2d^2 + \frac{\epsilon^2}{2} + 2d^2 + \frac{\epsilon^2}{2} - 4d^2 = \epsilon^2
\end{aligned}
$$

Therefore, $\{u_n\}$ is Cauch. Since H is complete, then $\exists v \in H$ s.t. $u_n \to v$. Since C is closed, $v \in C$.
$\|v\| = \lim_{n\to\infty} \|u_n\| = d$
Thus the existence of $v \in C$, $\|v\| = d = \inf_{u\in C} \|u\|$ is proved.

Now we show the uniqueness. Suppose $v, \bar{v} \in C$ s.t. $\|v\| = \|\bar{v}\| = d$. Then

$$
\begin{aligned}
\|v - \bar{v}\|^2 &= 2\|v\|^2 + 2\|\bar{v}\|^2 - 4\left\|\frac{v+\bar{v}}{2}\right\|^2 \\
&= 4d^2 - 4\left\|\frac{v+\bar{v}}{2}\right\| \leq 4d^2 - 4d^2 = 0
\end{aligned}
$$

Thus $v = \bar{v}$. □

Theorem: 3.19: Orthogonal Complement

If H is a Hilbert space, $W \subset H$ is a subspace, then $W^\perp = \{u \in H : \langle u, w\rangle = 0, \forall w \in W\}$ is a closed linear subspace of H. If W is closed, then $H = W \oplus W^\perp$ (*i.e.* $\forall u \in H$, $\exists! w \in W$, $w^\perp \in W^\perp$ s.t. $u = w + w^\perp$)

Proof. Note that $W^\perp$ is a subspace of H by linearity of inner product and $W \cap W^\perp = \{0\}$ by definiteness. Let $\{u_n\}_n$ be sequence in $W^\perp$ and $u \in H$ s.t. $u_n \to u$. Let $w \in W$. Then by Theorem 3.3 (continuity), $\langle u, w\rangle = \lim_{n\to\infty} \langle u_n, w\rangle = 0$. Thus $u \in W^\perp$, $W^\perp$ is closed. $W^\perp$ is therefore a closed linear subspace of H.

Now suppose W is closed, we show that $H = W \oplus W^\perp$.
If $W = H$, then $W^\perp = \{0\}$ and $H = W \oplus \{0\} = W \oplus W^\perp$
Suppose $W \neq H$. Let $u \in H \setminus W$. Define $C = u + W = \{u + w : w \in W\}$. Note $u \in C$, so $C \neq \emptyset$
Let $u + w_1 \in C$, $u + w_2 \in C$, for $w_1, w_2 \in W$ and $t \in [0, 1]$, then $t(u + w_1) + (1 - t)(u + w_2) = u + (tw_1 + (1 - t)w_2) \in C$, since W is a subspace. Thus C is convex.
Now suppose $u + w_n \to v \in H$. Then $w_n \to v - u$. Since w is closed, $v - u \in W$. Then $v = u + w$ for $w \in W \Rightarrow v \in C$. Thus C is closed.
Since C is closed and covex, by Theorem 3.18, $\exists! v \in C$ s.t. $\|v\| = \inf_{w\in W} \|u + w\|$.

Note that $v \in C \Rightarrow u - v \in W$ and $u = (u - v) + v$. We show that $v \in W^\perp$.
Let $w \in W$, $f(t) = \|v + tw\|^2 = \|v\|^2 + t^2 \|w\|^2 + 2t\text{Re}\,\langle v, w\rangle$. Then $f(t)$ has a min at $t = 0$, $f'(t) = 0 \Rightarrow \text{Re}\,\langle v, w\rangle = 0$.
Repeat the same argument with iw to get $\text{Re}\,\langle v, iw\rangle = \text{Im}\,\langle v, w\rangle = 0$. Thus $\langle v, w\rangle = 0$ and $v \in W^\perp$.

We have now decompsed $u \in H$ to $u = v + w$ for $w \in W, v \in W^\perp$. We need to show that the decomposition is unique.
If $u = w_1 + w_1^\perp = w_2 + w_2^\perp$. Then $w_2 - w_1 = w_1^\perp - w_2^\perp$. Since $W \cap W^\perp = \{0\}$, $w_1 = w_2$, $w_1^\perp = w_2^\perp$. ☐

Theorem: 3.20:

If $W \subset H$ is a subspace, then $(W^\perp)^\perp$ is the closure $\bar{W}$ of W. If W is closed, then $(W^\perp)^\perp = W$.

Definition: 3.9: Projection

A bounded operator $P : H \to H$ is a projection if $P^2 = P$.

Theorem: 3.21:

Let H be a Hilbert space, $W \subset H$ be a closed subspace s.t. $H = W \oplus W^\perp$. The map $\Pi_W = H \to H$, defined by if $v = w + w^\perp$, then $\Pi_W(v) = w$, is a projection.

Proof. Π_W is linear: If $v_1 = w_1 + w_1^\perp$, $v_2 = w_2 + w_2^\perp$, $\lambda_1, \lambda_2 \in \mathbb{C}$, then $\lambda_1 v_1 + \lambda_2 v_2 = (\lambda_1 w_1 + \lambda_2 w_2) + (\lambda_1 w_1^\perp + \lambda_2 w_2^\perp)$. $\Pi_W(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 \Pi_W(v_1) + \lambda_2 \Pi_W(v_2)$.

Π_W is bounded: If $v = w + w^\perp$, then $\|v\|^2 = \left\|w + w^\perp\right\|^2 \overset{\langle w, w^\perp\rangle = 0}{=} \|w\|^2 + \left\|w^\perp\right\|^2 \geq \|w\|^2$. Then $\|\Pi_W(v)\| = \|w\| \leq \|v\|$ and $\|\Pi_W\| \leq 1$.

Projection: $\Pi_W(\Pi_W(v)) = \Pi_W(w) = w = \Pi_W(v)$. ☐

Theorem: 3.22: Riesz Representation Theorem

If H is a Hilbert space, then $\forall f \in H'$, there is a unique $v \in H$ s.t. $f(u) = \langle u, v\rangle$ for $u \in H$.

Proof. v is unique: if $f(u) = \langle u, v\rangle = \langle u, \bar{v}\rangle$ for all u, then $\langle u, v - \bar{v}\rangle = 0$ $\forall u \in H$. Thus $v = \bar{v}$.

If $f = 0$, we can simply choose $v = 0$. So we suppose $f \neq 0$.
Then $\exists u_1 \in H$ s.t. $f(u_1) = \langle u_1, v\rangle \neq 0$.
Let $u_0 = \frac{u_1}{f(u_1)}$. $f(u_0) = \left\langle \frac{u_1}{f(u_1)}, v\right\rangle = \frac{1}{f(u_1)}\langle u_1, v\rangle = 1$.
Let $C = \{u \in H : f(u) = 1\} = f^{-1}(\{1\})$. C is a non-empty closed subset of H.
Let $u_1, u_2 \in C$, $t \in [0, 1]$, then $f(tu_1 + (1-t)u_2) = tf(u_1) + (1-t)f(u_2) = t + 1 - t = 1$. Thus C is convex.
Then by Theorem 3.18, $\exists v_0 \in C$ s.t. $\|v_0\| = \inf_{u\in C}\|u\|$
Let $v = \frac{v_0}{\|v_0\|^2}$, $N = f^{-1}(\{0\}) = \{w \in H : f(w) = 0\}$. Then $C = \{v_0 + w : w \in N\}$, so $\|v_0\| = \inf_{w\in N}\|v_0 + w\|$ and $v_0 \in N^{\perp}$.
Let $u \in H$. Then $f(u - f(u)v_0) = f(u) - f(u)f(v_0) = 0$. Thus $u = u - f(u)v_0 + f(u)v_0 \in N + N^{\perp}$.
Therefore, $\langle u, v\rangle = \frac{1}{\|v_0\|^2}\langle u, v_0\rangle = \frac{1}{\|v_0\|^2}\left[\langle u - f(u)v_0, v_0\rangle + f(u)\langle v_0, v_0\rangle\right] = f(u)$. ☐

3.4 Adjoint

Theorem: 3.23: Adjoint Operator

Let H be a Hilbert space, $A : H \to H$ be a bounded linear operator. Then there exists a unique bounded linear operator $A^* : H \to H$ (adjoint) s.t. $\forall u, v \in H$, $\langle Au, v\rangle = \langle u, A^*v\rangle$ and $\|A^*\| = \|A\|$.

Proof. Uniqueness of A^* follows from $\langle Au, v\rangle = \langle u, A^*v\rangle$.
Define $f_v : H \to \mathbb{C}$ s.t. $f_v(u) = \langle Au, v\rangle$. Then $\forall u_1, u_2 \in H$, $\lambda_1, \lambda_2 \in \mathbb{C}$,

$$\begin{aligned} f_v(\lambda_1 u_1 + \lambda_2 u_2) &= \langle A(\lambda_1 u_1 + \lambda_2 u_2), v\rangle = \langle \lambda_1 Au_1 + \lambda_2 Au_2, v\rangle \\ &= \lambda_1\langle Au_1, v\rangle + \lambda_2\langle Au_2, v_2\rangle \\ &= \lambda_1 f_v(u_1) + \lambda_2 f_v(u_2) \end{aligned}$$

Thus f_v is linear.
If $\|u\| = 1$, then $|f_v(u)| = |\langle Au, v\rangle| \overset{\text{Theorem 3.1}}{\leq} \|Au\|\,\|v\| \leq \|A\|\,\|v\|$ since A is bounded linear operator.
Thus $\|f_v\| \leq \|A\|\,\|v\|$ is a bounded linear operator, $f_v \in H'$.
By Theorem 3.22, there exists a unique $A^*v \in H$ s.t. $\forall u \in H$, $f_v(u) = \langle u, A^*v\rangle$
i.e. $\forall u \in H$, $\langle Au, v\rangle = \langle u, A^*v\rangle$.

$v \to A^*v$ is linear: Let $v_1, v_2 \in H, \lambda_1, \lambda_2 \in \mathbb{C}$, $\forall u \in H$,

$$\begin{aligned} \langle u, A^*(\lambda_1 v_1 + \lambda_2 v_2)\rangle &= \langle Au, \lambda_1 v_1 + \lambda_2 v_2\rangle = \bar{\lambda_1}\langle Au, v_1\rangle + \bar{\lambda_2}\langle Au, v_2\rangle \\ &= \bar{\lambda_1}\langle u, A^*v_1\rangle + \bar{\lambda_2}\langle u, A^*v_2\rangle = \langle u, \lambda_1 A^*v_1 + \lambda_2 A^*v_2\rangle \end{aligned}$$

Therefore, $A^*(\lambda_1 v_1 + \lambda_2 v_2) = \lambda_1 A^*v_1 + \lambda_2 A^*v_2$, $A^* : H \to H$ is a linear operator.

Suppose $\|v\| = 1$. If $A^*v = 0$, then $\|A^*v\| \leq \|A^*\|$.
Suppose $A^*v \neq 0$. Then $\|A^*v\|^2 = \langle A * v, A^*v\rangle = \langle AA^*v, v\rangle \leq \|AA^*v\|\,\|v\| \leq \|A\|\,\|A^*v\|$.
Then $\|A^*v\| \leq \|A\|$. $\|A^*\| \leq \|A\|$.
Note: $\forall u, v \in H$, $\langle u, (A^*)^*v\rangle = \langle A^*u, v\rangle = \overline{\langle v, A^*u\rangle} = \overline{\langle Av, u\rangle} = \langle u, Av\rangle$.
Thus $(A^*)* = A$, and $\|A\| = \|(A^*)*\| \leq \|A\|$.
Thus $\|A\| = \|A^*\|$. ☐

Example: For $u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \in \mathbb{C}^n$, define A s.t. $(Au)_i = \sum_{j=1}^{n} A_{ij} u_j$, where $A_{ij} \in \mathbb{C}$. Then

$$\begin{aligned} \langle Au, v \rangle &= \sum_{i=1}^{n} (Au)_i \overline{v_i} = \sum_{i,j} A_{ij} u_j \overline{v_i} \\ &= \sum_{j=1}^{n} u_j \overline{\sum_{i=1}^{n} \overline{A_{ij}} v_i} = \sum_{j=1}^{n} u_j \overline{(A^* v)_j}, \end{aligned}$$

where $(A^* v)_i = \sum_{j=1}^{n} \overline{A_{ji}} v_j$. Thus if $A = (A_{ij})$, then $(A^*)_{ij} = \overline{A_{ji}}$.

Example: Suppose $\{A_{ij}\}_{i,j=1}^{\infty}$ is a double sequence in $\mathbb{C}^n$ s.t. $\sum_{i,j} |A_{ij}|^2 = \lim_{N \to \infty} \sum_{i=1}^{N} \sum_{j=1}^{N} |A_{ij}|^2 < \infty$.
Define $A : l^2 \to l^2$ by $Aa = \sum_{j=1}^{\infty} A_{ij} a_j$, where $a = \{a_j\}_j \in l^2$.
Then $A \in B(l^2, l^2)$ and $\forall a, b \in l^2$, $\langle Aa, b \rangle = \sum_i \sum_j A_{ij} a_j \overline{b_i} = \sum_j a_j \overline{\sum_i \overline{A_{ij}} b_i} = \langle a, A^* b \rangle$, where $(A^* b)_i = \sum_{j=1}^{\infty} \overline{A_{ji}} b_j$.

Example: Suppose $K \in C([0,1] \times [0,1])$. Define $A : L^([0,1]) \to L^2([0,1])$ s.t. $Af(x) = \int_0^1 K(x,y) f(y) dy$.
Then $A^* g(x) = \int_0^1 \overline{K(y,x)} g(y) dy$

***Theorem:* 3.24: Range Null Space**

Suppose H is a Hilbert space and $A : H \to H$ is a bounded linear operator. Then $(\text{Range}(A))^\perp = \text{Null}(A^*)$, where $\text{Range}(B) = \{Bu : u \in H\}$, $\text{Null}(B) = \text{Ker}(B) = \{u \in H : Bu = 0\}$.

Proof. $v \in \text{Null}(A^*) \Leftrightarrow \langle u, A^* v \rangle = 0, \forall u \in H \Leftrightarrow \langle Au, v \rangle = 0 \Leftrightarrow v \in (\text{Range}(A))^\perp$ □

Remark 13. Suppose Range(A) is closed. Then $A : H \to H$ is surjective $\Leftrightarrow$ $A^* : H \to H$ is injective.

3.5 Compactness

***Definition:* 3.10: Compact Subset**

If X is a metric space, $K \subset X$ is compact if every sequence in K has a subsequence converging to an element in K.

Example: all finite subsets of any metric space are compact.

***Theorem:* 3.25: Heine-Borel**

A subsets $K \subset \mathbb{R}$ (or $\mathbb{R}^n$, $\mathbb{C}^n$) is compact if and only if K is closed and bounded.

Exampe: $[a, b]$, $\left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$ are compact.

Example: Suppose H is an infinite-dimensional Hilbert space, then $F = \{u \in H : \|u\| \leq 1\}$ is not compact.

Proof. Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal subset of H. Then $\forall n \neq k$, $\|e_n - e_k\|^2 = \|e_n\|^2 + \|e_k\|^2 + 2\text{Re}\,\langle e_n, e_k\rangle = 2$. Thus $\{e_n\}$ cannot be Cauchy, *i.e.* No converging subsequences. ☐

***Definition:* 3.11: Equi-small Tails**

Let H be a Hilbert space. A subset $K \subset H$ has equi-small tails w.r.t. a countable orthonormal subset $\{e_n\}_n$ if $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\forall v \in K$, $\displaystyle\sum_{k>N} |\langle v, e_k\rangle|^2 < \epsilon^2$.

Example: $K = \{v_1, ..., v_n\} \Rightarrow K$ has an equi-small tail w.r.t. any $\{e_k\}_k$.

***Theorem:* 3.26:**

Let H be a Hilbert space, $\{v_n\}_n$ be a sequence with $v_n \to v$. Let $\{e_k\}_k$ be a countable orthonormal subset. Then $K = \{v_n : n \in \mathbb{N}\} \cup \{v\}$ is compact and K has equi-small tails w.r.t. $\{e_k\}_k$.

Proof. We show the equi-small tails here.
Let $\epsilon > 0$, since $v_n \to v$, $\exists M \in \mathbb{N}$ s.t. $\forall n \geq M$, $\|v_n - v\| < \frac{\epsilon}{2}$.
Choose $N \in \mathbb{N}$ large s.t. $\displaystyle\sum_{k>N} |\langle v, e_k\rangle|^2 + \max_{1\leq n\leq M-1} \sum_{k>N} |\langle v_n, e_k\rangle|^2 < \frac{\epsilon^2}{4}$.
Then $\displaystyle\sum_{k>M} |\langle v, e_k\rangle|^2 < \frac{\epsilon^2}{4} < \epsilon^2$, and $\forall 1 \leq n \leq M-1$, $\sum_{k>N} |\langle v_n, e_k\rangle|^2 < \frac{\epsilon^2}{4} < \epsilon^2$.
If $n \geq M$, by Theorem 3.5,

$$\begin{aligned}\left(\sum_{k>N} |\langle v_n, e_k\rangle|^2\right)^{1/2} &= \left(\sum_{k>N} |\langle v_n - v, e_k\rangle + \langle v, e_k\rangle|^2\right)^{1/2} \\ &\leq \left(\sum_{k>N} |\langle v_n - v, e_k\rangle|^2\right)^{1/2} + \left(\sum_{k>N} |\langle v, e_k\rangle|^2\right)^{1/2} \quad \text{(By Theorem 1.6)} \\ &\overset{\text{Theorem 3.10}}{\leq} \|v_n - v\| + \frac{\epsilon}{2} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\end{aligned}$$

☐

***Theorem:* 3.27:**

Let H be a separable Hilbert space, and $\{e_k\}_k$ be an orthonormal basis of H. Then $K \subset H$ is compact if and only if K is and bounded and has equi-small tails.

Proof. $(\Rightarrow)$ Suppose K is compact, then K is closed and bounded by metric space theory.
Suppose K does not have equi-small tails w.r.t. $\{e_k\}_k$.
Then $\exists \epsilon_0 > 0$ s.t. $\forall N \in \mathbb{N}$, $\exists u_N \in K$ s.t. $\displaystyle\sum_{k>N} |\langle u_N, e_k\rangle|^2 > \epsilon_0^2$
Since $\{u_N\}_N$ is a sequence in K, then there exists a subsequence $\{v_n\}_n$ and $v \in K$ s.t. $v_n \to v$. Then

$\forall n \in \mathbb{N}, \sum_{k>n} |\langle v_n, e_k\rangle| \geq \epsilon_0^2$

Then $\{v_n : n \in \mathbb{N}\} \cup \{v\}$ does not have equi-small tails w.r.t, $\{e_k\}_k$. Contradiction to Theorem 3.26. Thus K must have equi-small tails w.r.t. $\{e_k\}_k$.

($\Leftarrow$) Suppose K is closed and bounded and has equi-small tails. Let $\{u_n\}_n$ be a sequence in K.
Since K is closed, we just need to show $\{u_n\}_n$ has a convergent subsequence.
Since K is bounded, then $\exists C \geq 0$ s.t. $\forall n$, $\|u_n\| \leq C$. Then $\forall k, n$, $|\langle u_n, e_k\rangle| \leq \|u_n\| \|e_k\| \leq C$. *i.e.* $\forall k \in \mathbb{N}$, $\{\langle u_n, e_k\rangle\}_n$ is a bounded sequence in $\mathbb{C}$.
Since $\{\langle u_n, e_1\rangle\}_n$ is bounded, there is a subsequence $\left\{\left\langle u_{n_1(j)}, e_1\right\rangle\right\}_j$ of $\{\langle u_n, e_1\rangle\}_n$ which converges in $\mathbb{C}$.
Since $\left\{\left\langle u_{n_1(j)}, e_2\right\rangle\right\}_j$ is bounded, there exists a subsequence $\left\{\left\langle u_{n_2(j)}, e_2\right\rangle\right\}_j$ of $\left\{\left\langle u_{n_1(j)}, e_2\right\rangle\right\}_j$ which converges.
Note $\lim_{j\to\infty} \left\langle u_{n_2(j)}, e_1\right\rangle$ exists and $\lim_{j\to\infty} \left\langle u_{n_2(j)}, e_2\right\rangle$ exists.
Then $\forall l$, there exists subsequence $\{n_l(j)\}_j$ of $\{n_{l-1}(j)\}_j$ s.t. $\forall 1 \leq k \leq l$, $\lim_{j\to\infty} \left\langle u_{n_l(k)}, e_k\right\rangle$ exists.
Pick $v_l = u_{n_l}(l)$ for $l = 1, 2, 3, \ldots$. Then $\{v_l\}_l$ is a subsequence of $\{u_n\}_n$ s.t. $\forall k$, $\{\langle v_l, e_k\rangle\}_l$ converges.

Now we show that $\{v_l\}_l$ is Cauchy. Let $\epsilon > 0$.
Since K has equi-small tails, $\exists N \in \mathbb{N}$ s.t. $\forall l \in \mathbb{N}$, $\sum_{k>N} |\langle v_l, e_l\rangle|^2 < \frac{\epsilon^2}{16}$.
Since the N sequences $\{\langle v_l, e_1\rangle\}_l, \ldots, \{\langle v_l, e_N\rangle\}_l$ converge, $\exists M \in \mathbb{N}$ s.t. $\forall l, m \geq M$,
we have $\sum_{k=1}^{N} |\langle v_l, e_k\rangle - \langle v_m, e_k\rangle|^2 < \frac{\epsilon^2}{4}$
Then $\forall l, m \geq M$,

$$
\begin{aligned}
\|v_l - v_m\| &= \left[\sum_{k=1}^{N} |\langle v_l - v_m, e_k\rangle|^2 + \sum_{k>N} |\langle v_l - v_m, e_k\rangle|^2\right]^{1/2} \\
&\leq \left[\sum_{k=1}^{N} |\langle v_l - v_m, e_k\rangle|^2\right]^{1/2} + \left[\sum_{k>N} |\langle v_l, e_k\rangle - \langle v_m, e_k\rangle|^2\right]^{1/2} \\
&< \frac{\epsilon}{2} + \left[\sum_{k>N} |\langle v_l, e_k\rangle|^2\right]^{1/2} + \left[\sum_{k>N} |\langle v_m, e_k\rangle|^2\right]^{1/2} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{4} = \epsilon
\end{aligned}
$$

Therefore $\{v_l\}_l$ is Cauchy, and thus converges. □

***Definition:* 3.12: Hilbert Cube**

$K = \left\{\{a_k\}_k \in l^2 : |a_k| \leq 2^{-k}\right\}$ is compact. K is the Hilbert cube.

***Theorem:* 3.28:**

A subset $K \subset H$ is compact if and only if K is closed and bounded, and $\forall \epsilon > 0$, there exists a finite dimensional subspace $W \subset H$ s.t. $\forall u \in K$, $\inf_{w\in W} \|u - w\| < \epsilon$.

3.6 Operators

Let H be a Hilbert space, the bounded linear operators set $B(H,H)$ will be denoted by $B(H)$.

3.6.1 Finite Rank Operators

***Definition:* 3.13: Finite Rank Operators**

$T \in B(H)$ is a finite rank operator if Range(T) (a subspace of H) is finite dimensional. Write $T \in R(H)$.

Example: $Ta = \left\{\frac{a_1}{1}, \frac{a_2}{2}, ..., \frac{a_n}{n}, 0, 0, ...\right\}$ for $a = \{a_k\}_k \in l^2$. Then T is finite rank.

***Theorem:* 3.29:**

$R(H)$ is a subspace of $B(H)$.

***Theorem:* 3.30: Matrix Representation of Finite Rank Operators**

$T \in R(H)$ if and only if there exists a finite orthonormal set $\{e_k\}_{k=1}^{l}$ and constants $\{C_{ij}\}_{i,j=1}^{L} \subset \mathbb{C}$ s.t. $Tu = \sum_{i,j=1}^{L} C_{ij}\langle u, e_j\rangle e_i$.

Proof. ($\Leftarrow$) By defintion, T is a finite rank operator.

($\Rightarrow$) Since Range(T) is finite dimensional, there exists an orthonormal basis $\{\bar{e}_k\}_{k=1}^{N}$ s.t.
$Tu = \sum_{k=1}^{N}\langle Tu, \bar{e}_k\rangle \bar{e}_k = \sum_{k=1}^{N}\langle u, T^*\bar{e}_k\rangle \bar{e}_k = \sum_{k=1}^{N}\langle u, v_k\rangle \bar{e}_k$, where $v_k = T^*\bar{e}_k$.
Let $\{e_1, ..., e_L\}$ be the orthonormal subset of H obtained by applying Gram-Schmidt to $\{\bar{e}_1, ..., \bar{e}_L, v_1, ..., v_L\}$
Then $\exists a_{ki}, b_{kj}$ s.t. $\bar{e}_k = \sum_{k=1}^{L} a_{ki}e_i$, $\bar{v}_k = \sum_{j=1}^{L} b_{kj}e_j$.
Then $Tu = \sum_{k=1}^{N}\langle u, v_k\rangle \bar{e}_k = \sum_{i,j=1}^{L}\left(\sum_{k=1}^{N} a_{ki}\overline{b_{kj}}\right)\langle u, e_j\rangle e_i$. We can thus define $C_{ij} = \sum_{k=1}^{N} a_{ki}\overline{b_{kj}}$. □

***Theorem:* 3.31:**

1. If $T \in R(H)$, then $T^* \in R(H)$.
2. If $T \in R(H)$, $A, B \in B(H)$, then $ATB \in R(H)$.

Proof. Write $Tu = \sum_{i,j=1}^{L} C_{ij} \langle u, e_j \rangle e_i$, for $u \in H$. Then $\forall u, v \in H$,

$$\begin{aligned}
\langle u, T^* v \rangle = \langle Tu, v \rangle &= \left\langle \sum_{i,j} C_{ij} \langle u, e_j \rangle e_i, v \right\rangle \\
&= \sum_{i,j} C_{i,j} \langle u, e_j \rangle \langle e_i, v \rangle \\
&= \left\langle u, \sum_{i,j} \overline{C_{ij} \overline{\langle e_i, v \rangle}} e_j \right\rangle \\
&= \left\langle u, \sum_{i,j} \overline{C_{ij}} \langle v, e_i \rangle e_j \right\rangle
\end{aligned}$$

Thus $\left\langle u, T^* v - \sum i, j \overline{C_{ji}} \langle v, e_i \rangle e_j \right\rangle = 0$ for all u, v

Therefore, $T^* v = \sum_{i,j=1}^{L} \overline{C_{ji} \langle v, e_i \rangle e_j}$ for all $v \in H$. Then $T^* \in R(H)$. ☐

3.6.2 Compact Operators

Notice that $R(H)$ is not a closed subset in $B(H)$. *i.e.* if $T_n \in R(H)$ and $\|T_n - T\| \to 0$, $T \in R(H)$ is not necesarily true.
Example: Take $T_n : l^2 \to l^2$ s.t. $T_n a = \left\{\frac{a_1}{1}, \frac{a_2}{2}, ..., \frac{a_n}{n}, 0, ...\right\}$ for $a = \{a_k\}_{k \in l^2}$. Then $T_n \in R(H)$ and $\|T_n - T\| \to 0$, where $Ta = \left\{\frac{a_1}{1}, \frac{a_2}{2}, ...\right\}$. ($\|T_n - T\| \leq \frac{1}{n+1}$). Then $Te_1 = e_1$, $Te_2 = \frac{1}{2}e_2$, $Te_n = \frac{1}{n}e_n$, but $T \notin R(H)$.

***Definition:* 3.14: Compact Operator**

A bounded linear operator $K \in B(H)$ is a compact operator if $\overline{K(\{u \in H : \|u\| < 1\})}$ is compact.

Example: $Ka = \left\{\frac{a_1}{1}, \frac{a_2}{2}, \frac{a_3}{3}, ...\right\}$, $a \in l^2$. Then K is a compact operator.

Example: If $K \in C([0,1] \times [0,1])$ and $Tf(x) = \int_0^1 K(x,y) f(y) dy$, $f \in L^2([0,1])$. T is a compact operator on $L^2([0,1])$. If $K(x,y) = \begin{cases} (x-1)y, 0 \leq y \leq x \leq 1 \\ x(y-1), 0 \leq x \leq y \leq 1 \end{cases}$, then $u = \int_0^1 K(x,y) f(y) dy$ solves $u'' = f$, $u(0) = u(1) = 0$ on $[0,1]$.

Example: I on l^2 is not a compact operator. Let e_n be the nth orthonormal basis vector. Then $\|e_n\| = 1$ and $\|Ie_n - Ie_m\|^2 = 2$, $\forall n \neq m$. Then $\{Ie_n\}$ does not have a convergent subsequence.

***Theorem:* 3.32:**

Let H be a separable Hilbert space. Then $T \in B(H)$ is a compact operator $\Leftrightarrow \exists \{T_n\}_n$ of finite rank operators s.t. $\|T - T_n\| \to 0$. *i.e.* the set of compact operators is the closure of $R(H)$.

Proof. ($\Rightarrow$) Let $\{e_k\}_k$ be an orthonormal basis for H. Since T is a compact operator $\overline{\{Tu : \|u\| \leq 1\}}$ is a compact set, then $\forall \epsilon > 0$, $\exists N \in \mathbb{N}$ s.t. $\sum_{k > N} \|\langle Tu, e_k \rangle\|^2 < \epsilon^2$, $\forall \|u\| \leq 1$ by Theorem 3.27.

For $n \in \mathbb{N}$, define $T_n u = \sum_{k=1}^{n} \langle Tu, e_k \rangle e_k$ for $u \in H$.
Then $T_n \in B(H)$ and Range$(T_n) \subset$ span $\{e_1, ..., e_n\}$, thus $T_n \in R(H)$.
Let $\epsilon > 0$, N as above. Let $n \geq N$. Then if $\|u\| = 1$,

$$\begin{aligned} \|T_n u - Tu\| &= \left\| \sum_{k=1}^{n} \langle Tue_k \rangle e_k - \sum_{k=1}^{\infty} \langle Tu, e_k \rangle e_k \right\|^2 \\ &= \left\| \sum_{k>n} \langle Tu, e_k \rangle e_k \right\|^2 \overset{\text{By Theorem 3.10}}{=} \sum_{k>N} |\langle Tu, e_k \rangle|^2 \\ &\leq \left\| \sum_{k>N} \langle Tu, e_k \rangle e_k \right\|^2 < \epsilon \end{aligned}$$

Thus $\|T_n - T\| < \epsilon$, $\|T_n - T\| \to 0$

($\Leftarrow$) Suppose $\|T_n - T\| \to 0$ with $T_n \in R(H), \forall n$, then $\overline{\{Tu : \|u\| \leq 1\}} \subset \{v : \|v\| \leq \|T\|\}$.
Then $\overline{\{Tu : \|u\| \leq 1\}}$ is closed and bounded.
Claim: $\forall \epsilon > 0$, there exists a finite dimensional subspace W s.t. $\forall \|u\| \leq 1$, $\inf_{w \in W} \|Tu - w\| < \epsilon$.
Since $\|T_n - T\| \to 0$, $\exists N \in \mathbb{N}$ s.t. $\|T_N - T\| < \epsilon$. Let $W =$ Range(T_N). W is a finite dimensional subspace.
Then $\forall \|u\| \leq 1$, $\|Tu - T_N u\| \leq \|T - T_N\| \|u\| \leq \|T - T_N\| < \epsilon$.
Thus, $\inf_{w \in W} \|Tu - w\| < \epsilon$. $T_N u \in W$. By Theorem 3.28, T is compact. □

Theorem: 3.33: Properties of Compact Operators

Let H be a separable Hilbert space, $K(H)$ be the set of compact operators on H. Then
1. $K(H)$ is a closed subspace of $B(H)$
2. If $T \in K(H)$, then $T* \in K(H)$
3. $\forall A, B \in B(H)$, if $T \in K(H)$, $ATB \in K(H)$

Proof. 1. clear because $K(H)$ is the closure of $R(H)$

2. If $T \in K(H)$, by Theorem 3.32, $\exists T_n \in R(H)$ s.t. $\|T_n - T\| \to 0$. Since $T_n^* \in R(H)$, $\|T_n^* - T*\| = \|T_n - T\| \to 0$. Thus $T^* \in K(H)$

3. $T_n \in R(H)$, so $\exists T_n \in R(H)$ s.t. $\|T_n - T\| \to 0$. $AT_nB \in R(H)$ by Theorem 3.31 and $\|AT_nB - ATB\| = \|A(T_n - T)B\| \leq \|A\| \|T_n - T\| \|B\| \to 0$. Thus $ATB \in K(H)$.

□

3.6.3 Spectrum

Theorem: 3.34:

Let $T \in B(H)$. If $\|T\| < 1$, then $I - T$ is invertible and $(I - T)^{-1} = \sum_{n=0}^{\infty} T^n$. (Analogous to $(1 - x)^{-1} = \sum_{n=0}^{\infty} x$ for $|x| < 1$)

Theorem: 3.35: Invertible Linear Operators

The set of invertible linear operators $GL(H) = \{T \in B(H) : T \text{ is bijective}\}$ is an open subset of $B(H)$.

Proof. Let $T_0 \in GL(H)$. Suppose $\|T - T_0\| < \left\|T_0^{-1}\right\|^{-1}$. Then $\left\|T_0^{-1}(T - T_0)\right\| \leq \left\|T_0^{-}1\right\| \|T - T_0\| < 1$. Thus $I - T_0^{-1}(T - T_0) \in GL(H)$. $T = T_0(I - T_0^{-1}(T - T_0)) \in GL(H)$.
i.e. $\left\{\|T - T_0\| < \left\|T_0^{-1}\right\|^{-1}\right\}$ is an open neighborhood of T_0 in $GL(H)$. $GL(H)$ isopen. ☐

Definition: 3.15: Spectrum

Let $A \in B(H)$. The *resolvent set* of A is $\text{Res}(A) = \{\lambda \in \mathbb{C} : A - \lambda I \in GL(H)\}$. The *spectrum* of A is the complement $\text{Spec}(A) = \mathbb{C} \setminus \text{Res}(A)$.

Example: Let $A : \mathbb{C}^2 \to \mathbb{C}^2$, $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$. Then $A - \lambda I = \begin{pmatrix} \lambda_1 - \lambda & 0 \\ 0 & \lambda_2 - \lambda \end{pmatrix}$. $A - \lambda I \in GL(\mathbb{C}^2) \Leftrightarrow \lambda \neq \lambda_1, \lambda_2$. Then $\text{Spec}(A) = \{\lambda_1, \lambda_2\}$, $\text{Res}(A) = \mathbb{C} \setminus \{\lambda_1, \lambda_2\}$.

Definition: 3.16: Eigenvalue and Eigenvector

If $A \in B(H)$ and $A - \lambda I$ is not injective, then $\exists u \in H \setminus \{0\}$ s.t. $Au = \lambda u$. Then $\lambda \in \text{Spec}(A)$ is an eigenvalue of A and u is an eigenvector.

Example: $Ta = \left\{\frac{a_1}{1}, \frac{a_2}{2}, \ldots\right\}$ for $a \in l^2$. Note $Te_n = \frac{1}{n}e_n$, *i.e.* $\left\{T - \frac{1}{n}\right\} e_n = 0$. Then $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}}$ are eigenvalues of T, so $\left\{\frac{1}{n}\right\}_{n\in\mathbb{N}} \subset \text{Spec}(T)$. $0 \in \text{Spec}(T)$ because $T - 0 = T$ is injective but not surjective and thus not invertible, $0 \notin \text{Res}(T)$.

Example: $T : L^2([0,1]) \to L^2([0,1])$ s.t. $Tf(x) = xf(x)$ has not eigenvalues and $\text{Spec}(T) = [0,1]$.

Theorem: 3.36:

Let $A \in B(H)$. Then $\text{Spec}(A)$ is a closed subset of $\mathbb{C}$ and $\text{Spec}(A) \subset \{\lambda \in \mathbb{C} : |\lambda| \leq \|A\|\}$. (Spectrum is a compact subset of $\mathbb{C}$)

Proof. We show that the complement $\text{Res}(A)$ is open and $\{|\lambda| > \|A\|\} \subset \text{Res}(A)$.
Let $\lambda_0 \in \text{Res}(A)$. Since $GL(H)$ is open, then $\exists \epsilon > 0$ s.t. $\|T - (A - \lambda I)\| < \epsilon$. $T \in GL(H)$.
Then if $|\lambda - \lambda_0| < \epsilon$, $\|(A - \lambda I) - (A - \lambda_0 I)\| = \|(\lambda - \lambda_0)I\| = |\lambda - \lambda_0| < \epsilon$. Thus $A - \lambda I \in GL(H)$. $\lambda \in \text{Res}(A)$. So $\{|\lambda - \lambda_0| < \epsilon\} \subset \text{Res}(A)$. $\text{Res}(A)$ is open.
Suppose $|\lambda| > \|A\|$, then $\left\|\frac{1}{\lambda}A\right\| < 1$. $I - \frac{1}{\lambda}A$ is invertible. $A - \lambda I = -\lambda\left(I - \frac{1}{\lambda}A\right) \in GL(H)$. Thus $\lambda \in \text{Res}(A)$. *i.e.* $\{|\lambda| > \|A\|\} \subset \text{Res}(A)$. ☐

Remark 14. Spectrum cannot be empty. If it is, then $\forall u, v \in H$, $f(\lambda) = \left\langle (A - \lambda I)^{-1}u, v\right\rangle$ is continuous, complex differentiable function in λ on $\mathbb{C}$. As $\lambda \to \infty$, $(A - \lambda I)^{-1} \to 0$, but Liouville's theorem tells us that $f(\lambda) \to 0$ as $|\lambda| \to \infty$, f must be identically 0. Then $(A - \lambda I)^{-1} = 0$. Contradiction.

3.6.4 Self-Adjoint Operators

Theorem: 3.37: Self-Adjoint Operators

If $A = A^* \in B(H)$ is a self-adjoint operator, then
1. $\forall u \in H$, $\langle Au, u\rangle$ is real
2. $\|A\| = \sup_{\|u\|=1} |\langle Au, u\rangle|$

Proof. 1. If $u \in H$, $\overline{\langle Au, u\rangle} = \langle u, Au\rangle \overset{A=A^*}{=} \langle u, A^*u\rangle = \langle Au, u\rangle$. Thus $\langle Au, u\rangle$ is real.

2. Let $a = \sup_{\|u\|=1} |\langle Au, u\rangle|$.

Note $\forall \|u\| = 1$, $|\langle Au, u\rangle| \overset{\text{By Theorem 3.1}}{\leq} \|Au\| \|u\| = \|Au\| \overset{\text{Definition 1.10}}{\leq} \|A\|$. Thus $a \leq \|A\|$.
Let $\|u\| = 1$ and $Au \neq 0$. Define $v = \frac{Au}{\|Au\|}$. Then $\|v\| = 1$.

$$\begin{aligned}
&\|Au\| = \langle Au, v\rangle = \mathrm{Re}\,\langle Au, v\rangle \\
&= \frac{1}{4}\mathrm{Re}\left[\langle A(u+v), (u+v)\rangle - \langle A(u-v), u-v\rangle + i\,\langle A(u+iv), u+iv\rangle - i\,\langle A(u-iv), u-iv\rangle\right] \\
&= \frac{1}{4}\left(\langle A(u+v), (u+v)\rangle - \langle A(u-v), u-v\rangle\right) \\
&\leq \frac{1}{4}\left(a\,\|u+v\|^2 + a\,\|u-v\|^2\right) \\
&= \frac{a}{4}\left(2\,\|u\|^2 + 2\,\|v\|^2\right) \text{ (By Theorem 3.4)} \\
&= a
\end{aligned}$$

Thus $\forall \|u\| = 1$, $\|Au\| \leq a \Rightarrow \|A\| \leq a$
Thus $a = \|A\|$

□

Remark 15. In quantum mechanics, observables (positions, momentum, etc) are modeled by self-adjoint unbounded operators. All things measured in nature (the eigenvalues) are real.

Theorem: 3.38: Spectrum of Self-Adjoint Operator

Suppose $A = A^* \in B(H)$. Then
1. $\mathrm{Spec}(A) \subset [-\|A\|, \|A\|] \subset \mathbb{C}$
2. At least one of $\pm\|A\| \in \mathrm{Spec}(A)$

Proof. 1. Since $\mathrm{Spec} \subset \{|\lambda| \leq \|A\|\}$, we just need to show $\mathrm{Spec}(A) \subset \mathbb{R}$.
We show that if $\lambda = s + it$, $t \neq 0$, then $\lambda \in \mathrm{Res}(A)$.
Suppose $\lambda = s + it$, $s, t \in \mathbb{R}$, $t \neq 0$, then $A - \lambda = (A - s) - it = \tilde{A} - it$, where $\tilde{A} = A - s = \tilde{A}^*$.
Then $\tilde{A} - it$ is bijective $\Leftrightarrow A - \lambda$ is bijective, so we only need to consider the case $s = 0$.
Since $\langle Au, u\rangle$ is real, then $\mathrm{Im}\left(\langle (A - it)u, u\rangle\right) = -t\,\|u\|^2$. Thus $(A-it)u = 0 \Leftrightarrow u = 0$. $\mathrm{Nnull}(A-it) = \{0\}$, so $A - it$ is injective.
Similarly, $(A-it)^* = A+it$ is injective. $\mathrm{Range}(A-it)^\perp \overset{\text{Theorem 3.24}}{=} \mathrm{Null}((A-it)^*) = \mathrm{Null}(A+it) = \{0\}$
So $\overline{\mathrm{Range}(A - it)} = \left(\mathrm{Range}(A - it)^\perp\right)^\perp = \{0\}^\perp = H$.

Now we show that Range$(A - it)$ is closed.
Suppose $(A - it)u_n \to v$. Then

$$\begin{aligned} |t| \|u_n - u_m\|^2 &= |\text{Im}\left(\langle (A - it)(u_n - u_m), u_n - u_m\rangle\right)| \\ &\le \|(A - it)u_n - (A - it)u_m\| \|u_n - u_m\| \end{aligned}$$

Thus, $\|u_n - u_m\| \le \frac{1}{|t|} \|(A - it)u_n - (A - it)u_m\|$.
Since $\{(A - it)u_n\}_n$ is Cauchy (converges), then u_n is Cauchy. $\exists u \in H$ s.t. $u_n \to u$.
Then $(A - it)u = \lim_{n\to\infty} (A - it)u_n = v$. Thus $v \in$ Range$(A - it)$. Range$(A - it)$ is closed. Therefore $A - it$ is bijective.

2. Since $\|A\| = \sup_{\|u\|=1} |\langle Au, u\rangle|$, then $\exists \|u_n\| = 1$ s.t. $\langle Au_n, u_n\rangle \to \|A\|$ or $-\|A\|$ as $n \to \infty$.
Then $\langle (A \pm \|A\|)u_n, u_n\rangle \to 0$ as $n \to \infty$. We want to show that $A \pm \|A\|$ is not invertible.
Suppose $A \pm \|A\|$ is invertible, then

$$\begin{aligned} 1 = \|u_n\| &= \left\|(A \pm \|A\|)^{-1}(A \pm \|A\|)u_n\right\| \\ &\le \left\|(A \pm \|A\|)^{-1}\right\| \|(A \pm \|A\|)u_n\| \to 0 \end{aligned}$$

Contradiction. Thus $A \pm \|A\|$ is not invertible. $\pm\|A\| \in$ Spec(A).

□

Theorem: 3.39:

If $A = A^* \in B(H)$, and $a_- = \inf_{\|u\|=1} \langle Au, u\rangle$, $a_+ = \sup_{\|u\|=1} \langle Au, u\rangle$, then $a_\pm \in$ Spec$(A) \subset [a_-, a_+]$.

Proof. Note that $|\langle Au, u\rangle| \le \|A\|$ for all $\|u\| = 1$. Then $-\|A\| \le a_- \le a_+ \le \|A\|$.
By definition of $a_\pm$, $\exists \|u_n^\pm\| = 1$ s.t. $\langle Au_n^\pm, u_n^\pm\rangle \to a_\pm$. *i.e.* $\langle (A - a_\pm)u_n^\pm, u_n^\pm\rangle \to 0$.
By the same argument as in Theorem 3.38, $a_\pm \in$ Spec(A).
Let $b = \frac{a_- + a_+}{2}$, $B = A - bI$. Then $B^* = B \in B(H)$, so by Theorem 3.38, Spec$(B) \subset [-\|B\|, \|B\|]$, and therefore, Spec$(A) \subset [-\|B\| + b, \|B\| + b]$ by linearity.
$\|B\| = \sup_{\|u\|=1} |\langle Bu, u\rangle| = \sup_{\|u\|=1} \left|\langle Au, u\rangle - \frac{a_- + a_+}{2}\right| = \frac{a_+ - a_-}{2}$, since $\langle Au, u\rangle \in [a_-, a_+]$ and $\frac{a_+ + a_-}{2}$ is the midpoint, the supremum is half of the length.
Thus Spec$(A) \subset [a_-, a_+]$ □

Theorem: 3.40:

Let $A^* = A \in B(H)$, then $\forall u, \langle Au, u\rangle \ge 0 \Leftrightarrow$ Spec$(A) \subset [0, \infty)$

Definition: 3.17: Eigenspace

If $A \in B(H)$, define the eigenspace $E_\lambda = \text{Null}(A - \lambda I) = \{u \in H : (A - \lambda I)u = 0\}$

Theorem: 3.41: Compact Self-Adjoint Operators

Suppose $A^* = A \in B(H)$ is a compact self-adjoint operator. Then

1. If $\lambda \ne 0$ is an eigenvalue of A, then dimE_λ is finite and $\lambda \in \mathbb{R}$
2. If $\lambda_1 \ne \lambda_2$ are eigenvalues of A, then E_{λ_1} and E_{λ_2} are orthogonal.
3. The set of nonzero eigenvalues of A is either finite or countable. If it is countably infinite, then $\lim_{n\to\infty} |\lambda_n| = 0$

Proof. 1. Suppose $\lambda \neq 0$ and $\dim E_\lambda = \infty$. Then by Gram-Schmidt process, there exists a sequence $\{u_n\}_n$ of orthonormal elements in E_λ.
Since A is a compact operator, $\{Au_n\}_n$ has a convergent subsequence $\{Au_{n_j}\}_j$.
Then $\{Au_{n_j}\}_j$ is Cauchy, but $\|Au_{n_j} - Au_{n_k}\|^2 = \|\lambda u_{n_j} - \lambda u_{n_k}\|^2 = |\lambda|^2 \|u_{n_j} - u_{n_k}\|^2 = 2|\lambda|^2$ does not converge to 0, since u_n are orthonormal. Contradiction.
If $\|u\| = 1$, $Au = \lambda u$, then $\lambda = \lambda \langle u, u\rangle = \langle \lambda u, u\rangle = \langle Au, u\rangle = \langle u, Au\rangle = \langle u, \lambda u\rangle = \bar{\lambda} \langle u, u\rangle = \bar{\lambda}$. Thus $\lambda \in \mathbb{R}$.

2. Suppose $\lambda_1 \neq \lambda_2$, $u_1 \in E_{\lambda_1}$ and $u_2 \in E_{\lambda_2}$.
Then $\lambda \langle u_1, u_2\rangle = \langle \lambda u_1, u_2\rangle = \langle Au_1, u_2\rangle = \langle u_1, Au_2\rangle = \langle u_1, \lambda_2 u_2\rangle = \lambda_2 \langle u_1, u_2\rangle$.
Then $(\lambda_1 - \lambda_2)\langle u_1, u_2\rangle = 0$, but $\lambda_1 \neq \lambda_2$, we must have $\langle u_1, u_2\rangle = 0$. *i.e.* E_{λ_1} and E_{λ_2} are orthogonal.

3. Let $\Lambda = \{\lambda \neq 0 : Au = \lambda u\}$ be the set of nonzero eigenvalues.
Claim: If $\{\lambda_n\}_{n=1}^\infty$ is a sequence of distinct nonzero eigenvalues of A, then $\lambda_n \to 0$.
Define $\Lambda_N = \left\{\lambda \in \Lambda : |\lambda| \geq \frac{1}{N}\right\}$. Λ_N is finite for all N, otherwise we can take a sequence in Λ_N that doesn't converge to 0. Then $\Lambda = \bigcup_{N\in\mathbb{N}} \Lambda_N$ is countable.
Let $\{u_n\}_n$ be a sequence in H s.t. $\|u_n\| = 1$ and $\forall n$, $Au_n = \lambda_n u_n$. Then $|\lambda_n| = \|\lambda_n u_n\| = \|Au_n\|$.
Assume $\|Au_n\| \not\to 0$. Then $\exists \epsilon_0 > 0$ and $\{Au_{n_j}\}_j$ s.t. $\forall j$, $\|Au_{n_j}\| \geq \epsilon_0$
Since A is a compact operator, there exists a subsequence $e_k = u_{n_{j_k}}$ of $\{u_{n_j}\}_j$ s.t. $\{Ae_k\}_k$ converges in H and $\|Ae_k\| \geq \epsilon_0$ for all k.
Note $\forall k \neq l$, $\langle e_k, e_l\rangle = \langle u_{n_k}, u_{n_l}\rangle = 0$.
Let $f = \lim_{k\to\infty} Ae_k$. Then $\epsilon_0^2 \leq \|f\|^2 = \langle f, f\rangle = \lim_{k\to\infty} \langle Ae_k, f\rangle = \lim_{k\to\infty} \langle e_k, Af\rangle$.
By Theorem 3.5, $\sum_k |\langle e_k, Af\rangle|^2 \leq \|Af\|^2 < \infty$.
Thus $\lim_{k\to\infty} \langle e_k, Af\rangle = 0$. Contradiction. Therefore, $|\lambda_n| = \|Au_n\| \to 0$.

□

3.6.5 Spectral Theorem

Theorem: 3.42: Fredholm Alternative

Let $A = A^* \in B(H)$ be a compact operator and $\lambda \in \mathbb{R} \setminus \{0\}$. Then $\mathrm{Range}(A - \lambda I)$ is closed and thus $\mathrm{Range}(A - \lambda I) = \left(\mathrm{Range}(A - \lambda I)^\perp\right)^\perp = \mathrm{Null}(A - \lambda I)^\perp$. Therefore, either $A - \lambda I$ is bijective or $\mathrm{Null}(A - \lambda I)$ is nontrivial and finite dimensional.

Remark 16. 1. $f \in \mathrm{Range}(A - \lambda I) \Leftrightarrow f \in \mathrm{Null}(A - \lambda I)^\perp$

2. Since $\mathrm{Spec}(A) \subset \mathbb{R}$, $\mathrm{Spec}(A) \setminus \{0\} = \{\text{eigenvalues of } A\}$.

Proof. Suppose $(A - \lambda I)u_n \to f \in H$. We want to show that $f \in \mathrm{Range}(A - \lambda I)$.
Let $v_n = \Pi_{\mathrm{Null}(A-\lambda I)^\perp} u_n$ (the projection of u_n onto $\mathrm{Null}(A - \lambda I)^\perp$).
Then $(A - \lambda I)u_n = (A - \lambda I)\left(\Pi_{\mathrm{Null}(A-\lambda I)} u_n + v_n\right) = (A - \lambda I)v_n$. Then $(A - \lambda I)v_n = (A - \lambda I)u_n \to f$.

Claim: $\{v_n\}_n$ is bounded.
Assume it is not bounded, then there exists a subseqeunce $\{v_{n_j}\}_j$ s.t. $\|v_{n_j}\| \to \infty$.
Then $(A - \lambda I)\frac{v_{n_j}}{\|v_{n_j}\|} = \frac{1}{\|v_{n_j}\|}(A - \lambda I)v_{n_j} \to 0f = 0$

Since A is a compact operator, there exists a subsequence $\{v_{n_k}\}_k$ of $\{v_{n_j}\}_j$ s.t. $\left\{A\left(\frac{v_{n_k}}{\|v_{n_k}\|}\right)\right\}_k$ converges.

Then $\frac{v_{n_k}}{\|v_{n_k}\|} = \frac{1}{\lambda}\left[A\left(\frac{v_{n_k}}{\|v_{n_k}\|}\right) - (A-\lambda I)\left(\frac{v_{n_k}}{\|v_{n_k}\|}\right)\right]$. Therefore, $\left\{A\left(\frac{v_{n_k}}{\|v_{n_k}\|}\right)\right\}_k$ converges to an element $v \in \text{Null}(A-\lambda I)^\perp$.
Then $\|v\| = \lim_{k\to\infty}\left\|\frac{v_{n_k}}{\|v_{n_k}\|}\right\| = 1$ and $(A-\lambda I)v = \lim_{k\to\infty}(A-\lambda I)\left(\frac{v_{n_k}}{\|v_{n_k}\|}\right) = 0$.
Therefore, $v \in \text{Null}(A-\lambda I)\cap\text{Null}(A-\lambda I)^\perp = \{0\}$, $v = 0$. Contradiction, since $\|v\| = 1$. Thus, $\{v_n\}_n$ must be bounded.

Since $\{v_n\}_n$ is bounded and A is a compact operator, then there exists a subsequence $\{v_{n_j}\}_j$ s.t. $\{Av_{n_j}\}_j$ converges.
Then, $v_{n_j} = \dfrac{1}{\lambda}\left(Av_{n_j} - (A-\lambda I)v_{n_j}\right)$ converges to an element v.
Then $f = \lim_{j\to\infty}(A-\lambda I)v_{n_j} = (A-\lambda I)v$, so $f \in \text{Range}(A-\lambda I)$. □

***Theorem:* 3.43:**

Let $A = A^* \in B(H)$ be a non-trivial compact operator. Then A has a non-trivial eigenvalue λ_1 with $|\lambda_1| = \sup_{\|u\|=1}|\langle Au,u\rangle| = |\langle Au_1,u_1\rangle|$, where $\|u_1\| = 1$ satisfies $Au_1 = \lambda_1 u_1$.

Proof. In Theorem 3.38, we have shown that at least one of $\pm\|A\| \in \text{Spec}(A)$ for a self-adjoint operator $A^* = A$.
Then $\pm\|A\|$ is an eigenvalue of A by Theorem 3.42, and $|\lambda_1| = \sup_{\|u\|=1}|\langle Au,u\rangle|$ from $\|A\| = |\pm\|A\|| = \sup_{\|u\|=1}|\langle Au,u\rangle|$.
And $|\langle Au_1,u_1\rangle|$ comes from the fact that eigenvalues are associated with eigenvectors. □

***Theorem:* 3.44: Maximum Principle**

Let $A = A^* \in B(H)$ be a compact operator. Then the nonzero eigenvalues of A can be ordered $|\lambda_1| \geq |\lambda_2| \geq \cdots$ (counted with multiplicity) with corresponding orthonormal eigenfunctions $\{u_k\}$ s.t. $|\lambda_j| = \sup_{\|u\|=1, u\in\text{span}\{u_1,\ldots,u_{j-1}\}^\perp}|\langle Au,u\rangle| = |\langle Au_j,u_j\rangle|$ and if the sequence does not terminate, $|\lambda_j| \to 0$ as $j\to\infty$.

Proof. The construction proceeds inductively.
Base case: $j = 1$ follows from Theorem 3.43.
Induction: Suppose we have $\lambda_1,\ldots,\lambda_n$ along with orthonormal eigenvectors $u_1,\ldots,u_n$ s.t. $|\lambda_1| \geq |\lambda_2| \geq \cdots$.
Case 1: $Au = \sum_{k=1}^{n}\lambda_k\langle u,u_k\rangle u_k$, we found all eigenvlaues and the process terminates. A is a finite rank operator
Case 2: $Au \neq \sum_{k=1}^{n}\lambda_k\langle u,u_k\rangle u_k$. Let $A_n u = Au - -\sum_{k=1}^{n}\lambda_k\langle u,u_k\rangle u_k \neq 0$
Then A_n is a self-adjoint compact operator and

1. $\forall u \in \text{span}\{u_1,\ldots,u_n\}$, $A_n u = 0$
2. $\forall u \in \text{span}\{u_1,\ldots,u_n\}^\perp$, $A_n u = Au$
3. $\forall u \in H$, $v \in \text{span}\{u_1,\ldots,u_n\}$, $\langle A_n u,v\rangle = \langle u,A_n v\rangle = 0$, so $A_n u \in \text{span}u_1,\ldots,u_n{}^\perp$. $\text{Range}(A_n) \subset \text{span}u_1,\ldots,u_n{}^\perp$

4. If $A_n u = \lambda u \neq 0$, then $u \in \text{Range}(A_n) \subset \text{span}\{u_1, ..., u_n\}^\perp$. Thus $Au = A_n u = \lambda u$, *i.e.* λ is an eigenvalue of A

By Theorem 3.43, A_n has a nonzero eigenvalue λ_{n+1} with unit eigenvector u_{n+1} s.t.

$$\begin{aligned}
|\lambda_{n+1}| = |\langle Au_{n+1}, u_{n+1}\rangle| &= \sup_{\|u\|=1} |\langle A_n u, u\rangle| \\
&= \sup_{\|u\|=1, u\in\text{span}\{u_1,...,u_n\}^\perp} |\langle A_n u, u\rangle| \\
&= \sup_{\|u\|=1, u\in\text{span}\{u_1,...,u_n\}^\perp} |\langle Au, u\rangle| \\
&\leq \sup_{\|u\|=1, u\in\text{span}\{u_1,...,u_{n-1}\}^\perp} |\langle Au, u\rangle| = |\lambda_n|
\end{aligned}$$

□

Theorem: 3.45: Spectral Theorem

Let $A = A^* \in B(H)$ be a compact operator on a separable Hilbert space H. Let $|\lambda_1| \geq |\lambda_2| \geq \cdots$ be the nonzero eigenvalues of A (counted with multiplicity) with corresponding orthonormal eigenvalues $\{u_k\}_k$. Then

1. $\{u_k\}_k$ is an orthonormal basis for Range(A).
2. $\{u_k\}_k$ is an orthonormal basis for $\overline{\text{Range}(A)}$ and $\exists$ orthonormal basis $\{f_j\}_j$ of Null(A) s.t. $\{u_k\}_k \cup \{f_j\}_j$ is an orthonormal basis for H.

Proof. 1. The process of obtaining $|\lambda_1| \geq |\lambda_2| \geq \cdots$ and eigenvectors $\{u_k\}_k$ terminates $\Leftrightarrow \exists n$ s.t. $Au = \sum_{k=1}^{n} \lambda_k \langle u, u_k\rangle u_k$. In this case Range$(A) = \text{span}\{u_1, ..., u_k\}$.
Suppose the process does not terminate, $\{\lambda_k\}_k$ is countably infinite, $\lambda_k \to 0$ by Theorem 3.44.
Claim: If $f \in \text{Range}(A)$ and $\forall k, \langle f, u_k\rangle = 0$, then $f = 0$
Suppose $f = A_k$ and $\langle f, u_k\rangle = 0, \forall k$.
Then $\forall k$, $\lambda_k \langle u, u_k\rangle = \langle u, \lambda_k u_k\rangle = \langle u, Au_k\rangle = \langle Au, u_k\rangle = \langle f, u_k\rangle = 0\ \forall k$.
By Theorem 3.44, $\|f\| = \|Au\| = \left\|\left(A - \sum_{k=1}^{n} \lambda_k \langle u, u_k\rangle u_k\right) u\right\| = \|A_n u\| \leq |\lambda_{n+1}| \|u\| \to 0$. Thus $\|f\| = 0$, $f = 0$.

2. By part 1, $\overline{\text{Range}(A)} \subset \overline{\text{span}\{u_k\}_k} = \left\{\sum_k c_k u_k : \sum_k |c_k|^2 < \infty\right\}$.
Thus u_k is an orthonormal basis for $\overline{\text{Range}(A)} = \left(\text{Range}(A)^\perp\right)^\perp = (\text{Null}(A))^\perp$.
Since H is separable, Null(A) is separable, $\exists$ an orthonormal basis $\{f_j\}_j$ of Null(A), so $\{f_j\}_j \cup \{u_k\}_k$ is an orthonormal basis for Null$(A) \oplus \text{Null}(A)^\perp = H$.

□

3.7 Dirichlet Problem

Let $V \in C([0,1])$ be a real valued function. Consider $\begin{cases} -u''(x) + V(x)u(x) = f(x) \\ u(0) = u(1) = 0 \end{cases}$, $x \in [0,1]$. Given $f \in C([0,1])$, does there exist a unique solution $u \in C^2([0,1])$ to the problem? If $V(x) \geq 0$, then yes. Otherwise, it depends on f.

Theorem: 3.46:

Let $V \geq 0$. If $f \in C([0,1])$, $u_1, u_2 \in C^2([0,1])$ solve the problem, then $u_1 = u_2$.

Proof. Ket $u = u_1 - u_2$. Then $\begin{cases} -u''(x) + V(x)u(x) = 0 \\ u(0) = u(1) = 0 \end{cases}$.

$$\begin{aligned}
0 &= \int_0^1 \left(-u''(x) + V(x)u(x)\right) \overline{u(x)} dx \\
&= -\int_0^1 u''(x)\overline{u(x)} dx + \int_0^1 V(x)|u(x)|^2 dx \\
&= -u'(x)\overline{u(x)}|_0^1 + \int_0^1 u'(x)\overline{u'(x)} dx + \int_0^1 V(x)|u(x)|^2 dx \text{ (IBP)} \\
&= \int_0^1 |u'|^2 + \int_0^1 V|u|^2 \text{ (Boundary Condition)} \\
&\geq \int_0^1 |u'|^2 \ (V \geq 0)
\end{aligned}$$

Thus $\int_0^1 |u'|^2 = 0$, $u' = 0$, u is constant, and $u = 0$, so $u_1 = u_2$. □

We now want to show the existence of solution, firstly consider $V = 0$ case

Theorem: 3.47:

Let $K(x,y) = \begin{cases} (x-1)y, 0 \leq y \leq x \leq 1 \\ (y-1)x, 0 \leq x \leq y \leq 1 \end{cases}$, $K \in C([0,1] \times [0,1])$. Define $Af(x) = \int_0^1 K(x,y)f(y)dy$. Then $A \in B(L^2([0,1]))$ is a compact self-adjoint operator and if $f \in C([0,1])$, then $u = Af$ is the unique solution for $\begin{cases} -u''(x) = f \\ u(0) = u(1) = 0 \end{cases}$ on $[0,1]$.

Proof. If $C = \sup_{[0,1]^2} |K| < \infty$, then by Theorem 3.1,

$$\begin{aligned}
|Af(x)| &= \left| \int_0^1 K(x,y)f(y)dy \right| \leq C \int_0^1 |f(y)| \, dy \\
&\leq C \left(\int_0^1 1^2 \right)^{1/2} \left(\int_0^1 |f|^2 \right)^{1/2} = C \, \|f\|_2
\end{aligned}$$

And $|Af(x) - Af(z)| \leq \sup_{y \in [0,1]} |K(x,y) - K(z,y)| \, \|f\|_2$.

These two estimates and Arzela-Ascoli theorem (sufficient condition for a sequence of functions to have a convergent subsequence) give that A is a compact operator on $L^2([0,1])$.

Let $f, g \in C([0,1])$. Then

$$\begin{aligned}\langle Af, g\rangle &= \int_0^1 \left(\int_0^1 K(x,y)f(y)dy\right)\overline{g(x)}dx \\ &= \int_0^1\int_0^1 K(x,y)f(y)\overline{g(x)}dydx \\ &= \int_0^1 f(y)\overline{\left(\int_0^1 \overline{K(x,y)}g(x)dx\right)}dy \\ &= \langle f, Bg\rangle ,\end{aligned}$$

where $Bg(x) = \displaystyle\int_0^1 \overline{K(y,x)}g(y)dy = \int_0^1 K(x,y)g(y)dy = Ag(x)$. *i.e.* $\langle Af, g\rangle = \langle f, Ag\rangle$, $\forall f, g \in C([0,1]) \subset L^2([0,1])$
Since $C([0,1])$ is dense in $L^2([0,1])$, $\langle Af, g\rangle = \langle f, Ag\rangle$, $\forall f, g \in L^2([0,1])$. Thus $A^* = A$ is a self-adjoint operator.
If $f \in C([0,1])$, then $u(x) = Af(x) = (x-1)\displaystyle\int_0^x yf(y)dy + x\int_x^1 (y-1)f(y)dy$.
By FTC, $u \in C^2([0,1])$ with $-u'' = f$. □

For $V \neq 0$, $\begin{cases} -u'' + Vu = f \\ u(0) = u(1) = 0 \end{cases}$ $\Leftrightarrow -u'' = f - Vu \Leftrightarrow u = A(f - Vu)$ by leeting $f - Vu = g$ and apply Theorem 3.47 $\Leftrightarrow (I + AV)u = Af$.
Write $u = A^{1/2}v$, then $A^{1/2}\left(I + A^{1/2}VA^{1/2}\right)v = Af$. Thus $\left(I + A^{1/2}VA^{1/2}\right)v = A^{1/2}f$.
Note $\left(A^{1/2}VA^{1/2}\right)^* = A^{1/2}VA^{1/2}$ is a compact self-adjoint operator.

***Theorem:* 3.48:**

Null$(A) = \{0\}$ and the orthonormal eigenvectors for A are given by $u_k(x) = \sqrt{2}\sin(k\pi x)$, $k \in \mathbb{N}$ with eigenvalues $\lambda_k = \frac{1}{k^2\pi^2}$.

Remark 17. By Theorem 3.45, $\left\{\sqrt{2}\sin k\pi x\right\}_{k=1}^{\infty}$ is an orthonormal basis for $L^2([0,1])$

Proof. We show that $\overline{\text{Range}(A)} = L^2([0,1])$.
Let u be a polynomial on $[0,1]$, $f = -u''$ with $u(0) = u(1) = 0$.
By Theorem 3.47, Af is the unique solution to Dirichlet problem with $V = 0$, *i.e.* $(-Af)'' = f$ and $Af(0) = Af(1) = 0$, so $Af = u$.
Since the set of polynomials on $[0,1]$ vanishing at $x = 0, 1$ is dense in $L^2([0,1])$ (from density of $C([0,1])$ and Weierstrass Approximation Theorem), Range(A) contains a dense subset of $L^2([0,1])$. Therefore, $\overline{\text{Range}(A)} = L^2([0,1])$.
Since $\text{Null}(A)^{\perp} = \overline{\text{Range}(A)}$, then Null$(A) = \{0\}$
Suppose $\lambda \neq 0$, $\|u\|_2 = 1$ and $Au = \lambda u$. Then because $Af \in C([0,1])$ by the bound of $|Af(x) - Af(z)|$, $u = \frac{1}{\lambda}Au \in C([0,1])$.
Thus $Au \in C^2([0,1])$, $u = \frac{1}{\lambda}Au \in C^2([0,1]) \Rightarrow -u'' = \frac{1}{\lambda}u$ gives that $u(x) = A\sin\left(\frac{1}{\sqrt{\lambda}}x\right) + B\cos\left(\frac{1}{\sqrt{\lambda}}x\right)$.
Since $u(0) = 0$, $B = 0$, $u(x) = A\sin\left(\frac{1}{\sqrt{\lambda}}x\right)$ and $A \neq 0$. $u(1) = 0 \Rightarrow \frac{1}{\sqrt{\lambda}} = n\pi$ for $n \in \mathbb{N}$.
Thus $u(x) = A\sin k\pi x$, and $A = \sqrt{2}$ from $\|u\|_2 = 1$. □

Definition: 3.18: Series Solution

If $f \in L^2([0,1])$, $f(x) = \sum_{k=1}^{\infty} c_k\sqrt{2}\sin k\pi x$, $c_k = \int_0^1 f(x)\sqrt{2}\sin k\pi x dx$. Define the operation $A^{1/2}f(x) = \sum_{k=1}^{\infty} \frac{1}{k\pi} c_k\sqrt{2}\sin k\pi x$. (Essentially, $A^{1/2}$ multiplies every term by $\frac{1}{k\pi}$)

Theorem: 3.49:

$A^{1/2}$ is a compact self-adjoint operator on $L^2([0,1])$ and $\left(A^{1/2}\right)^2 = A$.

Proof. Let $f(x) = \sum_{k=1}^{\infty} c_k\sqrt{2}\sin k\pi x$, $g(x) = \sum_{k=1}^{\infty} d_k\sqrt{2}\sin k\pi x$. Then

$$\left\|A^{1/2}f\right\|_2^2 = \left\|\sum_{k=1}^{\infty} \frac{c_k}{k\pi}\sqrt{2}\sin k\pi x\right\|_2^2 = \sum_{k=1}^{\infty} \frac{\left||c_k|^2\right|}{k^2\pi^2} \le \frac{1}{\pi^2}\sum_{k=1}^{\infty} |c_k|^2 = \frac{\|f\|_2^2}{\pi^2}$$

Then $\left\langle A^{1/2}f, g\right\rangle = \sum_{k=1}^{\infty} \frac{c_k}{k\pi}\overline{d_k} = \sum_{k=1}^{\infty} c_k \frac{\overline{d_k}}{k\pi} = \left\langle f, A^{1/2}g\right\rangle$. $A^{1/2}$ is self-adjoint.

$$\begin{aligned}
A^{1/2}\left(A^{1/2}f\right) &= A^{1/2}\left(\sum_{k=1}^{\infty} \frac{c_k}{k\pi}\sqrt{2}\sin k\pi x\right) \\
&= \sum_{k=1}^{\infty} \frac{c_k}{k^2\pi^2}\sqrt{2}\sin k\pi x \\
&= \sum_{k=1}^{\infty} c_k A\sqrt{2}\sin k\pi x \\
&= A\sum_{k=1}^{\infty} c_k\sqrt{2}\sin k\pi x = Af
\end{aligned}$$

To show that A is compact, it suffieces to show $\left\{A^{1/2}f : \|f\|_2 \le 1\right\}$ has equi-small tails.
Let $\epsilon > 0$. Choose $N \in \mathbb{N}$ s.t. $\frac{1}{N^2} < \epsilon^2$. Let $\|f\|_2 \le 1$.

$$\begin{aligned}
\sum_{k>N}\left|\left\langle A^{1/2}f, \sqrt{2}\sin k\pi x\right\rangle\right|^2 &= \sum_{k>N} \frac{|c_k|^2}{k^2\pi^2} \\
&\le \frac{1}{N^2}\sum_{k=1}^{\infty} |c_k|^2 = \frac{1}{N^2}\|f\|_2^2 \\
&\le \frac{1}{N^2} < \epsilon^2
\end{aligned}$$

Thus, A is compact. □

Theorem: 3.50:

Let $V \in C([0,1])$ be real valued, and define $m_V f(x) = V(x)f(x)$ for $f \in L^2([0,1])$. Then $m_V \in B(L^2([0,1]))$ is self-adjoint.

Theorem: 3.51:

Let $V \in C([0,1])$ be real valued. Then $T = A^{1/2} m_V A^{1/2}$ satisfies

1. T is a self-adjoint compact operator on $L^2([0,1])$
2. $T \in B(L^2([0,1]), C([0,1]))$

Proof. 1. follows from Theorem 3.49 and Theorem 3.50.

2. Since $m_V \in B(L^2([0,1]))$, it suffices to show $A^{1/2} \in B(L^2([0,1]), C([0,1]))$.
Let $f(x) = \sum_{k=1}^{\infty} c_k \sqrt{2} \sin k\pi x$. Then $A^{1/2} f(x) = \sum_{k=1}^{\infty} \frac{c_k}{k\pi} \sqrt{2} \sin k\pi x$ by Definition 3.18.
Since $\left| \frac{c_k}{k\pi} \sqrt{2} \sin k\pi x \right| \leq \frac{|c_k| \sqrt{2}}{k\pi} \leq \frac{|c_k|}{j}$ and $\sum_{k=1}^{\infty} \frac{|c_k|}{k} \leq \left(\sum_k \frac{1}{k^2} \right)^{1/2} \left(\sum_k |c_k|^2 \right)^{1/2} < \sqrt{\frac{\pi^2}{6}} \|f\|_2$ by Theorem 3.1, then by Weierstrass M-test, $A^{1/2} f \in C)[0,1]$ and $\left| A^{1/2} f(x) \right| \leq \sqrt{\frac{\pi^2}{6}} \|f\|_2$.

□

Theorem: 3.52:

Let $V \in C([0,1])$ with $V \geq 0$ and let $f \in C([0,1])$. Then there exists a unique $u \in C^2([0,1])$ solving
$$\begin{cases} -u'' + Vu = f \\ u(0) = u(1) = 0 \end{cases} \quad \text{on } [0,1]$$

Proof. The plan is to have $u = A^{1/2} \left(I + A^{1/2} m_V A^{1/2} \right)^{-1} A^{1/2} f$.
By Theorem 3.51, $A^{1/2} m_V A^{1/2}$ is a self-adjoint compact operator.
Then by Theorem 3.42, $\left(I + A^{1/2} m_V A^{1/2} \right)^{-1}$ exists $\Leftrightarrow$ Null $\left(I + A^{1/2} m_V A^{1/2} \right) = \{0\}$
Suppose $\left(I + A^{1/2} m_V A^{1/2} \right) g = 0$, then

$$\begin{aligned} 0 &= \left\langle \left(I + A^{1/2} m_V A^{1/2} \right) g, g \right\rangle = \|g\|_2^2 + \left\langle A^{1/2} m_V A^{1/2} g, g \right\rangle \\ &= \|g\|_2^2 + \left\langle m_V A^{1/2} g, A^{1/2} g \right\rangle \text{ (Self-adjoint)} \\ &= \|g\|_2^2 + \int_0^1 V \left(A^{1/2} g \right) \overline{\left(A^{1/2} \right) g} dx \\ &= \|g\|_2^2 + \int_0^1 V \left| A^{1/2} g \right|^2 \geq \|g\|_2^2 \end{aligned}$$

Thus $\|g\|_2 = 0$, $g = 0$. Then $\left(I + A^{1/2} m_V A^{1/2} \right)^{-1}$ exists.
Define $v = \left(I + A^{1/2} m_V A^{1/2} \right)^{-1} A^{1/2} f$, $u = A^{1/2} v$.
Thus $u + A(Vu) = A^{1/2} v + A^{1/2} \left(A^{1/2} m_V A^{1/2} \right) v = A^{1/2} \left(I + \left(A^{1/2} m_V A^{1/2} \right) \right) v = A^{1/2} A^{1/2} f = Af$.
Taking the derivatives gives $u'' - Vu = -f$, so $-u'' + Vu = f$. u solves the Dirichlet problem. □