MAT1856 Introduction to Mathematical Finance

0 Introduction

Given a bond, we can compound the interests

1. annually: $P_1 = \sum_i p_i(1+r)^{-t_i}$
2. n-times a year: $P_n = \sum_i p_i \left(1+\frac{r}{n}\right)^{-t_i n}$

As $n \to \infty$, we have $P_\infty = \sum_i p_i e^{-rt_i}$.

We call P the *dirty price*, it is the accrued interest + clean price, which is also equivalent to $\frac{n}{365}\times$ Annual coupon rate, where n is the number of days since last coupon payment.

Zero coupon bonds:
bonds that contain a single cash flow, *i.e.* a single payment at the time of maturity.

Bond has three variables:

1. Notional (payment to occur at maturity)
2. Price of the bond
3. Time to maturity

We can then calculate the yield $r(T) = -\frac{\log(P/N)}{T}$, where P is the bond price, N is the notional, and T is the time to maturity.

Cashflow valuation formula:
Yield curve can then be used to calculate the price of any series of future cash flows.
With dirty price $P = \sum_i p_i e^{-rt_i}$, we can do the following Bootstrapping to recover yield curve:

$$P = P_1 e^{-r(t_1)t_1} + P_2 e^{-r(t_2)t_2}, \text{ where } 0 < t_1 < \frac{1}{2} < t_2 < 1$$

If we have a bond maturity between 6 months and 1 year. The bond has a coupon payment within 6 months and another payment between 6 months and 1 year.

For maturity less than 6 months, all coupons are zero-coupon bonds.

We can extend to infinity, allowing us to calculate the yield curve for all maturities, assuming coupon bearing bonds for all maturities.

1 Fixed Income Mathematical

Yield to maturity in the usual convension: Given a bond with annual compounding, $P = \sum_i p_i(1+r)^{-t_i}$. We need a more robust mathematical construction.

In mathematical convention, a yield curve/zero curve/discount curve/spot curve is a curve $r(T)$ such that for all bonds, dirty price $P = \sum_i p_i e^{-r(t_i)t_i}$.
Example: for annual componding, this is equivalent to $P = \sum_i p_i(1+r(t_i))^{-t_i}$.
We want to find/build a curve that works for all bonds.

Questions:

- Does the curve exist?
- How do we calculate it?
- How does this curve change through time?
- What else does it give us?

1.1 Basic Definitions

Zero coupon bonds:

- It is bond that pays no coupons
- It contains a single cash flow: a single payment at the time of maturity
- It is characterized by 3 variables:
    1. The Notional (payment to occur at maturity)
    2. The Price of the bond
    3. The Time to maturity

***Definition:* 1.1: Yield**

Yield $r(T) = -\frac{\log(P/N)}{T}$, where P is the bond price, N is the notional, and T is the time to maturity.

Arbitrage-free Pricing

- If there were zero coupon bonds of all maturity, we can construct $r(t)$
- Then, all coupon bearing bonds need to be priced as $\sum_i p_i e^{-r(t_i)t_i}$, which is a linear function, sum of all possible bonds i with different time to maturity t_i
- Otherwise, there would be an *arbitrage* opportunity (making money for free).

Arbitrage is unstable, so we assume it does not exist. Therefore, the yield curve must exist and be unique.

Yield Curve Dynamics

Definition: 1.2: Bond Price and Rate

Consider a zero-coupon bond that, with a payment of $P(t,T)$ (bond price) at time t, pays \$1 at time T. $P(t,T) = e^{-r(T-t)}$, thus $r(t,T) = -\frac{\log P(t,T)}{T-t}$, where t is today, T is *term* which could be a year later.

As a function of T, r is smooth. As a function of t, it is random.

We asume non-negative interest rates, *i.e.* $0 \le r(t,T) \le 1$, and no transaction costs, *i.e.* $P(t,t) = 1$ (efficient market)

Definition: 1.3: Short Rate

Short rate is the instantaneous cost of borrowing.

$$r_t = r(t,t) = \lim_{T \to t} r(t,T) = -\left.\frac{\partial}{\partial T} \log P(t,T)\right|_{T=t}$$

It is used to be called LIBOR, the rate at which banks lend each other in overnight lendings.

Note: $P(t,t) = 1$ implies that $\lim_{T \to t} r(t,T) = \lim_{T \to t} -\frac{\log P(t,T)}{T-t} = \lim_{T \to t} -\frac{\log P(t,T) - \log P(t,t)}{T-t}$

1.2 Contracts

Definition: 1.4: Forward Contract

A forward contract (bilateral contract) is an obligation to purchase (supply) a certain asset at a precise time in the future, for a price fixed today, from a certain counter party. (Default risk)

Example: We agree we buy something with a fixed price now, but do the transaction in the future for bond, stock, rice, etc.

Definition: 1.5: Future Contract

A future contract is an obligation to purchase (supply) a certain asset at a precise time in the future, for a price fixed today, from an exchange. (No risk)

Example: Oil is only sold in a future market

Similarity: Two parties agree a price now, transact later.
Difference: In a forward contract, there is no guarantee that one party pays. (Can sue afterwards though.) In a future contract, it is guaranteed that the transaction goes through.

1.3 Pricing Futures

Example (Stock future prices):
Assume APPL stock is \$135 today, what is the price to deliver APPL one year from now? *i.e.* we want to find APPL(0,1)

We can build a *Replicating Portfolio*:
Borrow \$135 today, buy one stock of APPL, deliver in a year at \$$x$, pay back the loan \$$135e^{r(1)}$. Therefore, $\text{APPL}(0,1) = 135e^{r(1)}$.

Date	Today	T_1	T_2
Cashflow	0	$-P(t,T_1,T_2)$	1

Equity Futures Contracts:
For a stock with price given by S_t, the price of a futures contract at time t for delivery at time T is $S(t,T) = S_t e^{r(t,T)(T-t)}$.

Bond Futures Contracts:
Consider the following forward contract:

- Agreement date: Now (time t)
- Product to deliver: a zero-coupon bond B issued at T_1 paying \$1 at T_2
- Delivery date: T_1
- Price: $P(t,T_1,T_2)$ (Unknown. We want to find this price)
- Payment date: T_1 (We need to pay for the bond in order to receive the \$1 at T_2)

Cashflows of the future contract:

The Replicating Portfolio must contain a bond with maturity at T_2, so we get \$1. We need to buy that bond now at price $P(t,T_2)$, but we have \$0 cashflow now, so we need to short sell something to raise money to pay for the bond. That something is the bond that has maturity at time T_1 at price $P(t,T_1)$.

Consider a portfolio Π of 1 bond unit worth $P(t,T_2)$ each, and $-x$ bond units worth $P(t,T_1)$ each, with $x = \frac{P(t,T_2)}{P(t,T_1)}$. This gives enough money to buy the bond that matures at T_2. Thus $P(t,T_1,T_2) = \frac{P(t,T_2)}{P(t,T_1)}$.

The corresponding *forward yield* is

$$r(t,T_1,T_2) = -\frac{\log P(t,T_1,T_2)}{T_2 - T_1} = -\frac{\log P(t,T_2) - \log P(t,T_1)}{T_2 - T_1}.$$

Taking the limit as $T_1 \to T_2$, we get the *forward rate*:

$$f(t,T) = r(t,T,T) = -\frac{\partial}{\partial T}\log P(t,T).$$

The term structure is reconstructed from f as follows:

$$P(t,T) = \exp\left(-\int_t^T f(t,u)du\right),$$

where $f(t,u)$ is the *forward curve.*

Note: forward curve has no information loss as compared to yield curve.

Example: Consider the pension. We pay pension every month. As time goes on, the value should increase (by buying bonds). When retired, we get more back.
Example: We know exactly what we earn next month, so we can borrow money from someone now and buy the bonds that could have higher price next month.

1.4 Market Factors

Securities prices evolve in random ways, but with a strong internal dependence structure. That interdependence is an important market invariant.
It can be calculated using spectral theory:

- Simplest form: eigenvalues and eigenfunctions
- More compelx forms: neural embeddings and random forests

Note: Dollar values are autoregressive, so we need to look at the percentage returns.

Eigen Analysis:
For ρ a covariance matrix, consider the eigenvalues λ_i, where $\sum_i \lambda_i = \text{Tr}(\rho)$. The largest eigenvalue explains the most of the variance. The corresponding eigenvector describes the main direction of the market movement. If the rates/price changes move together, the eigenvector will be of the same sign. This eigenvector can be used as weights in *e.g.* S&P 500.

For rates, the second eigenvector is associated with how the rate tilts. The third eigenvector is associated with the change of complexity.

For hedge funds, the eigenvalues are mostly noisy and cannot explain much variance.

1.5 Regression Trees

Suppose we have two features: company size and volatility. We have some multidimensional data with points x_i according to marginal data.

***Definition:* 1.6: Residual Sum of Squares**

Group samples in rectangles (clustering) so that the fitting criterion (RSS) is minimized.

$$\text{RSS} = \sum_{j=1}^{J} \sum_{i \in R_j} |x_i - \hat{x}_{R_j}|^2,$$

where $\hat{x}_{R_j}$ us the mean of training observations in R_j.

Recursive splitting: For each pair (feature, value), define the pair of half hyperplanes $R_1(j,s) = \{X : X_j < s\}$ and $R_2(j,s) = \{X : X_j \geq s\}$ and select values of (j,s) to minimize $\sum_{i, x_i \in R_1(j,s)} |x_i - \hat{x}_{R_1}|^2 + \sum_{i, x_i \in R_2(j,s)} |x_i - \hat{x}_{R_2}|^2$

Cross-Validation: split the training set into possible validation sub-data sets
Selection of Optimal Tree: a sequence of optimal trees T_α, as α increases, the number of nodes decreases. Small α means overfitting. Large α means simplicity. Cross validation helps find optimal α.

2 Option Pricing

In Sec. 1.2, we see that contracts are obligations. Here we consider the *Options*. In options, we have the right, but we can choose to buy it or not.

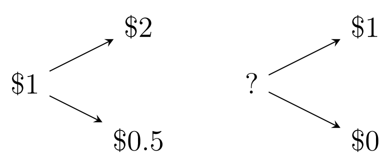
***Definition:* 2.1: Call Options**

In a call option, buyer has the right to buy an agreed quantity of underlying asset from the seller of the option on/before the expiration date for a certain price (strike price). Buyer pays a fee (premium) for the right.
The payoff is $f(S) = (S - K)_+ = \max(S - K, 0)$.

***Definition:* 2.2: Put Options**

In a put option, buyer has the right to sell an agreed quantity of underlying asset to the seller of the option on/before the expiration date for a certain price (strike price). Buyer pays a fee (premium) for the right.
The payoff is $f(S) = (K - S)_+ = \max(K - S, 0)$.

Example: a call option with payoff function $f_0(S) = (S - \$1)_+$. Given the stock tree on the left, and option tree on the right, What is the value at the question mark (price of the option)?



We can solve the following system of equations based on the first tree:

$$\begin{cases} p + q = 1 \\ 2p + \frac{q}{2} = 1 \end{cases}$$

and get $p = \frac{1}{3}$, $q = \frac{2}{3}$. The actual price of the option is $V = \$\frac{1}{3}$.

Replicating Portfolio: borrow $\$\frac{1}{3}$ and buy $\$\frac{2}{3}$ of S. Then we will exactly cover (hedge) the payoff, since it costs $\$\frac{1}{3}$ to purchase the portfolio, the price should be the same.

2.1 Discounted Value

Assume the existence of a bond with constant interest rate r. Build the porfolio: $\Pi = \frac{2}{3}$ stock units $+$ $\left(-\frac{1}{3}\right)$ bonds. No matter what p is, absence of arbitrage implies the option price $= \frac{2}{3} - \frac{1}{3}B = \frac{2}{3} - \frac{1}{3}e^{-rT}$, where T is the time to expiration and r is the constant interest rate.

Example: If option price $= \frac{1}{2}$, $T = 1$, then $r = \ln 2$ and the option is sold for \$0.5.

Implied Probabilities:
By selecting $p = \frac{2}{3}e^{rT} - \frac{1}{3}$, we can achieve option price $= \mathbb{E}(e^{-rT} f_0) = pe^{-rT}$.

In other words, we can construct a probability measure Q for the stock process, such that option price $= \mathbb{E}_Q(B_T^{-1} f_0)$.
Q is also called the risk netural measure. It can be obtained not only from prices dictated by arbitrage arguments, but also from market prices.

If we define the (arbitrage-free) price to equal to discounted pay-off $V = B_T^{-1} f_0$, then there exists a measure Q under which V is a martingale[1].

Incomplete Market:

Assume the stock is valued at \$1 today, and can be worth $S = \begin{cases} \$2 \\ \$1 \\ \$0.5 \end{cases}$ after a year. How can we price the option with strike price = \$1?

We cannot calculate the probabilities for sure with the system of equations:

$$\begin{cases} p + q + r = 1 \\ 2p + q + \frac{r}{2} = 1 \end{cases}$$

There are two probabilities:

1. Another derivative price is known
2. We can re-balance our hedge once before maturity.

2.2 Binomial Pricing Theory

Here, we consider a single period situation only, since the multi-period discrete time pricing can be solved using iterative single period pricing. However, the continuous time multi-period will be different and needs to be solved by stochastic calculus.

Elements:

1. **Payoff Matrix:** values of financial instruments in the future. Each column is an instrument. Each row is an event.
   Example: $D = \begin{pmatrix} 1 & 2 \\ 1 & 0.5 \end{pmatrix}$. In this payoff matrix, first column is bond, which doesn't depend on stock and the second column is stock. We have 2 events: stock goes up/down.
2. **Replicating Strategy:** a vector v, given by $Dv = p$, where p is the realized payoff vector. **Example:** with the above D, the realized payoff is $p = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then the replicating strategy $v = \begin{pmatrix} x \\ y \end{pmatrix}$ is given by $D \begin{pmatrix} x = & \text{bond units} \\ y = & \text{stock units} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$
3. **Cost Vector:** the current price of the instruments **Example:** if $q = (0.9, 1)$, this means that the stock price is \$1, and the bond price is \$0.9.

Then we can calculate the price of the option $= qD^{-1}p$, which is the expected discounted payoff.

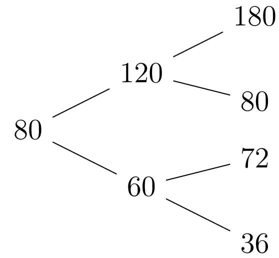
If we have $S = \begin{cases} \$2 \\ \$1 \\ \$0.5 \end{cases}$, then we get $D = \begin{pmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 0.5 \end{pmatrix}$, more rows than columns. The payoff can be $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and the cost vector can be $q = (0.9, 1, 1/3)$.

Side note: In financial market, a stock/bond is a *financial instrument*. A *derivative* is a derivative on a financial instrument. But from our perspective, we don't care. Everything is defined as the payoff in the future.

[1]A martingale is a a sequence of random variables such that the value today is the expected future value

Example (Multi-period):

Assume the stock price process can be:



Assuming $r = 0$, $q = (1, 1)$.

We buy a call option at strike price of \$75. Then the payoff vector is $\begin{pmatrix} 105 \\ 5 \\ 0 \\ 0 \end{pmatrix}$.

Focus on the 120-180,80 branch, we know the payoff is $p = \begin{pmatrix} 105 \\ 5 \end{pmatrix}$, the payoff matrix is $D = \begin{pmatrix} 1 & \frac{180}{120} \\ 1 & \frac{80}{120} \end{pmatrix} = \begin{pmatrix} 1 & 3/2 \\ 1 & 2/3 \end{pmatrix}$. Thus the price of the option at 120 can be calculated by $qD^{-1}p = 45$. For the 60-72,36 branch, since the payoffs are zero, the price of the option is simply 0.

Then we move back to 80-120, 60 branch. The payoff is $p = \begin{pmatrix} 45 \\ 0 \end{pmatrix}$. The payoff matrix is $D = \begin{pmatrix} 1 & \frac{120}{80} \\ 1 & \frac{60}{80} \end{pmatrix} = \begin{pmatrix} 1 & 3/2 \\ 1 & 3/4 \end{pmatrix}$. Thus the price of the option at the beginning is \$15.

2.3 Pricing Theory

Assume there is a probability space with measure μ[2] for the payoffs of N securites available for trading.

- A *security* is characterized by its cost now, and its payoff after one unit of time
- The *cost* of the ith security is q_i, $i = 1, ..., N$
- The *payoff* is given by the random variable $D_i(\omega)$
- The *expected payoff* of a security is $\mathbb{E}(D_i(\omega))$
- A *portfolio* is a vector $\theta = (\theta_1, ..., \theta_N) \in \mathbb{R}^N$, which represents the holdings of each security. θ_i can be positive or negative.
  - if $\theta_i < 0$, our position is said to be *long*
  - if $\theta_i > 0$, our position is said to be *short*
- The *payoff of the portfolio* is $\theta \cdot D(\omega)$
- A market is *complete* if $\text{Span}\{\theta \cdot D(\omega), \theta \in \mathbb{R}^N\} = L^2(\mu)$ (Entire sapce of L^2 payoff functions[3] on the probability space μ) and markets are usually assumed to be complete. In a complete market, for any payoff, there is a portfolio with that payoff.

[2] $\mu(B)$ = measure/probability of the set B

[3] $L^2 = \{f : \int |f|^2 < \infty\}$, *i.e.* the set of square integrable function.

- The cost of a portfolio θ is $q \cdot \theta$
- If a portfolio has non zero cost, $q \cdot \theta \neq 0$, the return is defined as $R_\theta(\omega) = \frac{\theta \cdot D(\omega)}{q \cdot \theta}$

In a real market, there are hedgers (people trying to minimize risk), speculators (people trying to maximize return) and arbitrageurs (people detecting market inefficienies).

Definition: 2.3: Arbitrage Opportunity

We say that there is an arbitrage opportunity if there is a portfolio θ such that $q \cdot \theta \leq 0$ (non-positive cost) and $D \cdot \theta \geq 0$ (non-negative payoff), a.e. and $D \cdot \theta > 0$ with non-zero probability. (making money at no cost)

Theorem: 2.1: Efficient Market Hypothesis

There is no arbitarge and there are no transaction costs.

Theorem: 2.2: Riesz Representation

If p_i are linear functionals of the payoffs $L^2(\mu)$, then there exists a random variable $\pi(\omega)$ s.t. $p \cdot \theta = \mathbb{E}(\theta\pi \cdot D)$ for all $\theta \in \mathbb{R}^N$. (price of the portfoli = corrected expectation of portfolio payoff, π changes/fixes weight of each event)
$\pi(\omega)$ is called the *state-price deflator*.
If markets are complete, π is unique. If there are no arbitrage opportunities, $\pi > 0$.

For all portfolios θ with returns R_θ, $\mathbb{E}(R_\theta \pi) = 1$.

Example: In the previous example with \$1 becoming \$2 and and \$0.5. $\pi = (2/3, 4/3)$.
Suppose we guess the probability wrong to be $(1/2, 1/2)$, then $\mathbb{E}(D_i(\omega)) = (2 + 0.5)/2 = 1.25$.

π helps us correct the guess.
For the bond, we guess 1 and 1 with probability $\frac{1}{2}$ and $\frac{1}{2}$. The price is $\frac{1}{2}\frac{2}{3} + \frac{1}{2}\frac{4}{3} = 1$.
For the stock, we guess 2 and 0.5 with probability $\frac{1}{2}$ and $\frac{1}{2}$. The price is $\frac{1}{2}\frac{2}{3}2 + \frac{1}{2}\frac{4}{3}\frac{1}{2} = 1$.
For the option, we guess 1 and 0 with probability $\frac{1}{2}$ and $\frac{1}{2}$. The price is $\frac{1}{2}\frac{2}{3}1 + \frac{1}{2}\frac{4}{3}0 = \frac{1}{3}$.
Assume $D_0(\omega)$ is constant for all $\omega \in \Omega$. This is a *savings account*.

Definition: 2.4: riskless-bond

A riskless bond is a portfolio θ_0 of constant payoff $\theta \cdot D(\omega) = \theta \cdot D(\omega')$ for all $\omega, \omega' \in \Omega$.

The riskless bond always exists, we just take $\theta = (1, 0, ..., 0)$. This gives $R^0 = E(R_{\theta 0}) = \frac{1}{\mathbb{E}(\pi)}$.

The *riskless interest rate* is given by $r = -\frac{1}{T}\ln(\mathbb{E}(R_{\theta 0}))$.

Theorem: 2.3: Price Deflator and Arbitrage

A price deflator exists if and only if there is no arbitrage.

Proof. ($\Rightarrow$) if a price deflator exists, then $\Pi(0) = \mathbb{E}(\pi\Pi(T))$.
Since π is positive as a functional on L. If $\Pi(T) > 0$, then $\Pi(0) > 0$
And if $\Pi(T) = 0$, then $\Pi(0) = 0$. Meaning that there is no arbitrage.

($\Leftarrow$) Suppose that there is no arbitrage. Consider the price-payoff vector space $V = \mathbb{R} \times L$.

The (cost, payoff) hyperplane is $M = \{(-\theta \cdot q, \theta \cdot P) : \theta \in \mathbb{R}^N\} \neq \mathbb{R} \times L$. $\theta \cdot q \in \mathbb{R}$ is the cost, $\theta \cdot P \in L$ is the payoff.
The cone $K = \mathbb{R}_+ \times L_+$ contains alll securities of non-positive price and non-negative payoff.
If there is no arbitrage, then $K \cap M = \{0\}$. The cone only touches the hyperplane at the origin.
Otherwise, there is an arbitrage opportunity.

By separating hyperplane theorem, there exists a functional $F : V \to \mathbb{R}$ s.t. $F(x) = 0$ for all $x \in M$ and $F(x) > 0$ for all $x \in K \setminus \{0\}$

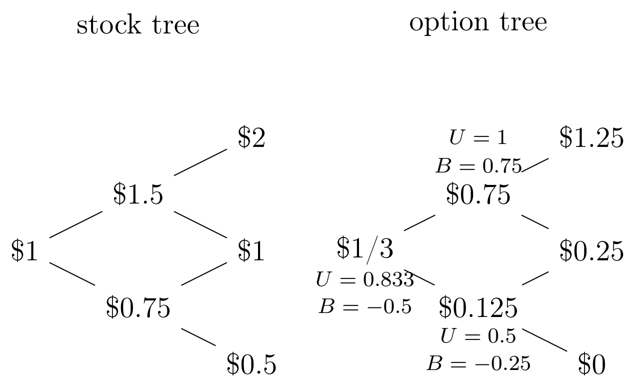
The Riesz representation of $F(x)$ is $F(v, c) = \alpha v + \mathbb{E}(\phi \cdot c)$.

In terms of α and ϕ, we have $-\alpha\theta \cdot q + \mathbb{E}(\phi(\theta \cdot P)) = 0$ for all $\theta \in \mathbb{R}^N$, since M is mapped to $\{0\}$.
$\alpha, \phi, \theta \cdot P$ are existence objects from Riesz representation.

Hence $\pi = \frac{\phi}{\alpha}$ is a price deflator. $\square$

3 Exotic Options

Example: Consider a European call option with strike $K = 0.75$.



> ***Definition:* 3.1: American Call Option**
>
> It is never optimal to exercise an American call option prior to maturity. Its value always exceeds its payoff. More specifically, an American option is only exercised when price$< S - K$.

At \$0.75 of stock, we do not redeem and keep the option, because
price$= 0.125 > S - K = 0.75 - 0.75 = 0$.
At \$1.5 of stock, price$= 0.75 = S - K = 1.5 - 0.75 = 0.75$. Keeping or redeeming is indifferent. But if the price is \$0.74, we redeem, and the American call option is exercised.
At \$1, it is worth at least $\$\frac{1}{3}$, we should not redeem, because $S - K = 1 - 0.75 = 0.25 < \frac{1}{3}$.

At each branch, if we exercise, we take the American option. Otherwise, we keep and take European option. That's why we do the comparison to $S - K$.

For call options, we rarely exercise. For put options, we sometimes exercise.

3.1 Swing Options

Swing opions (swing contracts, take-and-pay options, variable base-load factor contracts) are most commonly used for purchase of oil, natural gas, and electricity (energy sector).

This is because we cannot accumulate anything in the energy sector without cost. There is no arbitrage.

They can be used as hedging instruments by the option holder, to protect against price changes in these commodities.

Flight pass is also an example of swing option. We will have restrictions on the option.

2-up-swing option:
Pricing a k-up-swing option is like pricing k embedded American options.
Suppose we wish to price a 2-up-swing option with the stock tree above, we need to exercise the first up-swing when 2-up price - 1-up price $< U - K$.

Step 1: 1-up swing option

- At \$0.75, the 2-up swing price (\$0.75) and 1-up swing price (\$0.75) agreed
- Exercising at the $t = 1$ lower node (1-down movement) yields no money
- Exercising at the terminal nodes yields a discounted value of 0.125 at the $t = 1$ lower node

Step 2: 2-up swing option

The 0-node represents 2 possibilities

- Exercise one option: cash 0.25 and change tree
- Don't exercise: it makes the branch European and the option worth 0.5833.

Since both alternative yields the same result, we are indifferent whether we exercise or not.

4 Stochastic Calculus

In previous sections, we consider discrete models only. How do we extend the discrete time model to continous time?

Taking dt and dS discrete $\frac{1}{n} \to 0$ makes the option price converge to 0.

If the random outcomes are linked to independent identically distributed random variables X_i, then the path location after n steps is equal to $S_n = \sum_{k=1}^{n} X_k$.

Our interest is how S_n behaves as n grows to determine the location of the path after many random steps are taken.

Assuming step increments in the x-axis equal dx and step increments in the y-axis equal dy, the final location of the path is $(ndx, S_n dy)$.

Theorem: 4.1: Central Limit Theorem

$$\lim_{n\to\infty} P\left(\frac{S_n}{\sqrt{n}} \leq \lambda\right) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\lambda} e^{-x^2/2} dx$$

Equivalently, if $dx = \frac{1}{n}$, $dy = \frac{1}{\sqrt{n}}$, then S_n converges to a $\mathcal{N}(0,1)$ distribution on the vertical axis at $x = 1$.

Einstein's theory:
Consider a 1-D isotropic rod with an initial heat distribution. Temperature at a point x after some time dt is the average of the temperature around it now:

$$u(x, t + dt) = \frac{1}{2}[u(x + dx, t) + u(x - dx, t)].$$

Subtracting $u(x, t)$ from both sides, we get:

$$u(x, t + dt) - u(x, t) = \frac{1}{2}[u(x + dx, t) + u(x - dx, t) - 2u(x, t)].$$

This gives the heat equation $\frac{\partial u}{\partial t} dt = \frac{1}{2}\frac{\partial^2 u}{\partial x^2} dx^2$, but $dt = dx^2$ because of the stochastic movements.

4.1 Brownian Motion and Ito Process

In Brownian Motion, the value moves up or down with probability 0.5 by an amount of $\sqrt{dt}$:

$$dW_t = \pm\sqrt{dt}, \quad \mathbb{E}(dW_t) = 0$$

It is distributed at time t according to $P(x, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{x^2}{2t}\right)$.

An Ito process is a stochastic process that looks like the following:

$$X_t = X_0 + \int_0^t \sigma_s dW_s + \int_0^t \mu_s ds.$$

It will be written as:

$$dX_t = \sigma_s dW_t + \mu_t dt.$$

When σ and μ are dependent are X, we have a stochastic differential equation (SDE).

Ito's Lemma: the chain rule for stochastic processes

$$df(X_t, t) = \underbrace{\frac{\partial f}{\partial X} dX_t + \frac{\partial f}{\partial t} dt}_{\text{classical}} + \underbrace{\frac{1}{2}\sigma_t^2 \frac{\partial^2 f}{\partial X^2} dt}_{\text{stochastic}},$$

where the $\frac{\partial^2 f}{\partial X^2} dt$ comes from the $\frac{\partial^2 f}{\partial X^2}(dX)^2$ in the second order Taylor expanansion. It is non-negligible in stochastic process, because $(dX)^2 = dt$.

Take the stochastic process X_t, and map it with a smooth function f, we get a different Ito process.

Stochastic model for stocks: $\frac{\partial S_t}{S_t} = \mu dt + \sigma dW_t$.
Note if we simply have $dS = \mu dt + \sigma dW$, S (the stock price) can be negative.

Example: What is $d(\log S)$?
$dS = S\mu dt + S\sigma dW$, $(dS)^2 = S^2\sigma^2 dt$. Note that $(dS)^2$ here means the second partial derivative w.r.t. S.

Then
$d(\log S) = \frac{\partial}{\partial S} \log S ds + \frac{1}{2}\frac{\partial^2}{\partial S^2} \log S (dS)^2 = \frac{1}{S}(S\mu dt + S\sigma dW) - \frac{1}{2S^2}(S^2\sigma^2 dt) = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW$

Note: $d(\log S)$ is an Ito process independent of S if μ (trend), σ (volatility) are constant.

For stock price, the drift $S\mu$ and standard deviation $S\sigma$ are proportional to the stock price S, giving a controlling effect that prevents stock prices from being negative.

Stock price is a Geometric Brownian Motion (GBM), $\frac{dS_t}{S_t}$ is the geometric part, σdW_t is the Brownian Motion.

Usually σ (volatility) is another stochastic process $d\tilde{W}$ correlated with dW. This is then a Heston model.

5 Option Pricing

5.1 Types of Options

Definition: 5.1: European Options

European options expire at a preset future time.
Payoff depends on the price of the underlying S_T at expiration.
Call options with strike price K have payoff $P(S_T) = (S_T - K)_+$
Put options with strike price K have payoff $P(S_T) = (K - S_T)_+$

Definition: 5.2: American Options

American options can be exercised at any time prior to expiration T.
Payoff is a function of the value of the underlying S_t at the exercise time t with $t \leq T$.
Call options with strike price K have payoff $P(S_t, t) = (S_t - K)_+$
Put options with strike price K have payoff $P(S_t, t) = (K - S_t-)_+$

Definition: 5.3: Asian Options

Asian options can be issued with a European or American style.
Payoff depends on the average value of the underlying at certain times prior to expiration.
Example payoffs with $t_i \leq T$:

$$P = \frac{1}{n}\sum_{i=1}^{n}(S_{t_i} - K)_+$$

$$P = (\frac{1}{n}\sum_{i=1}^{n}S_{t_i} - K)_+$$

$$P = (K - \frac{1}{n}\sum_{i=1}^{n}S_{t_i})_+$$

Bermudan options are American options that can be exercised only at prescribed discrete future times. Remove protection, fewer rights, thus lower prices.

5.2 Fixed Income Derivatives

Recall that bonds are sold at a discount and pay a fixed amount at a future time. Their price determines the interest rates. They usually pay coupons every few months/every year.

Bond Options:

1. Bonds can be bought or sold any time before they expire.
2. Their price will fluctuate. Consequently, they can be used as financial underlying for options.
3. Similar to equity, except for the fact that at the time of expiry of the bond, options make no sense.

Types of Bond Options:

1. **Caps:** contracts that offer protection against time dependent interest rates rising over a certain ceiling by paying the corresponding exceeding interest on a fixed notional (like a call option, pay

when price > cap, protection against rising interest rates. *e.g.* Fixed rate mortgage=floating rate mortgage+cap)

2. **Floors:** charge the corresponding missing interest rates on a fixed notional. They have negative value (protection against lower interest rates)
3. **Collars:** A combination of a cap and a floor. By setting the ceiling and floor appropriately, they can be issued for free (zero collars)
4. **Swaps:** exploit the different interest rates that different parties will be charged for fixed and floating rate loans. A swap is a contract that exchanges future payments at fixed and floating rates (Derivative option)
5. **Swaptions:** When a swap is viewed as an underlying, options are issued on them (Derivative of a derivative. When swap is liquid, swaption gives a protection)
6. **Cross Currency Swaps:** same as swaps, but the exchange is between payments in two currencies.

Trading Instruments: Many other financial instruments are available for trade. Most of the time, they are designed with the objective of removing risk from uncertain future situations. They also offer risky speculative alternatives.

5.3 Black-Scholes Theory

Continuous Time Pricing:
Think of infinitesimal time intervals dt. Brownian motion moves up or down with proability 0.5 by an amount of $\sqrt{dt}$: $dW_t = \pm\sqrt{dt}$, with $\mathbb{E}(dW_t) = 0$
At time t, it is distributed according to $P(x,t) = \frac{1}{\sqrt{2\pi t}}\exp(-x^2/2t)$
The infinitesimal stock movements will be $dS_t = S_t(\mu dt + \sigma dW_t)$
Ito's lemma says: $d_t f(S,t) = \partial_S f(S,t)dS + \partial_t f(S,t)dt + \frac{1}{2}\sigma^2 S^2 \partial_S^2 f(S,t)dt$

Assume an option has price $f(S,t)$ at any given point in time, conditional on any possible value $S_t = S$ of underlying at time t, which is asumed to be known.

Arbitrage Free Argument:

- At time t, build a portfolio Π consisting of $a = -\partial_S f(S,t)$ units of stock and the option. $\Pi = f + aS$
  - At maturity, there is no derivative at $S = K$, but we can assume that it is well-defined before time of maturity
  - If $\partial_S > 0$, we are short
  - If $\partial_S < 0$, we are long
- Using Ito's lemma,
  $d_t\Pi = d_t f + adS = (0.5\sigma^2 S^2 \partial_S^2 f + \partial_t f)dt + \partial_S f dS + adS = (0.5\sigma^2 S^2 \partial_S^2 f + \partial_t f)dt.$

Note1: $d_{a_t} S = 0$ is the self-financing condition. a changes through time, but the change won't affect the portfolio now, the portfolio will change in next step.
Note2: There is no dW term, $d_t\Pi$ is deterministic, not stochastic. This is a replicating portfolio of a bond $d_t\Pi = r\Pi = r(f + aS)$.

This is a risk-free investment, hence it must earn risk-free interest and we obtain:

$$\frac{\partial f}{\partial t} = -0.5\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} - rS\frac{\partial f}{\partial S} + rf, \text{ where } f(S,T) = f_0(S) \text{ is the terminal condition.}$$

It is a backward parabolic equation (Note the minus sign in $-0.5\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2}$. It gives a time reversal.)

The solution is given by:

$$f(S,t) = e^{-r(T-t)} \int_{-\infty}^{\infty} f_0(Se^{(r-\sigma^2/2)(T-t)+x}) P_\sigma(x, T-t) dx, \text{ where } P_\sigma(x,t) = \frac{1}{\sqrt{2\pi t\sigma^2}} \exp\left(-\frac{x^2}{2t\sigma^2}\right)$$

$Se^{(r-\sigma^2/2)(T-t)+x}$ is the future value of S. $f(\cdot)$ gives the payoff. $P_\sigma(x, T-t)$ is the probability. The integral calculates the expected payoff. $e^{-r(T-t)}$ is the discount. Then, $f(S,t)$ is the value of the option at time t.

For a put option, we add a free boundary condition $f(S,t) > f_0(S,t)$

The price of a European call option on a stock S, valued today at S_0, maturing at time T with strike K, constant volatility σ and interest rate r is given by:

$$V(t, K, \sigma, r) = S_0 N(d_1) - Ke^{-r(T-t)} N(d_2),$$

where $N(d)$ is the cummulative normal $N(d) = \frac{1}{2\pi} \int_{-\infty}^{d} e^{-x^2/2} dx$, and

$$d_1 = \frac{\ln(S_0/K) + (r + 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$
$$d_2 = \frac{\ln(S_0/K) + (r - 0.5\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

6 Introduction to Risk Management

6.1 Risks

- Market: loss of NAV (Net Asset Value) due to changes in asset prices
- Credit: loss of NAV due to default events
- Liquidity: Delay turning asset value into cash
- Gap risk: an investment's price change from one level to another with no trading in between
- Legal risk: *e.g.* Bankers trust
- Operational risk: *e.g.* Fat fingers, fraud, cybersecurity

Risk Classification:

In the table, red is Regulated, blue is Revenue generator. Otherwise, it is unregulated.

	Banking	Insurance	Asset Management
Market Risk	Base I - VaR, Base III - FRTB	Solvency II - SCR	Concentration Exposures
Credit Risk	Base II - CVaR, Base III - CVA(MAR50)	SCR	DDQ-Credit premia
Operational Risk	Fraud, fat fingers, etc.	internal systems, personnel, procedures, or controls (IAIS)	Fraus, systems, etc
Liquidity Risk	LCR (UK), ILG (Basel), HQLA, NSFR (US)	Governance, roles and responsibilities not clearly defined	Ad Hoc (UCITS, 30-ACT, etc)
Gap risk	Asset valuation	ALM	$\emptyset$
Legal risk	Marginal to the risk derpartments		IAA-1933; MNPI, trading rules ESG
Cybersecurity	On-line banking	Disaster revenue generator	Data protection, disaster recovery

6.2 Value at Risk

***Definition:* 6.1: Value at Risk**

The Value at Risk (VaR) of a portfolio is α-confidence equal to x. *i.e.* The probability of losing more than VaR_α is $1-\alpha$

$$\int_{-\infty}^{-\text{VaR}_\alpha} \rho(r)dr = \text{Prob}(\text{losses} \leq \text{VaR}_\alpha) = 1-\alpha$$

Note: Value at Risk is not so much a measure of risk, but a measure of a regulatory capital.

Example: Suppose $\text{VaR}_{95\%} = \$100$, then the probability of losing at least \$100 is 5%.

Example: Issue 1000 lottery tickets, winner get \$1M

number of tickets sold	Prob(loss)	$\text{VaR}_{95\%}$
1	0.1%	0
49	4.9%	0
50	5%	\$1M
1000	100%	\$1M

Calculation in real life:

1. Generate scenarios
2. Similate P&L (profit and loss)
3. Calculate P&L statistics

Gaussian VaR - 1D:

$$\text{Prob}(\text{P\&L} \leq a) = \frac{1}{\sqrt{2\pi\sigma^2 T}} \int_{-\infty}^{a} \exp\left(-\frac{(x-\mu)^2}{2T\sigma^2}\right) dx$$

Denote $\phi(x)$ the cummulative distribution of $\mathcal{N}(0,1)$, then $\text{Prob}(\text{P\&L} \leq a) = \phi\left(\frac{a-\mu}{\sigma}\right)$.

Then $\text{VaR}_\alpha = \sigma Z_\alpha -$ Expected return, where $Z_{0.95} = 1.65$ and $Z_{0.99} = 2.33$.

Example: Assume a portfolio with one asset of \$5 million allocated to a stock. The price volatility of the stock is 2.98% for a one-day period.
Then one-day $\text{VaR}_{95\%} = Z_{0.95} S\sigma = \$1.65 \cdot 5 \cdot 0.0298 = \$246K$
one-year $\text{VaR}_{95\%} = \sqrt{365} \cdot \$246K = \$4.69M$

Multivariate Gaussian and VaR - nd:
If $X_1, ..., X_k$ are jointly Gaussian, with mean vector μ and covariance matrix Σ,

$$f_X(x_1, ..., x_k) = \frac{1}{\sqrt{(2\pi)^k \det(\Sigma)}} \exp(-0.5(x-\mu)^T \Sigma^{-1}(x-\mu))$$

A linear combination $\sum_{i=1}^k c_i X_i$ is a 1-D Gaussian, with $\mu = \sum_{i=1}^k c_i \mu_i$, $\sigma = c\Sigma c^T$

Application: Risk Metrics
Calculating an approximation to the VaR of a single investment or an investment portfolio. It assumes that investments returns follow a normal distribution overtime.

Delta Normal VaR:
Approximate the portfolio value P with future risk factor R (interest rate, stock price, etc) by Taylor approximation.

$$P(R) = P(R_0) + \underbrace{\left.\frac{\partial P}{\partial R}\right|_{R=R_0}}_{\delta}(R - R_0) + \underbrace{\left.\frac{1}{2}\frac{\partial P}{\partial R}\right|_{R=R_0}}_{\Gamma}(R - R_0)^2,$$

where R_0 is the risk factor today. δ and Γ are portfolio sensitivites.

For a bond $\delta = \left.\frac{\partial(N_0 e^{-R})}{\partial R}\right|_{R=R_0} = -N_0 e^{-R_0}$.

6.3 Portfolio Sensitivity and Greeks

P&L reflects changes in prices. Portfolio sensitives are efficient ways of trcking small price changes.

Delta (δ, Δ):

- Delta measures the rate of change of the portfolio w.r.t. changes in risk factors. It is the first derivative of the portfolio value w.r.t. the risk factors

$$\delta = \frac{\partial V}{\partial r}$$

- When there are several risk factors $(r_1, ..., r_k)$, delta becomes a vector $\delta = \left(\frac{\partial V}{\partial r_1}, ..., \frac{\partial V}{\partial r_n}\right)$

Gamma (Γ):

- Gamma measures the rate of change of the portfolio delta w.r.t. changes in risk factors. It is the second derivative of the portfolio value w.r.t. the risk factors

$$\Gamma = \frac{\partial^2 V}{\partial r^2}$$

- When there are several risk factors $(r_1, ..., r_k)$, gamma becomes a matrix $\Gamma = \left(\frac{\partial^2 V}{\partial r_i \partial r_j}\right)_{i,j=1,...,k}$

Vega (ν):

- Vega measures sensitivity to volatility. It is the first derivative of the portfolio value w.r.t. volatility of the underlying risk factors.

$$\nu = \frac{\partial V}{\partial \sigma}$$

- Often, the volatility of the risk factor is not observable, but implied.
- In asset management, long-vol ($\nu > 0$) and short-vol ($\nu < 0$) denote the sign of vega. Short-vol are common. Long-vol are desired.

Theta (θ): measures the sensitivity of portfolio value to the passage of time. $\theta = \frac{\partial V}{\partial t}$.

Rho (ρ): measures the sensitivity of portfolio value to interest rate. $\rho = \frac{\partial V}{\partial r}$.

6.4 PCA and Monte Carlo

Eigenvalues and Gaussians:

> ***Definition:* 6.2: Cholesky Decomposition**
>
> Suppose $\Sigma = ODO^{-1}$, where $D = \text{diag}(\lambda_1, ..., \lambda_n)$. $\Sigma = HH^\dagger$. Then $H = O\sqrt{D}$ is the Cholesky Decomposition.

Assume a multivariate Gaussian distribution with a $n \times n$ variance/covariance matrix A with dataset $x_i \in \mathbb{R}^n$. Then the dataset $H^{-1}x_i$ is Gaussian with $\Sigma = I$.

Principal Components:
Even when the dataset is not given by multivariate Gaussian distribution, we can still transform dataset by $H^{-1}x$. The result is uncorrelated marginals.

Then move on to analyze each marginal with 1-D methods, combine the results and transform back by H.

Monte Carlo:
Given a data distribution with probability density ρ, how can we produce a sample x_i so that the values $x_i, i = 1, ..., k$ follow the given distribution?

If F is the cumulative density corresponding to probability density ρ, and x_i is a uniformly distributed sample, then $F^{-1}(x_i)$ is distributed like ρ, where F^{-1} is the functional inverse of F. In 1-D, $\rho(x) = \frac{dF}{dx}$, $F(x) = \text{Prob}(X \leq x)$.

In multivariable case, $\rho(x_1, ..., x_n) = \frac{\partial^n F(x_1, ..., x_n)}{\partial x_1 ... x_n}$, where $F(x_1, ..., x_n) = \text{Prob}(X_1 \leq x_1, ..., X_n \leq x_n)$.

Definition: 6.3: Copula

If $(X_1, ..., X_n)$ is uniformly distributed, $F : [0,1]^n \to [0,1]$ is a copula.

Theorem: 6.1: Sklar's Theorem

Given $F(x_1, ..., x_n)$ with marginals $F_i(x_i)$, $(F_1^{-1}(x_1), ..., F_n^{-1}(x_n))$ is marginally uniformly distributed.

6.5 Sustainability

Substainability considers three aspects: Environment, Social and Governance (ESG).
There is a MSCI ESG rating. The ratings have impact on bonds. Green bonds (AAA, AA) are expensive. Others are cheaper. But lower rating bonds have higher forward zero curve.

7 Credit Risk

7.1 Review of Basic Concepts

Cashflow Valuation:
The present value of cashflows is given by the value equation:

$$\text{Value} = \sum_{i=1}^{n} p_i e^{-r_i t_i},$$

where n is the number of payments, p_i is the amount paid at time t_i, r_i is the continously compounded interest rate at time t_i.
Assumption: payments will occur with probability 1 (no default risk)
Credit Premium:
The discounted value of cash flows, when there is probability of default is:

$$\text{Value} = \sum_{i=1}^{n} p_i e^{-r_i t_i} q_i, \text{ where } 0 \leq q_i \leq 1$$

q_i denotes the probability that the counter-party is solvent at time t_i. A large default risk (small q) implies that

1. For a fixed set of p_is, the discounted present value will always be less than or equal to the value equation
2. To preserve the same present value of cashflows as in the equation, the cashflows $\{p_i\}_{i=1}^{n}$ need to be increased. The amount by which each payment is increased is q_i^{-1}. This it the *credit premium* at time t_i.

Since $q_i \leq 1$, we can write q_i as $q_i = e^{-h_i t_i}$ by defining $h_i = \frac{-\ln q_i}{t_i}$, the *credit spread (hazard rate)* at time t_i.

This gives the value function:

$$\text{Value} = \sum_{i=1}^{n} p_i e^{-(r_i+h_i)t_i}$$

Note: When bankrupt or get into default, the value becomes 0.

Example: (Default Yield Curve) A senior unsecured BB rated bond matures exactly in 5 years and is paying an annual coupon of 6%.
One-year forward zero curves for credit rating BB is 5.55% for Year 1, 6.02% for Year 2, 6.78% for Year 3, 7.27% for Year 4.
What is the value of the bond?

$$V_{BB} = 6 + \frac{6}{1.0555} + \frac{6}{1.0602^2} + \frac{6}{1.0678^3} + \frac{106}{1.0727^4} = 102.0063$$

Two Credit States:
Assumptions:

1. Assume only 2 possible credit states: solvency and default
2. Assume the probability of solvency in a period (*e.g.* 1 year), conditional on solvency at the beginning of the period is given by a fixed amount q. This is a Markov Chain:
   $\Pr(\text{Solvent at time } t_{i+1} | \text{Solvent at time } t_i) = q$

Consequence: $q_i = q^{t_i}$ and gives a constant credit spread $h_i = h = -\ln(q)$.
By Taylor approximation, the probability of the default is equal to the credit spread.

More formally, assume the default process follows a two-state Markov Chain with transition probability $P = \begin{pmatrix} q & 1-q \\ 0 & 1 \end{pmatrix}$.

Application to country risk:
Consider a one-year zero coupon Greek bond providing a spread h over a German risk-free bond with two assumptions:

1. Notional N
2. Risk Neutral: Probability of solvency after one year is q

When a country defaults, not all is lost. There is a recovery rate R which is the value we recover in the case of default. Assume $R = 50\%$
We can adjust the valuation equation: $V = \underbrace{Ne^{-r}q}_{\text{default}} + \underbrace{RNe^{-r}(1-q)}_{\text{recovery}}$
Using the spread $V = Ne^{-(r+h)}$, we get $q = \frac{e^{-h}-R}{1-R}$.

Credit Rating Agencies:

- Corporations whose business is to rate the credit quality of corporations, governments and specific debt issues.
- Examples include
  - Moody's Investors Service
  - Standard & Poors
  - Fitch IBCA
  - Duff and Phelps Credit Rating Co.

S&P Rating System

AAA	highest quality; capacity to pay interest and repay principal is extremely strong
AA	high quality
A	strong payment capacity
BBB	adequate payment capacity
BB	likely to fulfill obligations; onging uncertainty
B	High risk obligations
CCC	Current vulnerability to default
D	in bankruptcy or default or other market shortcomings

Markov Chain Default Models:
Generalize the setting to include more than one solvency state. Transition probabilities between states $1...n$ are constant overtime. p_ij is the conditional probability of changing from state i to state j.

Transition Probability:
Between state i and state j, in two time steps is given by $P_{ij}^{(2)} = \sum_k p_{ik}p_{kj}$
In other words, if denote by A the one-step conditional probability matrix, the two-step transition probability matrix is given by A^2.

If A denotes the transition probability matrix at one step (*e.g.* 1 year), the transition probability after n steps (30 is especially meaningful for credit risk) is given by A^n
Similarly, the quarterly transition probability matrix should be $A^{1/4}$.

Matrix Expansion:

$$A^\alpha = \sum_{k=0}^{\infty} \binom{\alpha}{k} (A-1)^k, \text{ where } \binom{\alpha}{k} = \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

7.2 Credit Loss

***Definition:* 7.1: Credit Exposure**

Credit Exposure is the maximum loss that a portfolio can experience at any time in the future, taken with a crtain level confidence (Market probability perspective).

In Goodrich swap, when we receive a payment, credit exposure drops, because money at risk drops. But as time goes by, the amount of money you could lose increases, because of uncertainty of how rates move. Decreases in credit exposure happens when a payment occurs.

The definition of credit exposure is independent of counter party and credit quality, but it is a conditional amount of money we may lose.

***Definition:* 7.2: Recovery Rate**

When default occurs, a portion of the portfolio value can usually be recovered. A recovery rate is always considered when evaluating credit losses. The recovery rate (R) represents the percentage value at which we expect to recover, given default.

***Definition:* 7.3: Loss Given Default (LGD)**

Loss Given Default is the percentage we expect to lose when default occurs. $R = 1 - \text{LGD}$.

R and LGD may be modelled as random variables. In simple exercises, we can assume they are constants.

For corporate bonds, there are two primary studies of recovery rates which arrive at similar estimates (Carty & Lieberman and Altman & Kishore)

There are several seniority classes deciding who collects payment of bonds first, including senior secured, senior unsecured, senior subordinated, subordinated, junior subordinated

- Senior gets paid before junior
- Secured has colateral
- Subordinated: at the end of line when it comes to collecting payments

Definition: 7.4: Default Process

The default process $\mathbb{I}_t$ is a random process which follows:

$$\mathbb{I}_t = \begin{cases} 1, \text{ counter party is at default at time } t \\ 0, \text{ counter party is solvent at time } t \end{cases}$$

The probability of default at time t is the expectation of $\mathbb{I}$ at time t, $1 - q_t = \mathbb{E}(\mathbb{I})$

Note: $\mathbb{I}$ depends on the credit quality of the counter party. Credit Exposure depends on the market risk of the instrument or portfolio. Loss Given Default is related to situation of the portfolio within in the capital structure of the counter party.

Definition: 7.5: Credit Loss

For a portfolio with several counter parties, the credit loss is defined by

$$\text{Credit Loss} = \sum_i \mathbb{I}(i) \times \text{Credit Exposure}_i \times \text{LGD}$$

The credit loss distribution is complex. As with Markowitz theory, we try to summarize its statistics with two numbers μ (expected value, or expected loss), and σ (standard deviation, or unexpected loss)

Worst Credit Loss (WCL): represents the credit loss which will not be exceeded with some confidence interval over a certain time horizon. This is a measure of risk. (quantile)

Credit VaR (CVaR): represents the credit loss which will not be exceeded in excess of the expected credit loss, with some confidence interval over a certain time horizon. This is a measure of regulatory capital. (quantile-expected loss)[4]

Example: A 95% WSL of \$5M on a certain portfolio means that the probability of losing more than \$5M in that particular portfolio is exactly 5%.
A daily CVaR of \$5M on a acertain portfolio with 95% CI means that the probability of losing more than the expected loss plus \$5M in one day in that particular portfolio is 5%

Economic Credit Capital:

Definition: 7.6: Capital

Capital is traditionally designed to absored *unexpected losses*. Credit VaR is the measure of capital, usually calculated within a one-year time horizon. Losses can come from either defaults or migrations

Definition: 7.7: Credit Reserves

Credit Reserves are set aside to absorb *expected losses*. Worst Credit Loss measures the sum of the capital and the credit reserves. Losses can come from either expected losses.

[4]This Credit VaR is different from the Conditional VaR (Expected Shortfall) which quantifies the amount of tail risk an investment portfolio has.

Definition: 7.8: Netting

When two counter parties enter into multiple contracts, the cashflows over all the contracts can be, by agreement, merged into one cashflow. This practice called Netting is equivalent to assuming that when a party defaults on one contract, it defaults in all contracts imultaneously.

Expected Credit Loss (ECL): The general framework for ECL is:

$$\text{ECL} = \mathbb{E}[\mathbb{I} \times \text{CE} \times \text{LGD}] = \iiint (\mathbb{I} \times \text{CE} \times \text{LGD}) \times f(\mathbb{I} \times \text{CE} \times \text{LGD}) d\mathbb{I} d\text{CE} d\text{LGD},$$

where $f(\mathbb{I} \times \text{CE} \times \text{LGD})$ is the joint probability density function of the default status ($\mathbb{I}$), Credit Exposure (CE) and Loss Given Defaults (LGD). The ECL is using the joint pdf of $\mathbb{I}$, CE and LGD.
Because calculating the joint probability distribution of all relevant variables is hard, assume the distributions are independent.
Then $\text{ECL} = \underbrace{\mathbb{E}(\mathbb{I})}_{\text{probability of default}} \times \mathbb{E}(\text{CE}) \times \mathbb{E}(\text{LGD})$

Example (Commercial Mortgage): Consider a commercial mortgage, with a shopping mall as collateral. Assume the exposure of the deal is \$100M, an expected probability of default of 20% (std of 10%) and an expected recovery of 50% (std of 10%). Calculate the expected loss.

1. Assume independence of recovery and default

   LGD$= 1 - 50\% = 50\%$, thus $\text{EL} = \$100M \cdot 20\% \cdot 50\% = \$10M$

2. Assume a -50% correlation between the default probability and the recovery rate

   Using tree-based model:

   - Two equally likely future credit states 30% and 10%
   - Two equally likely future recovery rate states 60% (LGD=40%) and 40% (LGD=60%).

   $$\begin{cases} p^{++} + p^{+-} = 0.5 \text{ (credit state goes up)} \\ p^{-+} + p^{--} = 0.5 \text{ (credit state goes down)} \\ p^{++} + p^{-+} = 0.5 \text{ (recovery state goes up)} \\ p^{++} - p^{+-} - p^{-+} + p^{--} = 0.5 \text{ (correlation)} \end{cases} \Rightarrow p^{++} = p^{--} = 0.125, p^{+-} = p^{-+} = 0.375$$

   $$\text{EL} = \$100M(0.125 \cdot 0.3 \cdot 0.4 + 0.375 \cdot 0.3 \cdot 0.6 + 0.125 \cdot 0.1 \cdot 0.6 + 0.375 \cdot 0.1 \cdot 0.4) = \$10.5M$$

Example (Goodrich-Rabobank): Consider the swap between Goodrich and MGT. Assuming a total exposure averaging \$10M (std=50%), a default rate averaging 10% (std=3%), fixed recovery 50%. Calculate the expected loss.

1. Assume independence of recovery and default

   LGD$= 1 - 50\% = 50\%$, thus $\text{EL} = \$10M \cdot 50\% \cdot 10\% = \$0.5M$

2. Assume a -50% correlation between the default probability and the recovery rate

   Using the tree method, we get $p^{++} = p^{--} = 0.125, p^{+-} = p^{-+} = 0.375$. However, this time, the changing values are the default and credit exposure.
   $\text{EL} = 0.5(0.125 \cdot \$15M \cdot 0.13 + 0.125 \cdot \$5M \cdot 0.07 + 0.375 \cdot \$15M \cdot 0.07 + 0.375 \cdot \$5M \cdot 0.13) = \$0.46M$

Example (FRM 1998 Q39): Calculate 1 year expected loss of a \$100M portfolio comprising 10 B-rated issuers. Assume probability of default of each issuer is 6%, and recovery rate for each issuer in the event of default is 40%.
LGD$= 1 - 0.4 = 0.6$, so EL $= \$100M \cdot 6\% \cdot 60\% = \$3.6M$.

Example (FRM 1998 Q39 modified): Assume now that the correlation between issuers is:

- 100% (They are all the same)
  Loss distribution: default (loss \$60M), no default (loss \$0)

  $\mathbb{E} \Rightarrow \sigma^2 = 6\% \cdot (60 - 3.6)^2 + 0.94 \cdot (0 - 3.6)^2 = 200$
  Thus, the unexpected loss is $\sigma = \sqrt{200} = \$14M$

- 0% (They are all different)
  Loss distribution is a sum of 10 random variables, each with 2 states: default (loss \$6M), no default (loss \$0)

  The expected value of each issuer is \$0.36M, and std of each issuer is \$1.4M

  Assuming the number of defauls is given by a Poisson distribution, then unexpected loss (std of the sum) is the sum of the stds $= \sqrt{10} \cdot 1.4 = \$5M$

- 50% (They are in the same sector)

  similar to the all different case, but the variance of the sum is

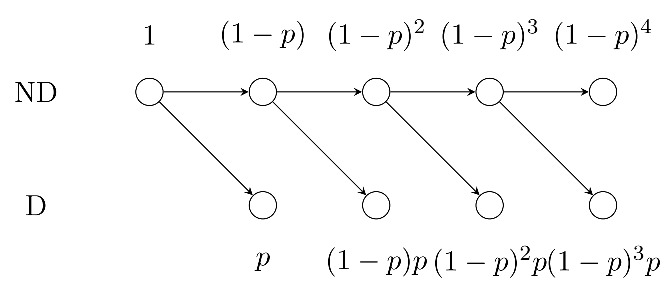
$$\begin{aligned}
\mathrm{Var}(\sum_i X_i) &= \sum_{i,j} \mathbb{E}[X_i X_j] - \mu_i \mu_j \\
&= \sum_i \sigma_i^2 + \sum_{i \neq j} \sigma_{i,j} \\
&= \sum_i \sigma_i^2 + \sum_{i \neq j} 0.5 \sigma_i \sigma_j \\
&= 110
\end{aligned}$$

  Thus, the unexpected loss is $\$\sqrt{110}M$.

7.3 Expected and Unexpected Loss Over Time

Consider a bond issued from a default-prone party, paying two \$5 coupons after the end of the second and forth years. We assume the interest rates are 0. Yearly default probability of 7%, with 0% recovery rate. Assme that the default-free party maintains a risk-capital to cover the standard deviation of losses that is adjusted annually and demands a certain return on this risk-capital.

We can build the following Markov Chain where D=Default, ND=Not Default



There are two ways to calculate the *expected loss*, which are equivalent in this case. Since the value of the contract is always non-negative to the default-free party, we don't need to discard any future events. (As every contract can be decomposed into contracts that always have non-negative or non-positive value)

- First way: compute expected cash flows, but also factor in the probabilities of default within these periods

  We get paid \$5 coupons at year 2 and year 4 each.

  $$\text{Expected cash flow} = 5 \cdot P_{ND\in(0,2]} + 5 \cdot P_{ND\in(0,4]} = 5(1-7\%)^2 + 5(1-7\%)^4 = \$8.065,$$

  where $P_{ND\in(i,j]}$ is the probability that the default prone party does not default in the time interval $(i,j]$.

  The expected loss is EL = 10 − EC = \$1.935

- Second way: expected loss is based on yearly exposure

  $$\text{EX}(1^-) = \$10, \text{EX}(2^-) = \$10, \text{EX}(3^-) = \$5, \text{EX}(4^-) = \$5,$$

  where no correction due to discounting was included, since the interest rates are 0%.

  $$\begin{aligned}\text{EL} &= \text{EX}(1^-)P_{D\in(0,1]} + \text{EX}(2^-)P_{D\in(1,2]} + \text{EX}(3^-)P_{D\in(2,3]} + \text{EX}(4^-)P_{D\in(3,4]} \\ &= \$10(7\%) + \$10(7\%)(1-7\%) + \$5(7\%)(1-7\%)^2 + \$5(7\%)(1-7\%)^2 = \$1.935\end{aligned}$$

Unexpected loss is the variance of the losses:

$$\begin{aligned}&\text{Var}(L_{[0,1]}) = \text{EX}(1^-)^2 P_{D\in(0,1]} - (\text{EX}(1^-)P_{D\in(0,1]})^2 = 6.51 \\ &\text{Var}(L_{[1,2]}) = 6.08 \\ &\text{Var}(L_{[2,3]}) = 1.42 \\ &\text{Var}(L_{[3,4]}) = 1.33\end{aligned}$$

Bank considers *unexpected loss*

Credit Reserve: If a risk-capital of two standard deviations is required, the default free party anticipates to use risk capital. A yearly return on such capital leads to an additional surcharge.
Note: A high enough return rate would lead to the possibility of arbitrage (an initial credit-risk premium of more than \$10 in the example)

Credit VaR: unexpected credit loss at some confidence level over a certain time horizon. If we note by $f(x)$ the distribution of credit loses over a time horizon (typically one year), denote C the confidence level (*e.g.* 95%), then the WCL is efined to be $\int_{WCL}^{\infty} f(x)dx = 1 - C$, with Credit VaR=WCL-ECL.

Example (FRM 1998): Credit VaR has CI of 99.9%, bond valued at \$1M one month forward with one year default probability of 2%. Assume no recovery. Calculate the Credit VaR.
Let d be the monthly probability of default. $(1-d)^{12} = 1 - 2\% \Rightarrow d = 0.001682$

$$\begin{aligned}\text{ECL} &= \$1M \cdot 0.001682 \cdot 100\% = \$1000 \\ \text{WCL}(99.9\%) &= \text{WCL}(1 - 0.001682) \approx \$1M\end{aligned}$$

$$\text{Credit VaR} = \$1M - 1682 = \$998,318$$

Example: Two bonds have credit VaR with CI of 99.9%, both of the bond valued at \$50K one month forward with one year default probability of 2% for each. Assume no recovery or correlation. Calculate the Credit VaR.

Let d be the monthly probability of default. $(1-d)^{12} = 1 - 2\% \Rightarrow d = 0.001682$

$$\text{ECL} = \$500K \cdot 0.001682 \cdot 100\% = \$841$$
$$\text{WCL}(99.9\%) = \text{WCL}(1 - 0.001682) \approx \$250K$$

$$\text{Credit VaR} = \$250K - 841 = \$249,159$$

7.4 Credit Risk Models

There are two main types of Credit Risk Models.

1. Credit Metrics
   - Introduced by JP Morgan
   - Based on bond prices
   - Consider several credit states
   - Modelled with Markov Chain
2. KMV
   - Based on the Merton credit model
   - Based on equity prices
   - Only two states: solvency and default
   - Modelled with Black-Scholes theorem

7.4.1 Credit Metrics

Key characteristics:

- Credit risk is driven by movements in bond ratings.
- Credit events are rating downgrades obtained through a matrix of migration probabilities.
- Each instrument is valued using the credit spread for each rating class
- Recovery rates are obtained from historical similarities
- Correlations between defaults are inferred from equity prices, assigning each obligor to a combination of 152 indices (factor decomposition)
- It does not integrate market and credit risk

To calculate the correlations between two counter-party in Credit Metrics

1. Models were generated using 152 indices, 28 country indices, 19 worldwide indices
2. Apply linear regression. E.g.:
   - $r_1 = 0.9 r_{US,Ch} + k_1 \epsilon_1$
   - $r_2 = 0.7 r_{GE,Ch} + 0.1 r_{GE,Ba} + k_2 \epsilon_2$
3. In the linear regressions, assume the residuals ϵ_1, ϵ_2 are uncorrelated.
4. Calculate the matrix multiplication of regression factors:
   $\rho_{def}(r_1, r_2) = 0.9 \cdot 0.74 \rho(r_{US,Ch}, r_{GE,Ch}) + 0.9 \cdot 0.15 \rho(r_{US,Ch}, r_{GE,Ba})$

If we want to simulate more than one asset

1. Consider a portfolio consisting of m counter parties and a total of n possible credit states
2. We need to simulate n^m states. Their multivariate distribution is given by marginal distributions and correlations are given by the regression models
3. To obtain accurate results, large simulations are needed, since many states have low probability
4. This does not integrate market and credit risk. Losses are assumed to be due to credit events alone.

7.4.2 Merton Model

In 1974, Merton introduced the view: equity value is a call option on the value of the assets of the firm with a strike price equal to the firm's debt.

Stock price embodies the forecast of the firm's default probabilities, in the same way that an option embodies an implied forecast of the option being exercised.

A *company* is a legal entity that has assets and liability. equity=value of assets-value of liability.

Consider a simple setting: Assume the firm's value is V. The firm issued a zero-coupon bond due in one time unit, equal to K.

- Solvency: The firm's value is higher than the bond ($V > K$)

  Bond holders get their bond payment. The remainder value of spread is distributed among the shareholders.

- Default: The value of the firm is less than the bond ($V < K$)

  Bond holders get the value of the firm V and equity value is 0.

Valud of equity= $(V-K)_+$. This is a call option on the value of the firm, with strike price equal to debt.

Equity Value:
The value of a firm can be determined at the time debt is due $S_T = \max(V_T - K, 0)$. Since the firm's value equals equity plus bonds: $B_T = V_T - \max(V_T - K, 0) = \min(V_T, K)$. Before debt is due, stocks are priced as a call option and bonds are priced as a short put.

Pricing Equity:
Assume the firm's assets are geometric Brownian: $dV = \mu V dt + \sigma V dz$. Also assume there is no transaction costs including bankruptcy costs, then $V = B + S$. We can price S with the Black-Scholes methodology, obtaining:

$$S = VN(d_1) - Ke^{-rt}N(d_2),$$

$$\text{where } d_1 = \frac{-\ln(Ke^{-r\tau}/V)}{\sigma\sqrt{\tau}} + \frac{\sigma\sqrt{\tau}}{2}, d_2 = d_1 - \sigma\sqrt{\tau}.$$

Note that $Ke^{-r\tau}$ is the present discounted value of liability, V is the value of the firm, $Ke^{-r\tau}/V$ is the leverage of the firm, $\sigma\sqrt{\tau}$ is the volatility over time. σ is the volatility of the assets, which is hard to calculate.

Asset Volatility:
In practice, only equity volatility is observed, but not asset volatility, which we can define as follows: The hedge ratio $dS = \frac{\partial S}{\partial V} dV$ yields a relationship between stochastic differential equations for S and V, from where we get

$$\sigma_S S = \sigma_V \frac{\partial S}{\partial V} \text{ and } \sigma_V = \sigma_S = \frac{S\partial V}{V\partial S}.$$

We observe dS, but not dV or σ_V. Note σ_S and V are both functions of σ_V. We need to solve for a fixed point to find σ_V.

Pricing Debt:
The value of the bond is given by $B = V - S$. Therefore, $B = Ke^{-rt}N(d_2) + N(1 - N(d_1))$ or $\frac{B}{Ke^{-r\tau}} = N(d_2) + \frac{V}{Ke^{-r\tau}}N(-d_1)$. Black-Schole tells us that $N(d_2)$ is the probability of exercising the all or probability that the bond will not default. $\frac{B}{Ke^{-r\tau}}$ is the riskless value of the bond. $\frac{V}{Ke^{-r\tau}}N(-d_1)$ takes the recovery into account.

Credit Loss:
The expected credit loss is the value of the risk-free bond minus the risky bond.

$$\begin{aligned}
\text{ECL} &= Ke^{-r\tau} - Ke^{-r\tau}N(d_2) - V[1 - N(d_1)] \\
&= N(-d2)\left[Ke^{-r\tau} - V\frac{N(-d1)}{N(-d2)}\right] \\
&= \text{Probability of default} \times \text{PV of face value of the bond} \times \\
&\text{PV of expected value of the firm in case of default}
\end{aligned}$$

Advantages:

- Relies on equity prices, not bond prices. More companies have stock prices.
- Correlation among equity prices can generate correlations among default probabilities
- Generates movements in EDP that can lead to credit ratings

Disadvantages:

- Cannot be used for counter parties without traded stocks (*e.g.* governments)
- Relies on a static model for the firm's capital and risk structure
- The firm could take on operations that will increase stock price, but also its volatility, further leads to increased credit spreads. In contradiction with basic premise: higher equity prices should be reflected in lower credit spreads

7.4.3 KMV Model

KMV was a firm founded by Kealhofer, McQuown and Vasicek and then sold to Moody's. The KMV model is a simplified version of Merton's model. The basic model inputs are:

- Value of the liabilities: liabilities (1 year) $+0.5\times$ long term debt
- Stock values
- Asset volatility
- Assets

Define the *distance to default* by the following equation:

$$\text{distance to default} = \frac{\text{market value of assets} - \text{default point}}{\text{market value of assets} \times \text{asset volatility}}$$

Example: A firm with \$100M assets, \$80M liabilities, annualized volatility of \$10M. Then the distance from default is $\frac{A-K}{\sigma} = \frac{100-80}{10} = 2$. The default probability is then 0.023 by $\mathcal{N}(0,1)$.

Capital Requirements under BIS:

$$K = \text{LGD} \times \left[N \left(\frac{N^{-1}(PD) - \sqrt{R} N^{-1}(0.999)}{\sqrt{1-R}} - PD \right) \right] \times \text{MF},$$

where N is the cumulative normal distribution, $\sqrt{R}$ is one-factor asset correlation. MF is the maturity function, which empirically adjusts for the maturity of the portfolio.

8 Introduction to Quantitative Investments

A *hedge fund* is an unregulated invetment structure. Investment skills was hard to exhibit looking only at returns.

Speculation v.s. arbitrage:

- Cannot obtain returns without taking risks (risks associated with risk premiums give profits)
- Risks can often be mitigated by hedging
- Invetment opportunities exist with lower-than-normal risks (Arbitrage)
- Market inefficiencies are a source of arbitrage

Hedge funds are investment vehicles deisgned to take advantage of arbitrage (in general), market inefficiency (a type of arbitrage) and risk mitigation strategies through hedging.

Example (Fund Cash Flows): Risk Transfer. Assume correlation of 50% between city and resort precipitation. With 75% probability, both swaps yield opposite flows, we collect the fee \$2M. With 12.5% probability, we receive payments from both \$22M. With 12.5% probability, we have to pay both -\$18M.

Investiment parameters:

- Investment amount: \$20M
- Average return: 10%
- Volatility: $\sigma^2 = 0.75 \cdot 0 + 0.125 \cdot 1.1^2 + 0.125 \cdot 0.9^2 = 0.25$, thus $\sigma = 0.5$
- Sharpe ratio: $10\%/50\% = 0.2 < 1$, bad deal

A diversified fund: same deal in multiple cities, we get volatility 5%, sharpe ratio 2>1, thus a good deal.

8.1 Funds

A *fund* is a way to collect money from investors s.t. an external company manages it.

- Management Company: licensed/registered with local authorities (*e.g.* OSC, FSA, etc). Required to create a fund
- Bank: store the money in an account under the name of the fund
- Administrator: Company who count the money (NAV of the funds): earning, lossing. Mutual fund is calculated daily. Hedge fund is calculated monthly. share price= $\frac{NAV_t}{\text{Number of shares}_t}$
- Auditor: check that the administrator didn't make a mistake, because they need to file the tax
- Broker (Custodian): needed when we have stock trading

All 5 companies should be conflict free.

Fees:

- Management fees: proportional to NAV, often paid monthly/quarterly, accured monthly
- Performance fees: proportional to P&L, often paid annually, accured monthly

- Hurdles: stops performance fees when P&L is below a hurdle rate or modifies performance fees to P&L over a hurdle rate. Performance fees will be discounted by management fees paid. Make the management fee a loan against the performance fee.
- First loss fees: Managers give investors protection against loses in exchange to hihger performance fees

Collateral: How much money do we need for a contract? Usually contract counter parties want cash or securities as collateral for future payments. This allows indirect borrowing to take place, as one can act as if you have money when you don't.
e.g. In a mortgage, the house is a collateral against future payments. If payments do not occur, the bank keeps the house.

Leverage: How much money do we need for the fund? Depends on the counter parties.

- Case 1 (no trust): \$20M per swap, \$2B in total.
- Case 2 (some trust): \$20xM per swap, \$2xB total where $0 < x < 1$. Possible if a bank lends difference or the city sign the swap without full collateral.
- Case 3 (Ignorant counter parties): swap with \$0.

Offering Memorandum (OM):
A legally binding document that stipulates the basic conditions for the proper management of the fund, including: liquidity provision, portfolio and investment guidelines, fees and valuation principles and methodologies. Investors must read it and invest only if in aggreement with the terms.

Credit Risk: investors have negative profit, while manager still have infinite profit.

Most Favored Nations (MFNs):
Different investors can pay different fees through the issuance of different share classes. *Most Favored Nations Clause* is a guarantee issued to a special investor that no other investor will be given better fees.

Prime broker:

- Fund trades can get executed through any execution broker
- All fund trades settle at the Prime Broker, a key service provider to the fund
- A fund can have one or more PBs
- Services include: security lending, leveraged trade executions and cash management
- PBs also act as the primary custodian of the assets
- In certain situations, also provide risk management services
- Sublease office space and provide access to other facility-based benefis
- The funds are the PB's clients and often times facilitate introduction to investors.

Capacity of Hedge Fund: Amount of Asset Under Management (AUM) they can manage before performance will suffer. Capacity is an issue for funds operating in markets with limited inefficiences, sizes or volumes.
e.g. If only 100 swaps are available to be negotiated, then capacity is \$2B. With < \$2B AUM, risk will rise but return continues at 10%. With > \$2B AUM, return will lower, since additional funds cannot be invested and P&L will have be shared among more investors.

Growth and Death of a Fund:

- Growth: fees, AUM growth, risks, capacity
- fraud, performance drag, blow-up, flows

8.2 Understanding Performance

Portfolio Returns:
Given S_k the share value timeseries.

- Return: $r_k = \frac{S_k - S_{k-1}}{S_{k-1}}$
- Log Return: $r_k^{\log} = \log \frac{S_k}{S_{k-1}} = \log(1 + r_k)$

We can collect a time series of portfolio returns

Portfolio Stats: Given return density ρ, can compute the cumulative return distribution function $F = \text{Prob}(\text{return} \leq x)$, $\rho = \frac{\partial F}{\partial x}$

Average/Mean Return: $\mu = \int_{-\infty}^{\infty} x\rho(x)dx = \mathbb{E}(x)$, estimated by $\bar{r} = \frac{1}{n}\sum_{i=1}^{n} r_i$.

Volatility/std: $\sigma^2 = \int_{-\infty}^{\infty}(x-\mu)^2\rho(x)dx = \mathbb{E}((x-\mu)^2)$, estimated by $\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(r_i - \bar{r})^2}$.

Returns over time:
Running mean/std will have noise because of outliers in the sample. We can compute averages over time μ/σ. Stocks are higher risk. Bonds are medium risk. Hedge funds are lower risk.

- Time-weighted Rate of Return: monthly returns can be compounded overtime:
  $1 + R = (1 + r_1)(1 + r_2)\cdots(1 + r_n)$
- Internal Rate of Return: Takes in to account the amounts invested overtime. If we make investments worth p_k at time t_k ago, and final value of the fund is V, then the internal rate of return is defined by $V = \sum_k p_k(1 + R)^{t_k}$.

Correlation: For two funds, with monthly returns given by random variables X and Y, the covariance is obtained as:

$$Cov(X,Y) = \mathbb{E}(X - \mathbb{E}(X))(Y - \mathbb{E}(Y)) = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})$$

Correlation can be calculated as:

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}$$

By Cauchy-Schwartz, $\rho \in [-1, 1]$.

Portfolio Volatility:
Consider a portfolio Π that allocates w_i to assets with returns given by random variables X_i, $i = 1, ..., n$. If the covariance matrix of X_i is given by $V = (\sigma_{i,j})$, then the portfolio volatility is given by:

$$\begin{aligned}
\sigma_\Pi^2 &= \langle \sum_{i=1}^{n} w_i X_i, \sum_{j=1}^{n} w_j X_j \rangle \\
&= \sum_{i,j} w_i w_j \langle X_i, X_j \rangle \\
&= wVw^T \\
\sigma_\Pi &= \sqrt{wVw^T}
\end{aligned}$$

σ_Π implies portfolio diversification. When correlations are less than 1 or negative, portfolio volatility decreases.

Marginal Value of Diversification:
Assume N assets with returns given by random variables X_i, $i = 1, ..., N$, and a portfolio Π with allocations w_i. Assume constant pairwise correlations C, equal asset allocations and also equal means and variances μ and σ. Then

$$\begin{aligned}
\sigma_\Pi^2 &= \sum_i w_i^2 \sigma_i^2 + \sum_{i \neq j} w_i w_j \sigma_i \sigma_j \\
&= C + \underbrace{\frac{\sigma^2 - C}{N}}_{\text{Rate of Diversification}}
\end{aligned}$$

8.2.1 Portfolio Risk and Return (Markowitz)

Portfolios are characterized by two parameters: expected return and standard deviation.
Bonds+stocks with some proportion (diversification) gives lowest risk and good returns, and is better than all bonds (lower return but higher risk).

Efficient Frontier: The optimal combination of risk and a return. It is the boundary between possible and impossible investments.

Sharpe Ratio:
Markowitz argue that we should invest in the efficient frontier, but does not specify where. Sharpe tells us where in the frontier we should invest.

- Suppose we seek a portfolio Π that maximizes the probability of exceeding a fixed benchmark r. If returns are normally distributed, then $\text{Prob}(\Pi \geq r) = 1 - \phi\left(\frac{r-\mu}{\sigma}\right)$. The rational decision is to maximize the Sharpe ratio with benchmark r with $\frac{\mu - r}{\sigma}$
- Suppose we seek a portfolio Π that maximizes the probability of exceeding a known but random benchmark $Y + r$. If returns are normally distributed, then $\text{Prob}(\Pi \geq Y + r) = \text{Prob}((\Pi - Y) \geq +r) = 1 - \phi\left(\frac{r - \mu_{\Pi - Y}}{\sigma_{\Pi - Y}}\right)$. The rational decision is to maximize the Sharpe ratio with benchmark r with $\frac{\mu_{\Pi - Y} - r}{\sigma_{\Pi - Y}}$, A portfolio manager will be paid a bonus if it makes a benchmark return of r above an index Y. $\sigma_{\Pi - Y}$ is the tracking error.

Portable Alpha:

- Portfolios with return independent of market returns deliver alpha
- Portfolios with returns dependend on markets, indices, etc deliver beta
- Optimal performance of portfolios benchmarked to indices with futures or forwards can be obtained via portable alpha strategies
  1. Construct an optimal portfolio P with cash benchmark.
  2. Add an index futres contract to portfolio P.
- This constitutes a direct application of absolute return strategies in the institutional portfolio

Examples:

- Alpha: hedge funds, absolute return strategies, active portfolio management, prop desks
- Beta: stocks, bonds, infrastructure, private equity, credit

9 Trading Strategies

Hedge Funds Characteristics:

- Turn risk into return: market inefficiencies, information advantage
- Return profile is not directly linked to market direction
- Exchange traded or OTC securities
- May have a limited opportunity set
- May have liquidity restrictions
- Often innovative

9.1 Equity Based Trading

Equity is the populous investment style. It has many different trading styles and characteristics (long only, long biased, short biased, short only, long short, equity market neutral, quant equity)

Equity Long

- Fundamental: traditional accounting. It is hard to analyze tech companies
  - P/B: Price to Book
  - EPS: Earnings per share (Price/Earnings ratio). The company is making $8.35 per $147.37 if the stock price is $147.37 and EPS is $8.35
  - Net income
  - Free cashflow: cash generated but not needed to fund the operation
  - EBITDA: Earnings before income tax depreciation asset (Description of free cash)
  - P/E ratio: price to equity ratio. It's the future earning
- Growth: Focus more on the future
  - Historical growth strength (use history to forecast future)
  - Strong forward Earnings growth based on analyst or company projections target of >10%
  - Efficiency, costs, revenue, etc
- GARP (Growth at a Reasonable Price): In between the previous two
- Momentum: look for stocks moving significantly in one direction on high volume and jump on board to ride the momentum to a desired profit (*e.g.* Netflix in 2013)
- Technical: Developing trading signals from charts and graph patterns

Equity Short

- Stock lending: before you can short a stock, someone needs to be willing to lend it. It has restrictions, may need to give back. Financing rate: Fed+spread
- Locate process
- Uptick rule

- Short squeeze: forced to return if recalled, typically when stock is expensive
- No right to keep the position indefinitely

If stock rises, stock owners want to sell, short seller were called on the stock loan, stock rose even more. Many hedge funds were short the auto industry in 2008.

Short Selling Strategy:

- Trades based on negative company information: SEC fillings, litigations, etc.
- Very risky: take overs, short squeeze, unlimited
- Short selling gurns
- Put is better than short, risk based decision
- Short interest ratio $= \frac{\text{Short Interest}}{\text{Average Daily Volume}}$

Equity Long Short:: >0 long, <0 short, can have single name short, index short (sector hedge on NASDAQ, portfolio hedge on SPY)

9.2 Convertible Arbitrage

Convertible bonds are securities companies with bad credits issue. It is issued when both bonds (debt) and stocks fail to raise funding. It is structured as bonds offering protection arising from an obligation to pay the coupons and the principal. It also offers upscale, allowing convertible bond holders to exchange their bonds into stocks inside a period of time.

Convertible bonds are usually issued by companies with volatile stock prices and lower credit quality. Investors want to profit from the upside presented by convertible bonds, but they want to hedge the default and market risks.

Hedging is achieved by shorting the underlying stock

- Stock volatility implies upside potential for stock valuation, but convertibility of the bonds mitigates this
- More or less of the underlying stock will be donw as a hedge, which could totally eliminate the upside potential of the bond

Example: convertible bond sold at $80 (20% discount) can be converted into 10 shares anytime. Stock price is $7. Annual coupon payment is $4. Interest rate is 4%. What are possible performance 1 year later?

	Now	Calm	Bankruptcy	Explosive Success
bond	$ 80	$ 80	$ 50 (recovery rate)	$140 (After conversion)
stock	-$70	-$70	0	-$140
coupon		$4		
T-Bill	$70	$72.8 (interest)	$72.8	$72.8
Fees		-$3.5 (broker 5% on stock)	-$3.5	-$3.5
Total	$80	$83.3	$119.3	$79.3
Performance		(4+0.125)%	(4+46)%	(4-4.875)%

Optimizing the table. Consider one convertible bond, short $10x$ stocks $(0 < x < 1)$

	Now	S1	S2	S3 Success
bond	80	140	80	30
stock	-70x	-140x	-70x	0
coupon		0	4	0
T-Bill	70x	72.8x	72.8x	72.8x
Fees		-3.5	-3.5	-3.5
Total	80	$140 - 70.7x$	$84 - 0.7x$	$30 + 69.3x$

Each event has probability $\frac{1}{3}$. Then we can calculate μ and σ for these 3 events. See what gives the highest Sharpe ratio. $(x = 8)$