# Limits & continuity

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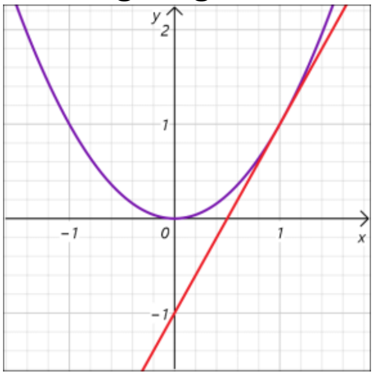
## 1. Limits

### a. Drawing tangents and a first limit




To find the tangent line to $y = x^2$ at point P(1,1), consider a nearby point Q(1+h,$1+h^2$), the line that goes through PQ is called the secant line. It has slope $\frac{\Delta y}{\Delta x} = \frac{(1+h)^2-1}{1+h-1} = h+2$, take the limit as h goes to 0, $\lim_{h \to 0} \frac{\Delta y}{\Delta x} = 2$, this is the slope of the tangent line($y = 2x - 1$)

### b. Another limit and computing velocity

**E.g.**$s(t) = 4.9t^2$, $s(t)$ is the distance travelled after $t$ seconds, average velocity between t=1s and t=1.1s is $\bar{v} = \frac{change\ in\ position}{change\ in\ time} = \frac{s(1.1)-s(1)}{1.1-1} = 10.29m/s$.

As interval becomes arbitrarily small, $\bar{v}$ approaches 9.8m/s which is the instantaneous velocity, also the slope of the tangent line to $s(t) = 4.9t^2$ at t=1

**Definition**: Let $s(t)$be the position as a function of time, the instantaneous velocity at t=a is $\lim_{h \to 0} \frac{s(a+h)-s(a)}{h}$

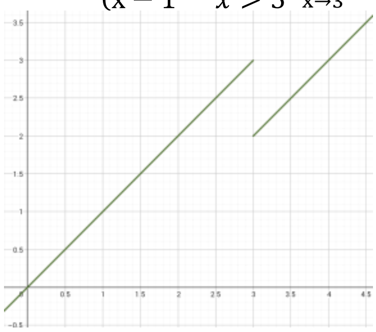
### c. The limit of a function

$\lim_{x \to a} f(x) = L\ or\ f(x) \to L\ as\ x \to a$

**Meaning**: as $x$ gets arbitrarily close to $a$, but not equal to $a$, $f(x)$ gets arbitrarily close to $L$

i. $f(x) = \begin{cases} x & x \neq 3 \\ 7 & x = 3 \end{cases}$, $\lim_{x \to 3} f(x) = 3$

ii. $f(x) = \begin{cases} x & x < 3 \\ x-1 & x > 3 \end{cases}$, $\lim_{x \to 3} f(x) DNE$, because $\lim_{x \to 3^-} f(x) \neq \lim_{x \to 3^+} f(x)$




**Definition(one-sided limits)**:

$\lim_{x \to a^-} f(x) = L$, $f(x)$ approaches $L$ as $x$ approaches $a$ from left

$\lim_{x \to a^+} f(x) = L$, $f(x)$ approaches $L$ as $x$ approaches $a$ from right

**Theorem**: $\lim_{x \to a} f(x) = L$ if and only if $\lim_{x \to a^-} f(x) = \lim_{x \to a^+} f(x) = L$

Limits can approach $\pm\infty$

### d. Calculating limits with limit laws

i. $\lim_{x \to a} c = c$, $\lim_{x \to a} x = a$

ii. Limits can interchange with basic arithmetic operations

Assume $\lim_{x\to a} f(x) = L, \lim_{x\to a} g(x) = K$ both exist, then $\lim_{x\to a} f(x) \pm g(x) = L \pm K$, $\lim_{x\to a} f(x) \times g(x) = L \times K$, $\lim_{x\to a} cf(x) = cL$, $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L}{K}$ (assuming K≠0)

iii. Limits and powers: $\lim_{x\to a} \left(f(x)\right)^n = \left(\lim_{x\to a} f(x)\right)^n$

iv. Suppose $f(x) = g(x)$ except when x=$a$, and $\lim_{x\to a} g(x)$ exists, then $\lim_{x\to a} f(x) = \lim_{x\to a} g(x)$

v. **Squeeze theorem**: let $f(x), g(x), h(x)$ be functions such that $g(x) \le f(x) \le h(x)$, except possibly at $x = a$, suppose $\lim_{x\to a} h(x) = \lim_{x\to a} g(x)$, then $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = \lim_{x\to a} g(x)$

### e. Limits at infinity

**Definition**: $\lim_{x\to \infty} f(x) = L$ $f(x)$ approaches $L$ as $x$ becomes arbitrarily large

**Remark**: when the limit exists, it is a horizontal asymptote

## 2. Continuity

**Definition:** a function is continuous at $a$, if $\lim_{x\to a} f(x) = f(a)$

i. $\lim_{x\to a} f(x)$ exists

ii. $a$ is in domain

iii. $\lim_{x\to a} f(x) = f(a)$

Left continuous: $\lim_{x\to a^-} f(x) = f(a)$

Right continuous: $\lim_{x\to a^+} f(x) = f(a)$

Continuous on $(a, b)$ ⇔continuous at every point in $(a, b)$

Continuous on $[a, b]$ ⇔continuous at every point in $(a, b)$ + right continuous at $a$ + left continuous at $b$

**Theorem:** Arithmetic operations (+-×÷) preserves continuity, providing that no zero-division

a. All elementary functions (polynomials, rational, trig, inverse, log, exponential) are continuous on their domain

b. **Continuity of composed functions:** $g(x)$ is continuous at $a$, $\lim_{x\to a} g(x) = b$, and $f(x)$ is continuous at $b$, then $f \circ g(x)$ is continuous at $a$

### c. Intermediate value theorem (IVT)

Let $f$ be a continuous function on $[a, b]$, $L$ be a constant between $f(a), f(b)$, then there is a point $c \in (a, b)$, so that $f(c) = L$

# Derivatives

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## 1. Derivative:

**Definition:** The derivative of a function at a point A is $f'(a) = \lim_{h \to 0} \frac{f(a+h)-f(a)}{h}$

**Meaning:**

i. the instantaneous rate of change
ii. Slope of the tangent line

### a. Differentiability

i. If $f'(a)$ exists (the definition of limit exists), then $f(x)$ is differentiable at $a$
ii. If $f(x)$ is differentiable at every point in an interval $(a, b)$, we say $f(x)$ is differentiable on $(a, b)$
iii. If $f(x)$ is differentiable at $a$, then $f(x)$ is continuous at $a$

### b. Higher order derivatives

$$f''(x) = \frac{\mathrm{d}}{dx} f'(x) = \frac{\mathrm{d}}{dx}\frac{\mathrm{d}f}{\mathrm{d}x} = \frac{\mathrm{d}^2 f}{\mathrm{d}x^2}$$

### c. Interpretation of derivatives

The general equation for tangent line to $f(x)$ at $x = a$ is $y = f(a) + f'(a)(x - a)$

## 2. Differentiation rules

If $s(x) = af(x) + bg(x)$, then $s'(x) = af'(x) + \mathrm{b}g'(x)$

$$\frac{\mathrm{d}}{\mathrm{d}x} x^r = rx^{r-1}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} f(x)g(x) = f'(x)g(x) + f(x)g'(x)$$

$$\frac{\mathrm{d}}{\mathrm{d}x}\frac{f(x)}{g(x)} = \frac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} a^x = a^x lna$$

$$\frac{\mathrm{d}}{\mathrm{d}x} sinx = cosx \quad \frac{\mathrm{d}}{\mathrm{d}x} cosx = -sinx \quad \frac{\mathrm{d}}{\mathrm{d}x} tanx = secx^2 \quad \frac{\mathrm{d}}{\mathrm{d}x} cotx = cscx^2$$

$$\frac{\mathrm{d}}{\mathrm{d}x} secx = secxtanx \quad \frac{\mathrm{d}}{\mathrm{d}x} cscx = -cscxcotx$$

a. **Chain rule:** $\frac{\mathrm{d}}{\mathrm{d}x} f(g(x)) = f'(g(x))g'(x)$

b. **Implicit differentiation:** $\frac{\mathrm{d}}{\mathrm{d}x} f(x)^2 = 2f(x)f'(x)$, $\frac{\mathrm{d}}{\mathrm{d}x} y^2 = 2y\frac{\mathrm{dy}}{\mathrm{d}x}$

### c. Inverse trigonometry functions:

$$\frac{\mathrm{d}}{\mathrm{d}x} arcsinx = \frac{1}{\sqrt{1-x^2}} \qquad \frac{\mathrm{d}}{\mathrm{d}x} arccosx = -\frac{1}{\sqrt{1-x^2}}$$

$$\frac{\mathrm{d}}{\mathrm{d}x} arctanx = \frac{1}{1+x^2}$$

## 3. Applications of derivative

### a. Optimization

#### i. Max and min values

**Definition**: Let $f(x)$ be a function with domain D,

$f(x)$ has a **global max** at $c \in \mathrm{D} \Leftrightarrow f(c) \geq f(x)$ for all $x \in \mathrm{D} \Leftrightarrow f(c)$ is the maximum of $f(x)$

$f(x)$ has a **global min** at $c \in \mathrm{D} \Leftrightarrow f(c) \leq f(x)$ for all $x \in \mathrm{D} \Leftrightarrow f(c)$ is the minimum of $f(x)$

$f(x)$ has a **local max** at $c \in \mathrm{D} \Leftrightarrow f(c) \geq f(x)$ for all $x$ near $c$

$f(x)$ has a **local min** at $c \in \mathrm{D} \Leftrightarrow f(c) \leq f(x)$ for all $x$ near $c$

**Theorem**: Every local max/min is a critical point or singular point. i.e. $f'(c) = 0$ if exists

ii. Finding max and min values

**Theorem**: if $f(x)$has a global max/min in $[a,b]$ at $x = c \in [a,b]$, there are three possibilities

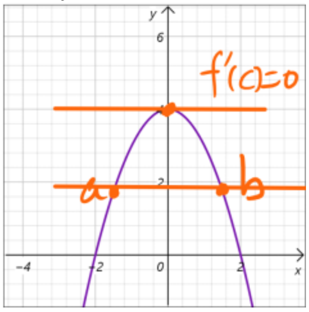
1) $f'(x) = 0$ critical point
2) $f'(x)\ DNE$ singular point
3) $c = a,\ c = b$ endpoint

Further, if $f(x)$ is continuous on $[a,b]$, it must have a global max and min on $[a,b]$

## b. Mean value theorem

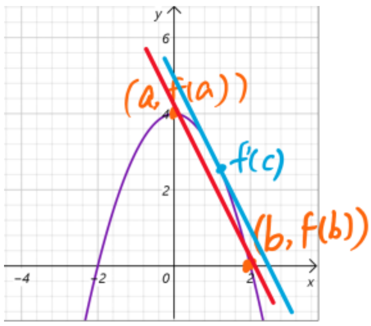
### i. Rolle's Theorem

Let $f(x)$ be a function satisfying: $f(x)$ is continuous on $[a,b]$, differentiable on $(a,b)$, and $f(a) = f(b)$. Then there exists at least one point $\left(c, f(c)\right)$, $c \in (a,b)$ with $f'(c) = 0$




### ii. Mean Value Theorem

Let $f(x)$ be a function satisfying: $f(x)$ is continuous on $[a,b]$ and differentiable on $(a,b)$. Then there exists at least one $c \in (a,b)$, such that $f'(c) = \frac{f(b)-f(a)}{b-a}$




### iii. Corollary

$f(x)$ and $g(x)$ are differentiable on $[a,b]$

1) If $f'(x) = 0$ on $[a,b]$, then $f(x)$ is constant on $[a,b]$
2) If $f'(x) = g'(x)$ on $[a,b]$, then $f(x) - g(x)$ is constant on $[a,b]$
3) if $f'(x) > 0$ on $[a,b]$, then $f(x)$ is increasing on $[a,b]$
4) if $f'(x) < 0$ on $[a,b]$, then $f(x)$ is decreasing on $[a,b]$

## c. Graph sketching

Domain, range, $x - int, y - int$

**Horizontal asymptotes**: $y = \lim_{x\to\infty} f(x)$ and/or $\lim_{x\to-\infty} f(x)$ if exist

**Vertical asymptotes:** $x = a$ if $\lim_{x\to a^-} f(x) = \pm\infty$ and/or $\lim_{x\to a^+} f(x) = \pm\infty$

**Monotonicity**: $f'(x) > 0$ increasing; $f'(x) < 0$ decreasing; $f'(x) = 0$ local max/min

**Concavity:** $f''(x) > 0$ concave up ($f(x)$ lies above all tangent lines); $f''(x) < 0$ concave down ($f(x)$ lies below all tangent lines); $f''(x) = 0$ point of inflection (if $f(x)$ is continuous and its concavity changes at $f''(x) = 0$ )

**Theorem**: $c$ is a critical point, if $f''(c) > 0$, $f(c)$ is a local min;if $f''(c) < 0$, $f(c)$ is a local max

**Symmetry:**

i. Even function f$(x) = f(-x)$
ii. odd function f$(x) + f(-x)$ = 0
iii. Periodic f$(x + T) = f(x)$

# 4. Applications of derivative in real world

a. **Velocity & acceleration**

$v(t) = s'(t), \qquad a(t) = v'(t) = s''(t)$

b. **Exponential growth & decay**

Quantity $y(t)$, whose rate of change is proportional to $y(t)$

$\frac{\mathrm{d}y}{\mathrm{d}t} = ky(t)$, then $y(t) = ce^{kt}$, $c$ is the initial value

General formula for doubling time: $t = \frac{\ln 2}{\mathrm{k}}$

c. **Carbon dating (half life problem)**

$y(t) = ce^{kt}, k = -\frac{\ln 2}{\text{half life}}$

d. **Newton's law of cooling**

Rate of change of temperature is proportional to the difference between temperatures

$\frac{\mathrm{d}T}{\mathrm{d}t} = k(T - A)$, $A$ is the environment temperature

$T(t) = ce^{kt} + A, c = T(0) - A$

e. **Related rates**

E.g. Air is being pumped into a spherical balloon at a constant rate of 100$cm^{3}/_{s}$. How fast is the radius r changing when r=25cm?

Solution: $V = \frac{4}{3}\pi r^3$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = 4\pi r^2 \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$100 = 4\pi \times 25^2 \frac{\mathrm{d}r}{\mathrm{d}t}$$

$$\frac{\mathrm{d}r}{\mathrm{d}t} = \frac{1}{25\pi}$$

## 5. Taylor polynomials

**Definition:** The nth degree Taylor Polynomial for $f(x)$ about $x = a$ is $T_n = f(a) + f'(a)(x-a) + \frac{f''(x)}{2!}(x-a)^2 + \cdots + \frac{f^n(a)}{n!}(x-a)^n = \sum_{k=0}^{n} \frac{f^k(a)}{k!}(x-a)^k$

Specially, when $a = 0$, it is an **Maclaurin polynomial**

a. **Lagrange remainder theorem:** suppose $f^{\mathrm{n}+1}(x)$ exists for all points in $[b, d]$, if $x, a \in [b, d]$, then the nth degree Taylor approximation around satisfies $R_n(x) = f(x) - T_n(x) = \frac{1}{(n+1)!} f^{n+1}(c)(x-a)^{n+1}$ for $c$ some between $x$ and $a$. $c$ is not specified.

Let $\left|f^{n+1}(c)\right| \le M$, then $\left|R_n(x)\right| \le \frac{M}{(n+1)!}\left|(x-a)^{n+1}\right|$

## 6. Indeterminant forms and L'Hopital's rule

**Definition:** consider $\lim\limits_{x \to a} \frac{f(x)}{g(x)}$

If $\lim\limits_{x\to a} f(x) = \lim\limits_{x\to a} g(x) = 0$, it's called an indeterminant form of type $\frac{0}{0}$

If $\lim\limits_{x\to a} f(x) = \pm\infty$ and $\lim\limits_{x\to a} g(x) = \pm\infty$, it's called an indeterminant form of type $\frac{\infty}{\infty}$

**Theorem: L'Hopital's rule**

Suppose $\lim\limits_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminant form, then $\lim\limits_{x\to a} \frac{f(x)}{g(x)} = \lim\limits_{x\to a} \frac{f'(x)}{g'(x)}$, provided right-hand side exists or =$\pm\infty$

## 7. Antiderivatives

(intro to integral)

**Definition:** a function $F$ is called an antiderivative of $f$ on an interval $I$ when $F'(x) = f(x)$ on $I$

# Integrals

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## 1. Summation notation $\sum$

If $j \leq k$ are integers and $a_j, a_{j+1}, \dots, a_k \in R$, then $\sum_{i=j}^{k} a_i = a_j + a_{j+1} + \dots + a_k$

a. $$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

b. $$\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

Proof: $(1+i)^3 - i^3 = 3i^2 + 3i + 1$

Sum both sides, we can get

$$(1+n)^3 - 1 = 3\sum_{i=1}^{n} i^2 + 3\frac{n(n+1)}{2} + n$$

$$\Rightarrow \sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$$

c. $\sum_{i=0}^{n} r^i = \frac{r^{n+1}-1}{r-1}$, for r≠1

### Least Upper Bound

**Definition:** Let A be a non-empty set in R bounded above, i.e. $\exists k \in R$, such that $\forall a \in A$, $a \leq k$

A real number $u^*$ is the least upper bound (supremum/sup) of A if and only if

a. $u^*$ is an upper bound
b. If $u$ is any upper bound of A, then $u^* \leq u$

Write $supA = u^*$

**Proposition:** If A has a least upper bound, then it is unique

Proof: let $u_1, u_2$ be the least upper bounds of A,

$u_2$ is an uper bound, $u_1$ is the least upper bound, by definition, $u_1 \leq u_2$

By symmetry, $u_2 \leq u_1$

Thus, $u_2 = u_1$, A has only one least upper bound

**Proposition:** let A be a non-empty set in R with a largest element M, then $supA = M$

### Greatest lower bound

**Definition:** Let A be a non-empty set in R bounded below, i.e. $\exists k \in R$, such that $\forall a \in A$, $a \geq k$

A real number $l^*$ is the least upper bound (infimum/inf) of A if and only if

a. $l^*$ is an lower bound
b. If $l$ is any lower bound of A, then $l^* \leq l$

Write $infA = l^*$

**Proposition:** If A is a non-empty set in R bounded below, then $infA$ exists and $infA = -\sup(-A)$

**Completeness Axiom (for real numbers):** if $A \neq \phi$, $A \subset R$, and $A$ is bounded above, then $A$ has a least upper bound. ($A$ is bounded below, then $A$ has a greatest lower bound) (Axiom does not follow from any other properties of R)

## 2. The Riemann Integral

Let $f: [a,b] \to R$ be bounded, i.e. $\exists k \in R$, such that $\forall x \in [a,b]$, $|f(x)| \leq k$

If $f \geq 0$ on $[a,b]$, the Riemann Integral finds and defines the area A between $f(x)$ and $y = 0$

If $f$ can be negative, A will be the signed area where $f < 0$ contributes negative area

**Definition:** A partition P of $[a,b]$ is a finite collection of points in $[a,b]$, $P = \{x_0, x_1, \dots, x_n\}$, where $a = x_0 < x_1 < \dots < x_n = b$

Let $\Delta x_i = x_i - x_{i-1} > 0,\ i = 1,2,\dots,n,\ \sum_{i=1}^{n} \Delta x_i = b - a$

Let $M_i = \sup\{f(x): x_{i-1} \leq x \leq x_i\},\ m_i = \inf\{f(x): x_{i-1} \leq x \leq x_i\}$

$M_i \Delta x_i = area\ of\ the\ larger\ rectangle\ (outer\ rectangle)$

$m_i \Delta x_i = area\ of\ the\ smaller\ rectangle\ (inner\ rectangle)$

Upper Riemann sum for P: $U(f,P) = \sum_{i=1}^{n} M_i \Delta x_i = total\ area\ of\ outer\ rectangles\ (if\ f > 0)$

Lower Riemann sum for P: $L(f,P) = \sum_{i=1}^{n} m_i \Delta x_i = total\ area\ of\ inner\ rectangles\ (if\ f > 0)$


**Area inequality:** However you define A, it must satisfy $L(f,P) \le \mathrm{A} \le U(f,P)$

Lemma: Let $P \subset Q$ be subdivisions of $[a,b]$, then $L(f,P) \le L(f,Q) \le \mathrm{A} \le U(f,Q) \le U(f,P)$

Proof for $L(f,Q) \le U(f,Q)$: $Q = \{x_0, x_1, \dots, x_n\}$, $m_i \le M_i$

$m_i \Delta x_i \le M_i \Delta x_i$, thus $\sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i (L(f,Q) \le U(f,Q))$

Proof for $U(f,Q) \le U(f,P)$:

start with special case $Q = P \cup \{\mathrm{y}\}$, choose j such that $y \in (x_{j-1}, x_j)$

$M_j = \sup\{f(x): x \in (x_{j-1}, x_j)\}$,

$M_j' = \sup\{f(x): x \in (x_{j-1}, \mathrm{y})\} \le M_j$

$M_j'' = \sup\{f(x): x \in (\mathrm{y}, x_j)\} \le M_j$

$M_j(x_j - x_{j-1}) = M_j\left[(x_j - y) + (y - x_{j-1})\right] = M_j(x_j - y) + M_j(y - x_{j-1})$

$\ge M_j'(x_j - y) + M_j''(y - x_{j-1})$

$$\sum_{i=1}^{j-1} M_i \Delta x_i + M_j \Delta x_j + \sum_{i=j+1}^{n} M_i \Delta x_i \ge \sum_{i=1}^{j-1} M_i \Delta x_i + M_j'(x_j - y) + M_j''(y - x_{j-1}) + \sum_{i=j+1}^{n} M_i \Delta x_i$$

$\Rightarrow U(f,P) \ge U(f,Q)$

In general case, we can construct $P = P_1 \subset P_2 \subset \cdots \subset P_m = Q$, by adding one point at a time.

**Correlation:** For any partitions P, P' of $[a,b]$, $L(f,P') \le U(f,P) \Rightarrow supL \le \mathrm{A} \le infU$

Proof: Let Q=$P \cup P'$ (still a partition of $[a,b]$), Apply Lemma $L(f,P') \le L(f,Q) \le \mathrm{A} \le U(f,Q) \le U(f,P')$

**Definition:** Let $f: [a,b] \to R$ be bound, and the Riemann(or definite) integred, then $f$ is Riemann integrable on $[a,b]$ if and only if $supL = \inf U$al of $f$ over $[a,b]$ is $\int_a^b f(x)\,\mathrm{d}x = supL = infU$.

It is the unique real number sub that for ∀P, $L(f,P) \le \int_a^b f(x)\,\mathrm{d}x \le U(f,P)$

If $f \ge 0$ on $[a,b]$, then $\int_a^b f(x)\,\mathrm{d}x$ is the area between $f(x)$ and x-axis

**Lemma:** let $\Delta \ge 0$, if $\forall \varepsilon > 0, \Delta < \varepsilon$, then $\Delta = 0$ (can be proved by contradiction)

**Theorem (Integral test):** Let $f: [a,b] \to R$ be bounded, $f$ is integrable if and only if $\forall \varepsilon > 0$, ∃subdivision P such that $U(f,P) - L(f,P) < \varepsilon$, and in this case:

a. $\left| U(f,P) - \int_a^b f(x)\,\mathrm{d}x \right| < \varepsilon$

b. $\left| L(f,P) - \int_a^b f(x)\,\mathrm{d}x \right| < \varepsilon$

Proof: let $\varepsilon > 0$, by hypothesis ∃P such that $U(f,P) - L(f,P) < \varepsilon$

$U(f,P) \ge \inf U, L(f,P) \le \sup L$

$\Rightarrow U(f,P) - L(f,P) \ge \inf U - \sup L \ge 0 \Rightarrow \inf U = \sup L$

$\Rightarrow 0 \le \int_a^b f(x)\,\mathrm{d}x - L(f,P) \le U(f,P) - L(f,P) < \varepsilon$

**Theorem (Additivity of domain):**

a. Let $f: [a,b] \to R$ be bounded and integrable on $[a,b]$, $a < \mathrm{c} < b$, then $f$ is integrable on $[a,c]$ and $[c,b]$, and $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$

Proof: let $\varepsilon > 0$, by integral test, ∃P such that $U(f,P) - L(f,P) < \varepsilon$

Let $P^* = P \cup \{c\}$ be partitions of $[a,c]$, $P^* = \{x_0, x_1, \dots, x_j, \dots, x_n\}$, $x_j = c$

$P_1 = \{x_0, x_1, \dots, x_j\}$, be a partition of $[a,c]$

$P_2 = \{x_j, x_{j+1}, \dots, x_n\}$, be a partition of $[c,b]$

Then $U(f,P^*) = U(f,P_1) + U(f,P_2)$ and $L(f,P^*) = L(f,P_1) + L(f,P_2)$

$\varepsilon > U(f,P) - L(f,P) \geq U(f,P^*) - L(f,P^*) \geq U(f,P_1) + U(f,P_2) - L(f,P_1) + L(f,P_2)$

$\Rightarrow \varepsilon > U(f,P_1) - L(f,P_1)$ and $\varepsilon > U(f,P_2) - L(f,P_2)$

By integral test, $f$ is integrable on $[a,b]$, $[a,c]$ and $[c,b]$

$L(f,P) \leq L(f,P^*) \leq \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x \leq U(f,P^*) \leq U(f,P)$ holds for all partitions

Since $\int_a^b f(x)\,\mathrm{d}x$ has the only real number, $\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$

b. If $f$ is integrable on $[a,c]$ and $[c,b]$, then $f$ is integrable on$[a,b]$

**Theorem (Squeeze Theorem):**

a. Assume three sequences $l_n \leq r_n \leq u_n$, and $l_n, u_n \to L$, then $r_n \to L$
b. Arithmetic of limits holds for sequences

## Riemann Sum

**Definition:** If $P = \{x_0, x_1, \dots, x_n\}$ is a partition of $[a,b]$, the norm of P is $\|P\| = \max\{\Delta x_i\}$. If $c_i \in [x_{i-1}, x_i]$ for all $1 \leq i \leq n$, call $c = (c_1, c_2, \dots, c_n)$ a choice vector for P, and $R(f,P,c) = \sum_{i=1}^{n} f(c_i)\Delta x_i$ is a Riemann sum. $m_i \leq f(c_i) \leq M_i$, $L(f,P) \leq \sum_{i=1}^{n} f(c_i)\Delta x_i \leq U(f,P)$

**Theorem:** Let $f: [a,b] \to R$ be bounded and continuous, if

a. $f$ is integrable on $[a,b]$
b. If $\{P_n\}$ is a sequence of partition such that $\|P_n\| \to 0$,

then $\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} L(f,P_n) = \lim_{n\to\infty} U(f,P_n)$; if $C^{(n)}$ is a choice function for $P_n$, then $\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} R(f,P_n,C^{(n)})$

## Monotonicity

**Definition:** $f$ is monotone if $f$ is always increasing/decreasing

**Theorem:** Let $f: [a,b] \to R$ be monotone,

a. $f$ is integrable on $[a,b]$
b. let $\{P_n\}$ is a sequence of partition such that $\|P_n\| \to 0$,
   i. $L(f,P_n) \to \int_a^b f(x)\,\mathrm{d}x$, $U(f,P_n) \to \int_a^b f(x)\,\mathrm{d}x$
   ii. If $C^{(n)}$ is a choice function for $P_n$, then $\int_a^b f(x)\,\mathrm{d}x = \lim_{n\to\infty} R(f,P_n,C^{(n)})$

**Remark:** if $f$ is integrable on $[a,b]$ then theorem b always holds;

If $f$ is monotone, it will be much easier to show that $\exists\{P_n\}$ such that $\mathrm{L}(f,P_n) \to \int_a^b f(x)\,\mathrm{d}x$, $U(f,P_n) \to \int_a^b f(x)\,\mathrm{d}x$

Proof: take $\varepsilon = \frac{1}{n}$, $\exists P_n$ such that $0 \leq U(f,P) - L(f,P) < \frac{1}{n} \to 0$

$\Rightarrow U(f,P) - L(f,P) \to 0$ by squeeze theorem

By integral test, $\mathrm{L}(f,P_n) \to \int_a^b f(x)\,\mathrm{d}x$, $U(f,P_n) \to \int_a^b f(x)\,\mathrm{d}x$

# 3. Properties of integral

**Theorem (linearity of integrals):** Let $f, g: [a,b] \to R$ and $\mathrm{A}, \mathrm{B} \in \mathrm{R}$. If $f, g$ are integrable, then $Af + Bg$ is integrable and $\int_a^b Af(x) + Bg(x)\,\mathrm{d}x = A\int_a^b f(x)\,\mathrm{d}x + B\int_a^b g(x)\,\mathrm{d}x$

Proof (Assume $f, g$ are continuous, $Af + Bg$ is continuous and intergrable):

By theorem of Riemann Sum, if $\{P_n\}$ satisfies $\|P_n\| \to 0$, then

$$\int_{\mathrm{a}}^{\mathrm{b}} \mathrm{Af}(x) + Bg(x)\,\mathrm{d}x = \lim_{n\to\infty} \sum_{i=1}^{N} Af(C_i^n)\Delta x_i^n + Bg(C_i^n)\Delta x_i^n$$

$$= A\lim_{n\to\infty} \sum_{i=1}^{N} f(C_i^n)\Delta x_i^n + B\lim_{n\to\infty} \sum_{i=1}^{N} g(C_i^n)\Delta x_i^n = A\int_a^b f(x)\,\mathrm{d}x + B\int_a^b g(x)\,\mathrm{d}x$$

**Remark**: Assume $Af + Bg$ is integrable, one can use monotonicity remark to make the above argument work and show $\int_a^b Af(x) + Bg(x)\,\mathrm{d}x = A\int_a^b f(x)\,\mathrm{d}x + B\int_a^b g(x)\,\mathrm{d}x$

**Theorem (order property of integral):**

a. If $f, g$ are integrable and $f \le g$ for all $x \in [a,b]$, then $\int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$

Proof: Assume $h(x) \ge 0$ is integrable and $\forall$partition P, $U(h,P) \ge 0, \inf U \ge 0, \int_a^b h(x)\,\mathrm{d}x \ge 0$,

Take $h(x) = g(x) - f(x) \ge 0$ on $[a,b]$ ($f \le g$ for all $x \in [a,b]$)

$$0 \le \int_a^b h(x)\,\mathrm{d}x = \int_a^b g(x) - f(x)dx = \int_a^b g(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x$$

$$\Rightarrow \int_a^b f(x)\,\mathrm{d}x \le \int_a^b g(x)\,\mathrm{d}x$$

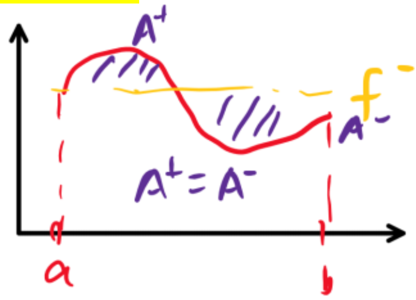
b. If $f$ is integrable, then $|f(x)|$ is integrable and $\left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$ (triangle inequality)

Proof: $-|f(x)| \le f(x) \le |f(x)|$ for all $x$

Both $\pm|f(x)|$ are integrable, by a, $\int_a^b -|f(x)|\,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x \le \int_a^b |f(x)|\,\mathrm{d}x$

$\Rightarrow \left|\int_a^b f(x)\,\mathrm{d}x\right| \le \int_a^b |f(x)|\,\mathrm{d}x$

**Definition (Mean Value)**: Let $f$ be integrable on $[a,b]$, the mean value of $f$ is $\bar{f} = \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a}$




**Theorem (Mean Value Theorem for Integrals)**: Assume $f: [a,b] \to R$ is continuous, then there is a $c \in [a,b]$ such that $\bar{f} = f(c)$.

Proof: By min-max theorem, $\exists c_{min}, c_{max}$ such that $\forall x \in [a,b], f(c_{min}) \le f(x) \le f(c_{max})$.

Then, by the order property of integrals, $\int_a^b f(c_{min})\,\mathrm{d}x \le \int_a^b f(x)\,\mathrm{d}x \le \int_a^b f(c_{max})\,\mathrm{d}x$

$\frac{\int_a^b f(c_{min})dx}{b-a} \le \frac{\int_a^b f(x)\,\mathrm{d}x}{b-a} \le \frac{\int_a^b f(c_{max})\,\mathrm{d}x}{b-a} \Rightarrow f(c_{min}) \le \bar{f} \le f(c_{max})$.

Because $f$ is continuous, by Intermediate Value Theorem, $\exists c \in [a,b]$, such that $\bar{f} = f(c)$

## Fundamental Theorem of Calculus:

a. Assume $f: [a,b] \to R$ is continuous, let $d \in [a,b]$, and $F(x) = \int_d^x f(t)\,\mathrm{d}t$, then $F'(x) = f(x)$, $\forall x \in [a,b]$

Proof: Let $F(x) = \int_a^x f(t)\,\mathrm{d}t$

Then $F'(x) = \lim_{h\to 0} \frac{\int_a^{x+h} f(x)dx - \int_a^x f(x)dx}{h}$ by definition of derivatives

$= \lim_{h\to 0} \frac{\int_x^{x+h} f(x)dx}{h}$ by additivity of domain

This Is the mean value on $[x, x+h]$

By mean value theorem for integrals,

$\exists c(h) \in [x, x+h]$, such that $\bar{f} = f(c(h))$

$F'(x) = \lim_{h\to 0} f(c(h)) = f(x)$ by squeeze theorem and continuity

b. Assume $f: [a,b] \to R$ is integrable, let $G$ be an antidetrivative of $f$, i.e. $G'(x) = f(x), \forall x \in [a,b]$, then $\int_a^b f(x)\,\mathrm{d}x = G(b) - G(a) = G|_a^b = G(x)|_{x=a}^{x=b}$

Proof: Let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$

$$G(b) - G(a) = \sum_{i=1}^{N} [G(x_i) - G(x_{i-1})]$$

By the ordinary mean value theorem, $\exists c_i \in [x_{i-1}, x_i]$ such that $G(x_i) - G(x_{i-1}) = G'(c_i)\Delta x_i = f(c_i)\Delta x_i$

$m_i \Delta x_i \le f(c_i)\Delta x \le M_i \Delta x_i$

$\sum_{i=1}^n m_i \Delta x_i \le \sum_{i=1}^n f(c_i)\Delta x_i \le \sum_{i=1}^n M_i \Delta x_i$

$\Rightarrow \sum_{i=1}^{n} m_i \Delta x_i \le \sum_{i=1}^{n} [G(x_i) - G(x_{i-1})]\Delta x_i \le \sum_{i=1}^{n} M_i \Delta x_i$

$L(f,P) \le \sum_{i=1}^{n} [G(x_i) - G(x_{i-1})]\Delta x_i \le U(f,P)$

Since $\int_a^b f(x)\,dx$ is the only real number that is in $[L(f,P), U(f,P)]$ for all P, $\int_a^b f(x)\,dx = G(b) - G(a)$

**Remark**: differentiation and integration are inverse operations. Write the general antiderivative of $f$ as $\int f(x)\,dx = G(x) + C$. Call $\int f(x)\,dx$ the indefinite integral.

### Integrability of continuous functions

**Definition:** $f: I \to R$ is continuous ($I$ is an interval), if and only if $\forall x_0 \in I, \forall \varepsilon > 0, \exists \delta = \delta(x_0, \varepsilon) > 0$, such that $\forall x \in I, |x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$;

$f: I \to R$ is continuous ($I$ is an interval), if and only if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$, such that $\forall x_0, x \in I$, $|x - x_0| < \delta \Rightarrow |f(x) - f(x_0)| < \varepsilon$

Uniform continuity requires that there is a $\delta = \delta(\varepsilon) > 0$ which works for $\forall x_0 \in I$ simultaneously.

**Proposition:** $f: I \to R$ is differentiable and $f'$ is bounded on $I \Rightarrow f$ is uniformly continuous on $I$

Proof: Let $M = \{|f'(x)| : x \in I\}$ be bounded, then $|f'(c)| \le M$ for $\forall c \in I$,

Let $\varepsilon > 0$, $\delta = \frac{\varepsilon}{M}$, $x_0, x \in I$ satisfy $|x - x_0| < \delta$

$|f(x) - f(x_0)| = |f'(c)||x - x_0|$ for some $c \in (x, x_0)$ by MVT

$$\le M|x - x_0| < M\frac{\varepsilon}{M} = \varepsilon$$

**Theorem (uniform continuity):** $f: [a,b] \to R$ is continuous$\Rightarrow f$ is uniformly continuous on $[a,b]$

**Proof (continuous functions are integrable):**

Let $\varepsilon > 0$, $f: [a,b] \to R$ is continuous$\Rightarrow f$ is uniformly continuous on $[a,b]$

So $\exists \delta > 0$ such that (1) $x, x' \in [a,b]$, $|x - x'| < \delta$, $|f(x) - f(x')| < \frac{\varepsilon}{4(b-a)}$

Let P be a partition such that (2) $\|P_n\| < \delta$, $P = \{x_0, x_1, \dots, x_n\}$, $m_i, M_i$ defined as usual

Let $x \in [x_{i-1}, x_i]$, then $|x - x_i| \le \Delta x_i \le \|P_n\| < \delta$

By (1), $|f(x) - f(x_i)| < \frac{\varepsilon}{4(b-a)}$

this means $\forall x \in [x_{i-1}, x_i]$, $f(x_i) - \frac{\varepsilon}{4(b-a)} < f(x) < f(x_i) + \frac{\varepsilon}{4(b-a)}$

$$\Rightarrow f(x_i) - \frac{\varepsilon}{4(b-a)} \le m_i \le M_i \le f(x_i) + \frac{\varepsilon}{4(b-a)}$$

$$\Rightarrow M_i - m_i \le \frac{\varepsilon}{2(b-a)}$$

$$\Rightarrow U(f,P) - L(f,P) = \sum_{i=1}^{n} (M_i - m_i)\Delta x_i \le \frac{\varepsilon}{2(b-a)} \sum_{i=1}^{n} \Delta x_i = \frac{\varepsilon}{2} < \varepsilon$$

By integrability test, $f: [a,b] \to R$ is integrable

## 4. Techniques of finding integrals

$\int x^r\,dx = \frac{x^{r+1}}{r+1} + C (r \ne -1)$ $\int x^{-1}\,dx = \ln|x| + C$

$\int e^{ax}\,dx = \frac{1}{a}e^{ax} + C$ $\int b^{ax}\,dx = \frac{1}{a\ln b}b^{ax} + C\ (b > 0)$

$\int \sin ax\,dx = -\frac{1}{a}\cos ax + C$ $\int \cos ax\,dx = \frac{1}{a}\sin ax + C$

$\int (\sec ax)^2\,dx = \frac{1}{a}\tan ax + C$ $\int (\csc ax)^2\,dx = -\frac{1}{a}\cot ax + C$

$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax + C$ $\int \csc ax \cot ax\,dx = -\frac{1}{a}\csc ax + C$

$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \arcsin\frac{x}{a} + C$ $\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\arctan\frac{x}{a} + C$

### a. Substitution (Chain rule):

$$\int F'(g(x))g'(x)\,dx = F(g(x)) + C$$

**Theorem** (substitution for definite integrals):

Let $g: [a,b] \to R, g', f$ are both continuous, $f \circ g$ is well defined, then $\int_a^b f'(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$

Integrating $\int (\sin x)^m (\cos x)^n\,dx$:

i. If m is odd, let $u = \cos x, du = -\sin x dx$; If n is odd, let $u = \sin x, du = \cos x dx$

ii. If m and n are both even, use $\cos^2 x = \frac{1+\cos 2x}{2}$ or $\sin^2 x = \frac{1-\cos 2x}{2}$ to reduce m or n to odd

Integrating $\int (\sec x)^m (\tan x)^n \, dx$ if m is even or n is odd:

i. Use $1 + \tan^2 x = \sec^2 x$, $\tan' x = \sec^2 x$, $\sec' x = \sec x \tan x$

ii. If n is odd, reduce n to 1, let $u = \sec x$, $du = secx\ tanx\ dx$

iii. If m is even, let $u = \tan x$, $du = \sec^2 x\, dx$

## b. Integration by parts (Product rule):

**Theorem:** Assume $u, v: [a, b] \to R$ have continuous derivatives

i. $\int uv' \, dx = uv - \int u'v \, dx$

ii. $\int_a^b uv' \, dx = u\,v\Big|_a^b - \int_a^b u'v \, dx$

Since $dv = v' \, dx, du = u' \, dx$, we can write $\int udv = uv - \int v \, du$

## c. Reduction formula (extended from integrating by parts)

$I_0 = \ln|\sec x + \tan x|, I_m = \int \sec^{2m+1} x \, dx = \frac{1}{2m} \sec^{2m-1} x \tan x + \frac{2m-1}{2m} I_{m-1}$

$I_1 = \frac{1}{a} arc \tan\left(\frac{x}{a}\right), I_n = \int \frac{1}{(x^2+a^2)^n} dx = \frac{1}{a^{2n-1}}\left[\frac{1}{2n-2} \frac{\frac{x}{a}}{\left(\frac{x^2}{a^2}+1\right)^{n-1}} + \frac{2n-3}{2n-2} I_{n-1}\right]$

$I_0 = -e^{-x}, I_n = \int x^n e^{-x} \, dx = -x^n e^{-x} + nI_{n-1}$

$I_0 = x, I_1 = \ln|\sec x|, I_n = \int \tan^n x \, dx = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}$

$\int \csc x \, dx = \ln|\csc x - \cot x|, \int \csc^m x \, dx = -\frac{1}{m-1} \csc^{m-2} x \cot x + \frac{m-2}{m-1} \int \csc^{m-2} x \, dx$

## d. Integration of rational functions

Definition: A polynomial is a function of the form $P(x) = a_0 + a_1 x + \cdots + a_n x^n$, $a_i \in$ R, If $a_n \neq 0, \deg(P) = n$; A rational function $f$ is a function of the form $f = \frac{P(x)}{Q(x)}$ $D = \{x: Q(x) \neq 0\}$, where $P(x), Q(x)$ are polynomials

Theorem(Factor a Polynomial): Let $Q(x)$ be a polynomial, then $\exists c, \alpha_i, \beta_i, \gamma_i \in$ R, $m_i, n_i \in$ N such that $Q(x) = c(x - \alpha_1)^{m_1} \bullet \cdots \bullet \left(x - \alpha_k\right)^{m_k} \bullet \left(x^2 + \beta_1 x + \gamma_1\right)^{n_1} \bullet \cdots \bullet (x^2 + \beta_i x + \gamma_i)^{n_i}$, where ${\beta_i}^2 - 4\gamma_i < 0$

To find $\int f \, dx$ for a rational function $f$:

i. Do long division of polynomials to reduce to the case where $\deg(P) < \deg(Q)$

ii. Factor $Q(x)$

iii. Find the partial fraction decomposition of $\frac{P(x)}{Q(x)}$

In practice you will find the PFD by solving N linear equations in N unknowns

iv. Integrate each term.

## e. Inverse substitutions:

Instead of substituting $u = g(x)$, try $x = g(u)$, $dx = g'(u)du$, $\int f dx = \int f\big(g(u)\big)g'(u)du$,

i. Integrals involving $\sqrt{a^2 - x^2}$, try $x = a \sin\theta$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, $dx = a \cos\theta d\theta$

ii. Integrals involving $\sqrt{x^2 - a^2}$, try $x = a \sec\theta$, $dx = sec\theta\ tan\theta\ d\theta$

Be cautious with the signs

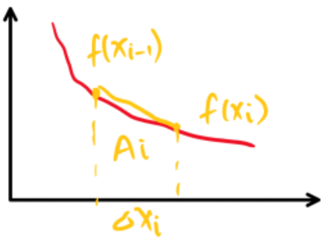
iii. Integrals involving $\sqrt{x^2 + a^2}$ or $\frac{1}{x^2+a^2}$, try $x = a \tan\theta$, $dx = \sec^2\theta\, d\theta$

iv. For integrals like $\int \frac{d\theta}{3+\sin\theta}$, try $x = \tan\frac{\theta}{2}$, $d\theta = \frac{2\,dx}{1+x^2}$, $\sin\theta = \frac{2x}{1+x^2}$, $\cos\theta = \frac{1-x^2}{1+x^2}$

## f. Numerical Methods

Often $\int_a^b f(x) \, dx$ cannot be expressed in terms of elementary functions, we can approximate $\int_a^b f(x) \, dx$ by Riemann sums/ trapezoid method/midpoint method

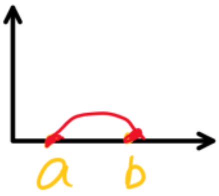
**i. Trapezoid method:**



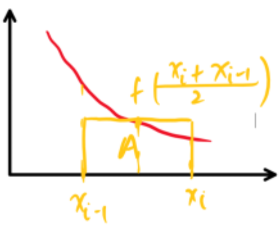

$\int_a^b f(x)\,\mathrm{d}x \approx$ Nth trapezoidal approximation area $T_n = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} \frac{f(x_{i-1})+f(x_i)}{2} \Delta x_i$

**Theorem (Trapezoidal rule)**: Let $f: [a,b] \to R$ such that $f''$ is continuous and $k = \sup\{|f''(x)|, x \in [a,b]\}$. Then $\left|\int_a^b f(x)\,\mathrm{d}x - T_n\right| \le \frac{k}{12}(b-a)(\Delta x)^2$

**Lemma:** $f: [a,b] \to R, f''$ is continuous and $f(a) = f(b) = 0$, then $-2\int_a^b f(x)\,\mathrm{d}x = \int_a^b (x-a)(b-x)f''\,\mathrm{d}x$




ii. **Midpoint method:**




$$\int_a^b f(x)\,\mathrm{d}x \approx M_n = \sum_{i=1}^{n} A_i = \sum_{i=1}^{n} f\left(\frac{x_{i-1}+x_i}{2}\right)\Delta x_i$$

Midpoint rule: $\left|\int_a^b f(x)\,\mathrm{d}x - M_n\right| \le \frac{k}{24}(b-a)(\Delta x)^2$

# 5. Improper integrals

## a. Type 1 improper integral:

**Definition:** Let $F: [a,\infty) \to R$, $\lim_{x\to\infty} F(x) = L$, if and only if $\forall \varepsilon > 0, \exists x_0 \ge a$, such that $x > x_0 \Rightarrow |F(x) - L| < \varepsilon$ (converge $F(x) \to L$ as $R \to \infty$)

$\lim_{x\to\infty} F(x) = \infty$, if and only if $\forall M \in \mathrm{R}, \exists x_0 \ge a$, such that $x > x_0 \Rightarrow F(x) > M$ ($F(x)$ diverges)

Let $f: [a,\infty) \to R$ be such that $\forall \mathrm{R} > a$, $f$ is integrable on $[a, \mathrm{R}]$

$\int_a^\infty f(x)\,\mathrm{d}x = \lim_{\mathrm{R}\to\infty} \int_a^{\mathrm{R}} f(x)\,\mathrm{d}x \in [-\infty, \infty]$ if the limit exists

$\int_a^\infty f(x)\,\mathrm{d}x$ converges if and only if $\lim_{\mathrm{R}\to\infty} \int_a^{\mathrm{R}} f(x)\,\mathrm{d}x \ne \pm\infty$

$\int_a^\infty f(x)\,\mathrm{d}x$ diverges if and only if $\lim_{\mathrm{R}\to\infty} \int_a^{\mathrm{R}} f(x)\,\mathrm{d}x = \pm\infty$

**Theorem (p-integral):**

$$p > 1, \int_1^\infty \frac{1}{x^p} = \frac{1}{p-1}; \qquad 0 < p \le 1, \int_1^\infty \frac{1}{x^p} = \infty$$

## b. Type 2 improper integral:

**Definition:** Let $F: (a, \mathrm{b}] \to R$, $\lim_{x\to a^+} F(x) = L$, if and only if $\forall \varepsilon > 0, \exists \delta > 0$ such that $0 < x - a < \delta \Rightarrow |F(x) - L| < \varepsilon$

$\lim_{x\to a^+} F(x) = \infty$, if and only if $\forall M \in \mathrm{R}, \exists \delta > 0$ such that $0 < x - a < \delta \Rightarrow F(x) > M$

Let $f: (a,b] \to R$ be such that $\forall c \in (a,b)$, $f$ is integrable on $[c,b]$ and $f$ is unbounded on $(a,b]$

$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to a^+} \int_c^{\mathrm{b}} f(x)\,\mathrm{d}x \in [-\infty, \infty]$ if the limit exists

$\int_a^b f(x)\,\mathrm{d}x$ converges if and only if $\lim_{c\to a^+} \int_c^b f(x)\,\mathrm{d}x \neq \pm\infty$
$\int_a^b f(x)\,\mathrm{d}x$ diverges if and only if $\lim_{c\to a^+} \int_c^b f(x)\,\mathrm{d}x = \pm\infty$
Note: if $f: [a,b) \to R$ is integrable on $[c,b]$, $\forall c \in (a,b)$ and $f$ is unbounded on $[a,b)$, then
$\int_a^b f(x)\,\mathrm{d}x = \lim_{c\to b^-} \int_c^b f(x)\,\mathrm{d}x \in [-\infty,\infty]$

**Theorem (p-integral):**

$$0 < p < 1, \int_0^1 \frac{1}{x^p} = \frac{1}{p-1}; \qquad p \geq 1, \int_0^1 \frac{1}{x^p} = \infty$$

## c. $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ type

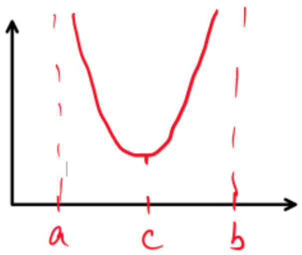
**Definition:** Let $f: R \to R$ be integrable on every bounded interval $[a,b]$, Then $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = \int_{-\infty}^{0} f(x)\,\mathrm{d}x + \int_{0}^{+\infty} f(x)\,\mathrm{d}x$, provided that this is not $\infty - \infty$ or $-\infty + \infty$, in which case $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ does not exist.

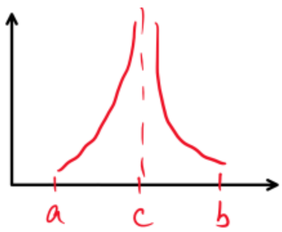
Note that $\int_{-\infty}^{+\infty} x\,\mathrm{d}x$ does not exist even though $\lim_{c\to\infty} \int_{-c}^{c} f(x)\,\mathrm{d}x = 0$

**Definition (Probability density):** Let $f: R \to [0,\infty)$ satisfy $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x = 1$, call $f(x)$ a probability density, the mean value of this density is $\int_{-\infty}^{+\infty} xf(x)\,\mathrm{d}x$

## d. More improper integrals:

$\int_a^b f(x)\,\mathrm{d}x = \int_a^c f(x)\,\mathrm{d}x + \int_c^b f(x)\,\mathrm{d}x$ this can extends to more singularities, given that it is not $\infty - \infty$ or $-\infty + \infty$,

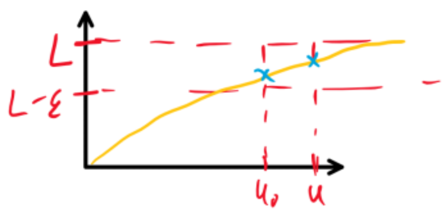



**Theorem:** Let $F: [a,\infty) \to R$ be increasing

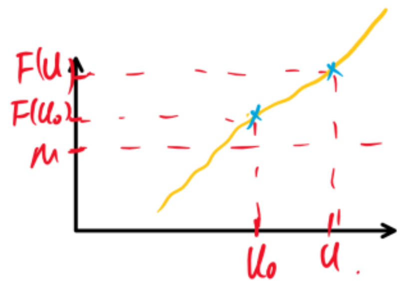
a. if $F$ is bounded above, then $F(u) \to \sup R$ as $u \to \infty$

Proof: By completeness Axiom, $L = \sup R \in$ R (because $F$ is bounded above)
Let $\varepsilon > 0$, $\exists u_0 \geq a$ such that $L - \varepsilon < F(u_0) < L$
Let $u > u_0$, $L - \varepsilon < F(u_0) \leq F(u) \leq \sup R = L$
$\Rightarrow |F(x) - L| < \varepsilon \Rightarrow F(u) \to \sup R$ as $u \to \infty$




b. if $F$ is not bounded above, then $F(u)$ diverges to $\infty$ as $u \to \infty$

Proof: let $M \in$ R,
$F$ is not bounded above$\Rightarrow \exists u_0 \geq a$ such that $F(u_0) > M$
Take $u > u_0$,then $F(u) \geq F(u_0) > M$ because $F$ is increasing
$F(u)$ diverges to $\infty$




Note: let $F(\infty) = \lim_{u\to\infty} F(u) \in (-\infty, \infty]$, in either case, $\forall u \in [a, \infty)$, $F(u) \le F(\infty)$, write $F(u) \nearrow F(\infty)$

**Theorem (Comparison test for Type 1 Integrals):** Assume $f, g: [a, \infty) \to [0, \infty)$, $f \le g$, and $f, g$ are integrable on $[a, R]$ for all $R > a$

a. If $\int_a^\infty g(x)\,\mathrm{d}x$ converges, then $\int_a^\infty f(x)\,\mathrm{d}x$ converges and $\int_a^\infty f(x)\,\mathrm{d}x \le \int_a^\infty g(x)\,\mathrm{d}x$

Proof: let $R > a$, $\int_a^R f(x)\,\mathrm{d}x \le \int_a^R g(x)\,\mathrm{d}x$ (order property)

$$\le \int_a^\infty g(x)\,\mathrm{d}x < \infty$$

Then $\int_a^R f(x)\,\mathrm{d}x \nearrow \int_a^\infty f(x)\,\mathrm{d}x < \infty$

$\le \int_a^\infty g(x)\,\mathrm{d}x$ (an upper bound for $\int_a^R f(x)\,\mathrm{d}x$)

b. If $\int_a^\infty f(x)\,\mathrm{d}x$ diverges, then $\int_a^\infty g(x)\,\mathrm{d}x$ diverges (the contrapositive of part a)

c. Same applies to type 2 integrals

## Piecewise Continuous Functions:

**Lemma:** let $h_{x_0}(x) = \begin{cases} 1 & x = x_0 \\ 0 & x \ne x_0 \end{cases}$, $\forall a < b, \forall x_0$, $h_{x_0}$ is integrable on $[a, \mathrm{b}]$, and $\int_a^{\mathrm{b}} h_{x_0}(x)\,\mathrm{d}x = 0$

**Proposition (singular point does not affect integration):** Let $g: [a, b] \to \mathrm{R}$ be integrable, assume $f: [a, b] \to \mathrm{R}$ is such that $\{x: f(x) \ne g(x)\} = \{x_1, x_2, \dots, x_n\}$ is finite. Then $f$ is integrable on $[a, b]$ and $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$

Proof: let $c_i = f(x_i) - g(x_i), i = 1, 2, \dots, k$

Then $f(x) = g(x) + \sum_{i=1}^k c_i h_{x_i}(x)$, which is integrable

And $\int_a^b f(x)\,dx = \int_a^b g(x)\,dx + \sum_{i=1}^k c_i \int_a^{\mathrm{b}} h_{x_i}(x)\,\mathrm{d}x = \int_a^b g(x)\,dx$

**Definition:** $f: [a, b] \to \mathrm{R}$ is piecewise continuous if and only if $\exists a = c_0 < c_1 < \cdots < c_k = b$ and there exist continuous functions $g_i: [c_{i-1}, c_i] \to \mathrm{R}$ such that $f(x) = g_i(x)$ for $\forall x \in (c_{i-1}, c_i)$

**Fact:** $f$ is piecewise continuous $\Rightarrow f$ is bounded and $\sup f(x) = \max\{\sup g_i, \sup f\}$

**Proposition (piecewise continuous functions are integrable):** Let $f: [a, b] \to \mathrm{R}$ be piecewise continuous and $g_i$,$c_i$ are as in the definition. Then $f$ is integrable and $\int_a^b f(x)\,dx = \sum_{i=1}^k \int_{c_{i-1}}^{c_i} g_i(x)\,\mathrm{d}x$

Proof: $g_i$ is continuous and integrable on $[c_{i-1}, c_i]$

$\{x: x \in [c_{i-1}, c_i], f(x) \ne g_i(x)\}$ is finite

$f$ is integrable on $[c_{i-1}, c_i]$ and $\int_{c_{i-1}}^{c_i} f(x)\,dx = \int_{c_{i-1}}^{c_i} g(x)\,dx$ since a singular point does not affect an integration

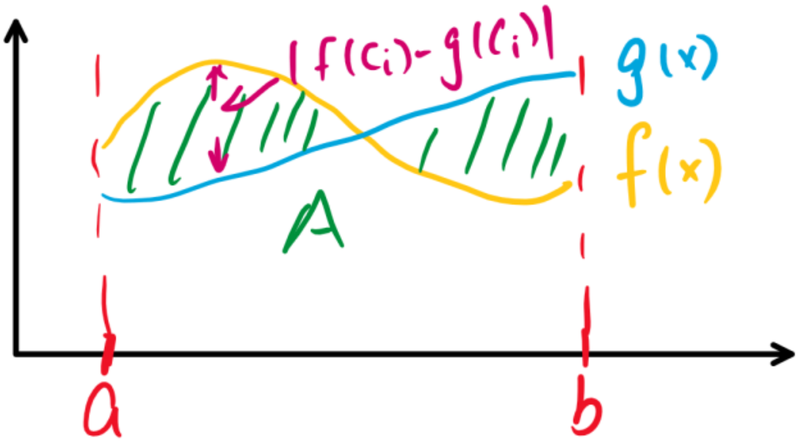
By additivity of domain, $f$ is integrable on $[a, b]$ and $\int_a^b f(x)\,dx = \sum_{i=1}^k \int_{c_{i-1}}^{c_i} g_i(x)\,\mathrm{d}x$

# 6. Application of integrals

## a. Area

To find area between $f$ and $g$ $x \in [a, b]$ using Riemann sum, the height of ith rectangle is $|f(c_i) - g(c_i)|$, $Area = \int_a^b |f(c_i) - g(c_i)|\,dx$.

To find the solution, split up into intervals where $f \ge g$ and $f < g$

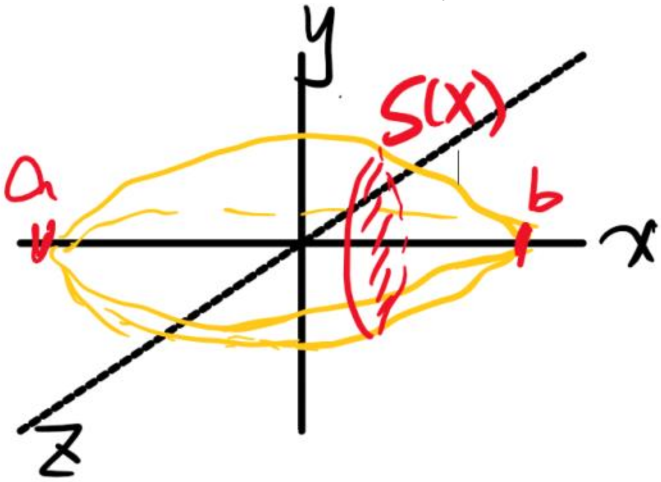



## b. Volumes

### i. Method of slices

Assume $V(S)$ = volume of a solid in $R^3 = \{(x, y, z): x, y, z \in \mathrm{R}\}$ is well-defined satisfying reasonable properties and formula

Let $S$ be a bounded solid in $R^3$ between planes $x = a$ and $x = b$ To find the volume:




$S(x) = intersection\ of\ S\ with\ the\ plane\ perpendicular\ to\ x - axis\ at\ (x, 0,0)$

$A(x) = area\ of\ S(x)$, $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$

Let $S_i$ = slice of S between the planes $x = x_{i-1}$ and $x = x_i$

$\Delta V_i = V(S_i)$ $M_i = \sup\{A(x): x \in [x_{i-1}, x_i]$, $m_i = \inf\{A(x): x \in [x_{i-1}, x_i]$

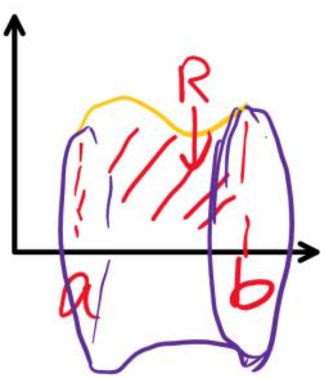
Then $m_i \Delta x_i \le \Delta V_i \le M_i \Delta x_i, \sum_{i=1}^{N} m_i \Delta x_i \le \sum_{i=1}^{N} \Delta V_i \le \sum_{i=1}^{N} M_i \Delta x_i$

$\Rightarrow L(A, P) \le V(S) \le U(A, P)$ for all P

Assume $A(x)$ is integrable on $[a, b]$ we know that $\int_a^b A(x)\,dx$ is the unique real number such that $L(A, P) \le \int_a^b A(x)\,dx \le U(A, P)$

Thus, we have the method of slices: $V(S) = \int_a^b A(x)\,dx$

### ii. Solid of revolution (disk method)

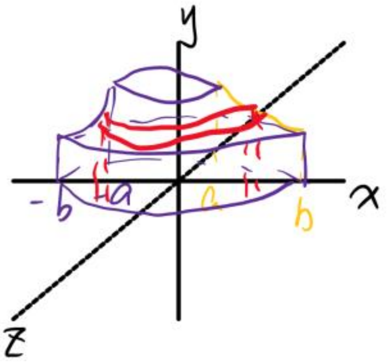



Let $f: [a, b] \to [0, \infty)$ integrable, $R = \{(x, y): a \le x \le b, 0 \le y \le f(x)\}$

Rotate $R$ about $x - axis$ to form a solid $S$

$S(x)$=disk of radius $f(x)$, $A(x) = \pi f(x)^2$, $V(S) = \int_a^b \pi f(x)^2 \, dx$

iii. **Solids of Revolution (cylindrical shell)**




Let $0 \le a < b$ $f: [a, b] \to [0, \infty)$ integrable, $R = \{(x, y): a \le x \le b, 0 \le y \le f(x)\}$
Rotate $R$ about $y - axis$ to form a solid $S$
$P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a, b]$, $R_i = \{(x, y): x_{i-1} \le x \le x_i, 0 \le y \le f(x)\}$
$C_i$ =cylindrical shell obtained by rotating $R_i$ by the $y - axis$

Unroll the shell, we get a thin rectangular solid $\Delta V_i = V(C_i) \approx 2\pi x_i f(x_i)\Delta x_i$

$$V(S) = \sum_{i=1}^{N} \Delta V_i = \sum_{i=1}^{N} 2\pi x_i f(x_i)\Delta x_i \to \int_a^b 2\pi x f(x) \, dx$$

## c. Mass, center of mass and centroid

i. **Mass**

Definition: let $B \subset R^d$ $(d = 1,2,3)$, the density of $B$ at $P \in B$ is $\rho(P)$ where the density function $\rho: B \to [0, \infty)$ is continuous. Then the mass of $B$ is $m(B) = \int_B \rho \, dV$
If $\rho = 1$, this defines the volume of $B$, $V(B) = \int_B dV$
If $d = 1$, $B = [a, b]$ then $m(B) = \int_a^b \rho \, dx$

ii. **Moment**

In 3-D, the $x$ −moment of $B$ is $M_x = \int_B x\rho(x, y, z) \, dV$
$y$ −moment of $B$ is $M_y = \int_B y\rho(x, y, z) \, dV$
$z$ −moment of $B$ is $M_z = \int_B z\rho(x, y, z) \, dV$

iii. **Center of mass**

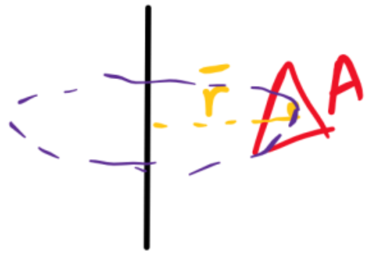
In 3-D, the center of mass of $B$ is $(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int_B x\rho \, dV}{\int_B \rho \, dV}, \frac{\int_B y\rho \, dV}{\int_B \rho \, dV}, \frac{\int_B z\rho \, dV}{\int_B \rho \, dV}\right) = \left(\frac{M_x}{m}, \frac{M_y}{m}, \frac{M_z}{m}\right)$

iv. **Centroid**

If $\rho = 1$, the center of mass becomes the centroid, which depends on the geomery of $B$ only,
$(\bar{x}, \bar{y}, \bar{z}) = \left(\frac{\int_B x \, dV}{V}, \frac{\int_B y \, dV}{V}, \frac{\int_B z \, dV}{V}\right)$

## d. Pappus Theorem

**Definition:** A plane region lie on one side of a line $L$ in $R^3$, $R$ is rotated aound line $L$ to form a solid of revolution, then the volume=distance travelled by the centroid of $R \times$ Area=$2\pi\bar{r}A$




**Remark:** it is related to volume by shells; Pappus theorem is more general
Proof: WOLOG, let L be the y-axis, $R$ lies to the right of y-axis

Centroid of $R$, $\bar{r} = \frac{\int_R x \, \mathrm{d}A}{A} = \frac{\int\int_R x \, \mathrm{d}x \, \mathrm{d}y}{A}$

Consider the volume swept out by a little box, $\Delta V = 2\pi x \Delta x \Delta y$

$$V = \sum_{x,y} \Delta V = \sum_{x,y} 2\pi x \Delta x \Delta y = \sum_{x,y} \frac{2\pi x \Delta x \Delta y A}{A} = \frac{\int\int_R 2\pi x \, \mathrm{d}x \, \mathrm{d}y \, A}{A} = 2\pi \bar{r} A$$

# Parametric and polar curves

2019年7月21日 9:54

## 1. Parametric curve

**Definition:** A parametric curve is a function $\gamma: [a,b] \to \mathbb{R}^2$ if $f: [a,b] \to \mathbb{R}$, derive $\gamma: [a,b] \to \mathbb{R}^2$ by $\gamma = (x, f(x))$

### a. Arc length

**Definition**: $\gamma: [a,b] \to \mathbb{R}^2$ is a parametric curve let $P = \{t_0, t_1, \dots, t_n\}$ be a partition of $[a,b]$. Let $D(\gamma, P) = \sum_{i=1}^{N} |\gamma(t_{i-1})\gamma(t_i)|$=length of the piecewise linear approximation of $\gamma$. The arc length of $\gamma$ is $l(\gamma) = \sup\{D(\gamma,P): P \text{ is a partition of } [a,b]\} \in [0,\infty]$

($l(\gamma)$ is the distance travelled by the particle whose position at time $t_i$ is $\gamma(t_i)$)

**Lemma** (Triangle inequality): if $P, Q, R \in \mathbb{R}^2$, then $|PR| \le |PQ| + |QR|$

**Lemma:** Let $P' \subset P$ be a partition of $[a,b]$ and $\gamma: [a,b] \to \mathbb{R}^2$, then $D(\gamma, P') \le D(\gamma, P)$

Proof by triangular inequality

**Lemma:** $\gamma: [a,b] \to \mathbb{R}^2$, $\exists$ a sequence $\{P_n : n \in \mathrm{N}\}$ such that $\|P_n\| \to 0$ and $D(\gamma, P) \to l(\gamma)$

Proof: $\forall n \in \mathrm{N}$, $\exists P_n{}'$ such that $l(\gamma) - \frac{1}{n} < D(\gamma, P_n{}') \le l(\gamma)$

We can find $Q_n$ such that $\|Q_n\| < 2^{-n} \to 0$,

let $P_n = P_n{}' \cup Q_n$, $\|P_n\| \to 0$

$$\Rightarrow l(\gamma) - \frac{1}{n} < D(\gamma, P_n{}') \le D(\gamma, P_n) \le l(\gamma)$$

$\Rightarrow D(\gamma, P) \to l(\gamma)$ by squeeze theorem

**Theorem:** let $f: [a,b] \to \mathbb{R}$ and $f'$ is continuous, let $\gamma = (x, f(x))$ $x \in [a,b]$,

Then $l(x) = \int_a^b \sqrt{1 + f'(x)^2}\, \mathrm{d}x < \infty$ is the arc length of the graph $y = f(x)$

Proof: let $P = \{x_0, x_1, \dots, x_n\}$ be a partition of $[a,b]$, $D(\gamma, P) = \sum_{i=1}^{N} |\gamma(t_{i-1})\gamma(t_i)|$

$$D(\gamma, P) = \sum_{i=1}^{N} \sqrt{(x_i - x_{i-1})^2 + (f(x_i) - f(x_{i-1}))^2}$$

$$= \sum_{i=1}^{N} \sqrt{1 + \left(\frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}\right)^2}\, \Delta x_i$$

$$= \sum_{i=1}^{N} \sqrt{1 + (f'(c_i))^2}\, \Delta x_i \text{ (by mean value theorem)}$$

$$l(x) = \lim_{N\to\infty} \sum_{i=1}^{N} \sqrt{1 + (f'(c_i))^2}\, \Delta x_i = \int_a^b \sqrt{1 + f'(x)^2}\, \mathrm{d}x$$

**Definition:** let $\gamma(t) = (x(t), y(t))$ be a parametric curve, $\gamma: [a,b] \to \mathbb{R}^2$ is $c^1$ (differentiable and its first derivative is continuous) if and only if $\frac{\mathrm{d}x}{dt}$ and $\frac{\mathrm{d}y}{dt}$ are continuous on $[a,b]$, the velocity is $\gamma'(t) = (x'(t), y'(t))$, the speed is

$$|\gamma'(t)| = \sqrt{(x'(t))^2 + (y'(t))^2}$$

**Theorem:** let $\gamma: [a,b] \to \mathbb{R}^2$ be $c^1$, then $\gamma$ has finite arc length $l(\gamma) =$

## 2. Polar Coordinates

**Definition:** the polar coordinates of a point $P = (x, y)$ are $r, \theta$, where $r = \sqrt{x^2 + y^2}$, and $\theta$ is the angle between $\overrightarrow{OP}$ and $+x - axis$ if $P = (0,0)$, $\theta$ is arbitrary. Let $P = [r, \theta]$ denote the point in the cartesian plane with polar coordinates $r, \theta$


If we restrict $\theta \in [0,2\pi)$, then $\theta$ is unique

**Note:** $[r, \theta] = [r, \theta + 2\pi k], k \in \mathrm{Z}$; $[0, \theta] = [0,0]$;$[r, \theta] = (r\cos\theta, r\sin\theta)$; $[-r, \theta] = [r, \theta + \pi] = -[r, \theta]$

We call the set of $[r, \theta]$ such that $r = f(\theta)$, $\theta \in [\alpha, \beta]$ the polar graph of $f$

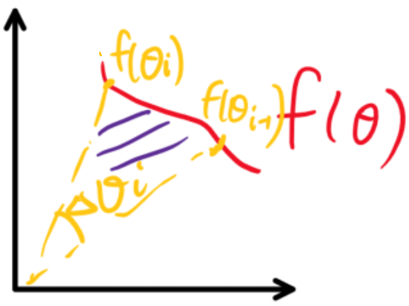
### Areas of polar graphs

Let $S(r, \Delta\theta)$ =sector of a circle with radius $r$ subtending angle $\Delta\theta$

Let $A(r, \Delta\theta)$ = area of $S(r, \Delta\theta) = \frac{1}{2} r^2 \Delta\theta$

Let $P = \{\theta_0, \theta_1, \dots, \theta_n\}$ be a partition of $[\alpha, \beta]$, $\Delta A_i$ be the area swepted by $r = f(\theta)$, $\theta \in [\theta_{i-1}, \theta_i] \approx A(f(\theta_i), \Delta\theta_i) = \frac{1}{2} f(\theta_i)^2 \Delta\theta_i$

Total area $A = \lim_{N\to\infty} \sum_{i=1}^{N} \Delta A_i = \lim_{N\to\infty} \sum_{i=1}^{N} \frac{1}{2} f(\theta_i)^2 \Delta\theta_i = \int_\alpha^\beta \frac{(f(\theta))^2}{2} \mathrm{d}\theta$




### Arclength of polar graphs

Let $f: [\alpha, \beta] \to R$ be $c^1$, then the polar graph $r = f(\theta)$ can be viewed as a $c^1$ parametric graph $r(\theta) = (r\cos\theta, r\sin\theta)$, we can then use the arclength formula for parametric curve to derive the arc length $l = \int_\alpha^\beta \sqrt{f(\theta)^2 + f'(\theta)^2}\, d\theta$

# Sequences and series

2019年7月21日 9:55

## 1. Basics

**Definition (Sequence)**: A sequence is a function $a:\{n_0, n_0+1, \dots\} \to R$ for some $n_0 \in Z$, denote $a$ by $\{a_n: n \geq n_0\}$ or $\{a_n\}$, usually $n_0 = 0 \; or \; 1$

**Definition (Converges/diverges):** $\{a_n\}$ converges to $L \in R$ if and only if $\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that $n > N \Rightarrow |a_n - L| < \varepsilon$ (write $a_n \to L$ or $\lim_{n\to\infty} a_n = L$); $\{a_n\}$ diverges if and only if $\forall M, \exists N \in \mathbb{R}$ such that $n > N \Rightarrow a_n > M$ ($a_n \to \infty$ or $\lim_{n\to\infty} a_n = \infty$)

**Theorem (Algebra of limits):** Assume $a_n \to L_a$, $b_n \to L_b$ and $L_a, L_b \in \mathbb{R}$

$\forall c \in R$, $a_n + cb_n \to L_a + cL_b$; $a_n b_n \to L_a L_b$; $\frac{a_n}{b_n} \to \frac{L_a}{L_b}$ (given that $L_b \neq 0$); $\lim_{n\to\infty} c = c$; $a_n \leq b_n$ ultimately, then $L_a \leq L_b$

## 2. Sequences

### a. Limits and sequential limits

**Theorem:** Assume $\lim_{x\to c} f(x) = L$, $c, L \in [-\infty, \infty]$, if $x_n \to c$, and $x_n \in \mathrm{Dom}(f)$ and $x_n \neq c$ ultimately, then $f(x_n) \to L$

Proof: let $\varepsilon > 0$, $\exists \delta > 0$ such that $0 < |x - c| < \varepsilon$ and $x \in \mathrm{Dom}(f)$, then $|f(x) - L| < \varepsilon$ ( since $\lim_{x\to c} f(x) = L$)

$x_n \to c$ so $\exists N_1$ such that $n > N_1 \Rightarrow |x_n - c| < \delta$

The ultimate hypothesis on $\{x_n\}$ implies that $\exists N_2$ such that $n > N_2 \Rightarrow x_n \neq c$ and $x_n \in \mathrm{Dom}(f)$

Let $n > \max(N_1, N_2)$, then $0 < |x_n - c| < \delta$ and $x_n \in \mathrm{Dom}(f)$

Let $x = x_n$, $|f(x_n) - L| < \varepsilon$

**Theorem:** let $f$ be continuous at $c$, if $x_n \to c$ and $x_n \in \mathrm{Dom}(f)$ ultimately, then $f(x_n) \to f(c)$ ($\lim(f(x_n)) = f(\lim(x_n))$)

Proof: let $\varepsilon > 0$, by continuity at $c$, $\exists \delta$ such that $|x - c| < \delta$, $x \in \mathrm{Dom}(f)$, $|f(x_n) - f(c)| < \varepsilon$, then substitute $L$ with $f(c)$ in the previous proof

**Definition:** $\{a_n\}$ is bounded if and only if $\exists M$ such that $\forall n$, $|a_n| \leq M$

**Theorem:**

i. $\{a_n\}$ is convergent $\Rightarrow$ $\{a_n\}$ is bounded

Proof: take $\varepsilon = 1$, $\exists N$ such that $n > N \Rightarrow |a_n - L| < 1$ ($\lim_{n\to\infty} a_n = L$)

By triangular inequality, $|a_n| \leq |L| + 1$

Let $M = \max\{|a_n|: n \leq N\} + |L| + 1$,

Then $|a_n| \leq M$ for $\forall n$

ii. $\{a_n\}$ is bounded $\not\Rightarrow$ $\{a_n\}$ is convergent (e.g. $a_n = (-1)^n$)

### b. Monotone sequences

**Definition:** $\{a_n\}$ is an increasing sequence if and only if $\forall n$, $a_{n+1} \geq a_n$ and decreasing if and only if $\forall n$, $a_{n+1} \leq a_n$; $\{a_n\}$ is monotone if and only if it is increasing or decreasing all the time

**Theorem (Increasing sequence theorem)**: Let $\{a_n\}$ be an increasing sequence, $L = \sup\{a_n: n \in \mathrm{N}\} \in (-\infty, \infty]$, then $a_n \to L$

i.e. if $\{a_n\}$ is bounded above, $a_n \to L \in \mathrm{R}$, if $\{a_n\}$ is not bounded above $a_n \to \infty$

Questions can be solved by induction.

## 3. Series

**Definition:** Let $\{b_k: k \in \mathrm{N}\}$ be a sequence, and set $S_n = \sum_{k=1}^{n} b_k (n \geq 1)$

The series $\sum_{k=1}^{\infty} b_k$ converges if and only if $\lim_{n\to\infty} S_n = L \in \mathbb{R}$, write $\sum_{k=1}^{\infty} b_k = L$

The series $\sum_{k=1}^{\infty} b_k$ diverges if and only if $\{S_n\}$ diverges $\sum_{k=1}^{\infty} b_k = \pm\infty$

**Proposition:**

a. For any sequence $\{a_n\}$, $\{a_n\}$ converges $\Rightarrow$ $\lim_{n\to\infty} a_{n+1} - a_n = 0$

Proof: let $\lim_{n\to\infty} a_n = L$, $then \lim_{n\to\infty} a_{n+1} = L$ by definition of limits

$$\lim_{n\to\infty} a_{n+1} - a_n = \lim_{n\to\infty} a_{n+1} - \lim_{n\to\infty} a_n = L - L = 0$$

b. If $\sum_{k=1}^{\infty} b_k$ converges, then $\lim_{n\to\infty} b_k = 0$, but $\lim_{n\to\infty} b_k = 0$ does not imply $\sum_{k=1}^{\infty} b_k$ converges

Proof: apply (a) to $\{S_n\}$, $S_n = \sum_{k=1}^{n} b_k \to L$, so $S_{n+1} - S_n \to 0 \Rightarrow \lim_{n\to\infty} b_{n+1} = 0 \Rightarrow \lim_{n\to\infty} b_n = 0$

**Theorem (Algebra of series):** $\sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} b_k$ are convergent, and $c_1, c_2 \in \mathbb{R}$, then $\sum_{k=1}^{\infty} c_1 a_k + \sum_{k=1}^{\infty} c_2 b_k = c_1 \sum_{k=1}^{\infty} a_k + c_2 \sum_{k=1}^{\infty} b_k$ and both sides are convergent

**Theorem (Positive series dichotomy):** assume $a_k \geq 0$ for all $k \in \mathrm{N}$, let $S_n = \sum_{k=1}^{n} a_k$

a. If $\{S_n\}$ is bounded above, then $\sum_{k=1}^{\infty} a_k = \sup\{S_n\}$ is convergent
b. If $\{S_n\}$ is not bounded above, then $\sum_{k=1}^{\infty} a_k = \infty$

Proof by increasing sequence theorem, since $a_{n+1} = S_{n+1} - S_n \geq 0$, $S_n$ is increasing

# 4. Convergence test

## a. Integral test

**Theorem:** let $f: [1, \infty) \to [0, \infty)$ be decreasing, then $\sum_{n=1}^{\infty} f(n)$ is convergent $\Leftrightarrow \int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent

Proof: Assume $\int_1^{\infty} f(x)\,\mathrm{d}x$ is convergent, apply the following lemma with $N = 1$,

$$\sum_{k=2}^{n} f(k) \leq \int_1^{n} f(x)\,dx \leq \int_1^{\infty} f(x)\,\mathrm{d}x \in \mathrm{R}$$

$\forall n \in \mathrm{N}$, $\sum_{k=1}^{n} f(k) \leq \int_1^{\infty} f(x)\,\mathrm{d}x + f(1)$

By positive series dichotomy, $\sum_{k=1}^{\infty} f(k)$ converges

Assume $\sum_{k=1}^{\infty} f(k)$ is convergent, apply the following lemma with $N = 0$,

$\forall n \in \mathrm{N}$, $\int_1^{n+1} f(x)\,\mathrm{d}x \leq \sum_{k=1}^{n} f(k) \leq \sum_{k=1}^{\infty} f(k) = \sup\{S_n\}$

$F(R) = \int_1^{R} f(x)\,\mathrm{d}x$ is increasing in R and $F(R) \leq \sum_{k=1}^{\infty} f(k)\ \forall R > 1$

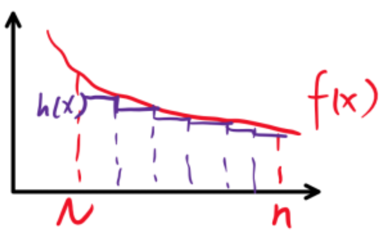
$\Rightarrow F(R) \to \int_0^{\infty} f(x)\,\mathrm{d}x$ (a finite number) as $R \to \infty$

**Correlation:** for $p > 0$, $\sum_{n=1}^{\infty} \frac{1}{n^p} < \infty \Leftrightarrow p > 1$

**Lemma:** let $f: [1, \infty) \to [0, \infty)$ be increasing, $\forall n > N \in \mathrm{N}$, $\int_{N+1}^{n+1} f(x)\,\mathrm{d}x \leq \sum_{k=N+1}^{n} f(k) \leq \int_N^{n} f(x)\,\mathrm{d}x$

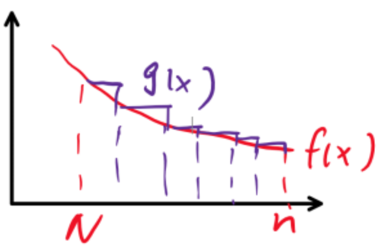
Proof: let $h(x) = f(k+1), x \in [k, k+1), k \in \mathrm{N}$,

$$h(x) \leq f(x) \Rightarrow \int_N^{n} f(x)\,\mathrm{d}x \geq \int_N^{n} h(x)\,\mathrm{d}x = \sum_{k=N}^{n-1} \int_k^{k+1} h(x)\,\mathrm{d}x = \sum_{k=N}^{n-1} f(k+1) = \sum_{k=N+1}^{n} f(k)$$




Let $g(x) = f(k), x \in [k, k+1), k \in \mathrm{N}$,

$$f(x) \leq g(x) \Rightarrow \int_{N+1}^{n+1} f(x)\,\mathrm{d}x \leq \int_{N+1}^{n+1} g(x)\,\mathrm{d}x = \sum_{k=N+1}^{n} \int_k^{k+1} h(x)\,\mathrm{d}x = \sum_{k=N+1}^{n} f(k)$$




Note: this lemma gives an error bound on the approximation of $S = \sum_{n=1}^{\infty} f(k)$ using $S_n = \sum_{k=1}^{n} f(k)$

**Remark:** $\sum_{n=1}^{\infty} a_n$ converges if and only if $\sum_{n=n_0}^{\infty} a_n$ converges

## b. Comparison test

**Theorem:** let $a_n, b_n \geq 0$ assume $k > 0$ and $a_n \leq kb_n$ ultimately,then

i. $\sum_{n=1}^{\infty} b_n$ converges$\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

ii. $\sum_{n=1}^{\infty} a_n$ diverges$\Rightarrow$ $\sum_{n=1}^{\infty} b_n$ diverges

Proof: choose $n_0$ such that $n \geq n_0, a_n \leq kb_n$

$\forall n \geq n_0, \sum_{i=n_0}^{n} a_i \leq k \sum_{i=n_0}^{n} b_i \leq k \sum_{i=n_0}^{\infty} b_i$

By positive series dichotomy, $\sum_{i=n_0}^{n} a_i$ converges, by the last remark, $\sum_{n=1}^{\infty} a_n$ converges

## c. Limit comparison test

**Theorem:** let $a_n, b_n \geq 0$ assume $\frac{a_n}{b_n} \to L \in [0, \infty]$

i. $L$ is finite, then if $\sum_{n=1}^{\infty} b_n$ is convergent, so is $\sum_{n=1}^{\infty} a_n$

Proof: $\frac{a_n}{b_n} \to L$ is finite, take $\varepsilon = 1$, $\exists n \geq n_0 \Rightarrow \left|\frac{a_n}{b_n} - L\right| < 1$

$\Rightarrow \frac{a_n}{b_n} < L + 1 \Rightarrow a_n < (L+1)b_n$ ultimately

By comparison test, if $\sum_{n=1}^{\infty} b_n$ is convergent, so is $\sum_{n=1}^{\infty} a_n$

ii. $L > 0$, then if $\sum_{n=1}^{\infty} a_n$ is divergent, so is $\sum_{n=1}^{\infty} b_n$

Proof: $\frac{a_n}{b_n} \to L \in (0, \infty]$, take inverse $\frac{b_n}{a_n} \to \frac{1}{L} \in [0, \infty)$, apply (i) with $a_n, b_n$ reversed

## d. Root test

**Theorem:** let $a_n \geq 0$, assume $a_n^{\frac{1}{n}} \to \rho \in [0, \infty]$

i. $\rho < 1$, $\sum_{n=1}^{\infty} a_n$ converges

ii. $\rho > 1$, $\sum_{n=1}^{\infty} a_n$ diverges

iii. $\rho = 1$, $\sum_{n=1}^{\infty} a_n$ may converge or diverge

## e. Ratio test

**Theorem:** let $a_n \geq 0$, assume $\frac{a_{n+1}}{a_n} \to \rho \in [0, \infty]$

i. $\rho < 1$, $\sum_{n=1}^{\infty} a_n$ converges

ii. $\rho > 1$, $\sum_{n=1}^{\infty} a_n$ diverges

iii. $\rho = 1$, $\sum_{n=1}^{\infty} a_n$ may converge or diverge

**Remark:** ratio test tends to be easier to implement arithmetically than root test (especially with n!); root test implies ratio test, but ratio test does not imply root test

**Lemma:** $a_n \geq 0, \frac{a_{n+1}}{a_n} \to \rho \in [0, \infty] \Rightarrow a_n^{\frac{1}{n}} \to \rho$ converse fails

# Absolute convergence

**Definition:** a series $\sum_{n=1}^{\infty} a_n$ converges absolutely if and only if $\sum_{n=1}^{\infty} |a_n|$ converges

**Theorem:**

a. $\sum_{n=1}^{\infty} a_n$ converges absolutely ( $\sum_{n=1}^{\infty} |a_n|$ converges)$\Rightarrow$ $\sum_{n=1}^{\infty} a_n$ converges

Proof: let $a_n^+ = \max\{a_n, 0\}$ $a_n^- = \min\{a_n, 0\}$

Then $a_n = a_n^+ - a_n^-$, $|a_n| = a_n^+ + a_n^-$, and $0 \leq a_n^{\pm} \leq |a_n|$

Since $\sum_{n=1}^{\infty} |a_n|$ converges, by comparison test, $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ converges

By algebra of series, $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} a_n^+ + a_n^-$ converges

b. However, inverse is false

**Lemma:** let $\{l_n : n \in \mathrm{N}\}$ be a sequence, if $l_{2n} \to L$ and $l_{2n-1} \to L$, then $l_n \to L$

**Theorem (Alternating Series Test):** let $b_n \searrow 0$ ($b_n \geq 0$ for $\forall n$) then $\sum_{n=1}^{\infty} (-1)^{n-1} b_n$ converges

Proof: let $S_n = \sum_{i=1}^{n} (-1)^{i-1} b_i$, $a_i = (-1)^{i-1} b_i$

$S_{2(n+1)} - S_{2n} = a_{2n+2} + a_{2n+1} = -b_{2n+2} + b_{2n+1} \geq 0$ (since $b_{2n+1} \geq b_{2n+2}$)$\Rightarrow S_{2n}$ is increasing

$S_{2n+1} - S_{2n-1} = a_{2n+1} + a_{2n} = b_{2n+1} - b_{2n} \leq 0 \Rightarrow S_{2n-1}$ is decreasing

$S_{2n-1} - S_{2n} = b_{2n} \geq 0$ $\Rightarrow S_{2n-1} \geq S_{2n}$ for $\forall n$

$S_{2n}$ is increasing and it has an upper bound of $S_1$, thus limit $S_{even}$ exists

$S_{2n-1}$ is decreasing and it has an lower bound $S_2$, thus limit $S_{odd}$ exists

And $S_{odd} - S_{even} = \lim_{n\to\infty} S_{2n-1} - S_{2n} = \lim_{n\to\infty} b_{2n} = 0$

By the previous lemma, $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges to $S = S_{odd} = S_{even}$

**Remark (Alternating Series Bounds):** $\forall n, S_{2n} \le S \le S_{2n+1} \le S_{2n-1}$, then $0 \le S - S_{2n} \le S_{2n+1} - S_{2n} = b_{2n+1}$, and $0 \le S_{2n-1} - S \le S_{2n-1} - S_{2n} = b_{2n}$, so $\forall m \in \mathrm{N}$, $|S_m - S| < b_{m+1}$ is the approximation error by the mth sum.

## Conditional convergence

**Definition:** a series $\sum_{n=1}^{\infty} a_n$ converges conditionally if and only if $\sum_{n=1}^{\infty} a_n$ converges, but $\sum_{n=1}^{\infty} |a_n|$ does not converge

**Proposition:** let $\sum_{n=1}^{\infty} a_n$ be convergent, $\sum_{n=1}^{\infty} a_n$ converges conditionally $\Leftrightarrow$ $\sum_{n=1}^{\infty} a_n^+ = \infty$ and $\sum_{n=1}^{\infty} a_n^- = \infty$

Proof:(1) $|a_n| = a_n^+ + a_n^- = a_n^+ + a_n^+ - (a_n^+ - a_n^-) = 2a_n^+ - a_n$

suppose $\sum_{n=1}^{\infty} a_n^+$ converges, by algebra of series $\sum_{n=1}^{\infty} |a_n|$ converges

But this contradicts, so $\sum_{n=1}^{\infty} |a_n| = \infty \Rightarrow \sum_{n=1}^{\infty} a_n^+ = \infty$

Similarly, $|a_n| = a_n^+ + a_n^- = a_n^- + a_n^- + (a_n^+ - a_n^-) = 2a_n^- + a_n$

We can get that $\sum_{n=1}^{\infty} |a_n| = \infty \Rightarrow \sum_{n=1}^{\infty} a_n^- = \infty$

(2) $\sum_{n=1}^{\infty} a_n^+ = \infty$, $|a_n| \ge a_n^+$, by comparison test, $\sum_{n=1}^{\infty} |a_n| = \infty$ $\sum_{n=1}^{\infty} a_n = \infty$, $\sum_{n=1}^{\infty} a_n$ converges conditionally

**Remark:** Assume $\sum_{n=1}^{\infty} a_n$ converges conditionally, by adding a lot of positive terms and then a few negative terms and a lot of positive terms and keeping going, as long as $a_n \to 0$, $\sum_{n=1}^{\infty} a_n$ can be $\pm\infty$ or any number

# 5. Power series

**Definition:** A power series centered at $c \in \mathrm{R}$ is a series of the form $\sum_{n=0}^{\infty} a_n(x-c)^n$, where $a_n \in \mathrm{R}$, and $x$ is an independent variable.

Let $C_a = \{x: \sum_{n=0}^{\infty} a_n(x-c)^n\}$ be a set of convergence of the power series, $C_a = (c-R, c+R)$(end points may be included), $c$ is the center of convergence and $R$ is the convergence radius

If $f(x) = \sum_{n=0}^{\infty} a_n(x-c)^n$, $x \in C_a$, then it is a power series representation of $f(x)$

**Theorem:** a power series $\sum_{n=0}^{\infty} a_n(x-c)^n$, there exists $R \in [0,\infty]$ such that

a. $|x-c| < R$, $\sum_{n=0}^{\infty} a_n(x-c)^n$ converges absolutely
b. $|x-c| > R$, $\sum_{n=0}^{\infty} a_n(x-c)^n$ diverges
c. $|x-c| = R$, $\sum_{n=0}^{\infty} a_n(x-c)^n$ may converge or diverge

Proof: (1)WOLOG, let $c = 0$ (let $x' = x - c$ if not)

$$\sum_{n=0}^{\infty} a_n(x-c)^n = \sum_{n=0}^{\infty} a_n(x')^n$$

By result for $c = 0$, $\exists\, R \in [0,\infty]$ such that $|x'| < R \Rightarrow \sum_{n=0}^{\infty} a_n(x')^n$ converges

(2)Let $R = \sup\{|x|: x \in C_a\}$, $|x| < R$, if $R < \infty$, $\exists x_0 \in C_a$ such that $|x| < |x_0| < R$

$x_0 \in C_a \Rightarrow \sum_{n=0}^{\infty} a_n(x_0)^n$ converges$\Rightarrow \lim_{n\to\infty} a_n(x_0)^n = 0 \Rightarrow \exists k$ such that $|a_n(x_0)^n| \le k$

$|a_n(x_0)^n| = |a_n||(x_0)^n|\left(\frac{|x|}{|x_0|}\right)^n \le kr^n$ where $r = \frac{|x|}{|x_0|} < 1$

By comparison test, since $\sum_{n=0}^{\infty} kr^n$ is convergent, then $\sum_{n=0}^{\infty} |a_n(x_0)^n|$ converges, $\sum_{n=0}^{\infty} a_n(x_0)^n$ converges absolutely

**Theorem:** Let $R$ be the convergence radius of $\sum_{n=0}^{\infty} a_n(x-c)^n$

a. If $|a_n|^{\frac{1}{n}} \to \sigma \in [0,\infty]$, then $R = \frac{1}{\sigma}$
b. If $\left|\frac{a_{n+1}}{a_n}\right| \to \sigma \in [0,\infty]$, then $R = \frac{1}{\sigma}$

**Lemma:** $H > 0$, then $\forall |h| \le H\ \forall x \in \mathrm{R}$, $|(x+h)^n - x^n - nx^{n-1}h| \le \left|\frac{h}{H}\right|^2 (|x| + H)^n$

**Remark:** if $|x| < r$, then $\sum_{n=0}^{\infty} |a_n(x)^n|$ converges

**Theorem (differentiation and integration of power series):** Assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < r \le$ radius of convergence, then

a. $f'(x) = \sum_{n=0}^{\infty} na_n x^{n-1}$ for $|x| < r$

Proof: (1)first check convergence

let $0 < |x| < r$, claim $\sum_{n=0}^{\infty} |na_n x^{n-1}| < \infty$

Choose $r_0$ such that $|x| < r_0 < r$, then $\sum_{n=0}^{\infty} a_n r^n$ converges$\Rightarrow a_n r^n \to 0$

$\Rightarrow \exists k$ such that $|a_n r_0^n| \le k$ for all $n$

$$|na_n x^{n-1}| = \frac{n|a_n r_0^n|}{|x|}\frac{|x|^n}{r_0^n} \quad (|a_n r_0^n| \le k, \alpha = \frac{|x|}{r_0} \in (0,1))$$

Then $|na_n x^{n-1}| \le \frac{k}{|x|} n\alpha^n$

Recall that $\sum_{n=0}^{\infty} n\alpha^n$ is convergent , so $\sum_{n=0}^{\infty} |na_n x^{n-1}|$ converges by comparison

(2)Let $g(x) = \sum_{n=0}^{\infty} na_n x^{n-1}, |x| < r$

$$\left|\frac{f(x+h)-f(x)}{h} - g(x)\right| = \left|\sum_{n=0}^{\infty}\left(\frac{a_n(x+h)^n - a_n x^n}{h} - na_n x^{n-1}\right)\right|$$

$$= \lim_{N\to\infty}\left|\frac{1}{h}\sum_{n=0}^{\mathrm{N}} a_n\left((x+h)^n - x^n - nx^{n-1}h\right)\right| \le \left|\frac{1}{h}\right| \lim_{N\to\infty}\left|\sum_{n=0}^{\mathrm{N}} a_n \left|\frac{h}{H}\right|^2 (|x|+H)^n\right|$$

(by triangular inequality and previous lemma)

Since $\sum_{n=0}^{\infty} a_n \left|\frac{h}{H}\right|^2 (|x|+H)^n$ converge to 0.

By squeeze theorem,$\left|\frac{f(x+h)-f(x)}{h} - g(x)\right| \to 0$ as $h \to 0$

By definition $\lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = g(x) = f'(x)$

b. $\int_0^x f(t)\,\mathrm{d}t = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ for $|x| < r$

Proof: $\left|\frac{a_n}{n+1} x^{n+1}\right| \le |x||a_n x^n|$ RHS converges

By comparison test, $\sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ converges absolutely for $|x| < r$

Let $h(x) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$

By (a) $h'(x) = \sum_{n=0}^{\infty} a_n x^n = f(x)$

$\int_0^x f(t)\,\mathrm{d}t = \int_0^x h'(t)\,\mathrm{d}t = h(x) - h(0) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} x^{n+1}$ by FTC

c. Note: with this theorem, we can generate new power series representations from old ones like $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$

**Correlation:** Assume $f(x) = \sum_{n=0}^{\infty} a_n (x)^n$ for $|x| < r \le$ radius of convergence, then $f(x)$ is infinitely differentiable for $|x| < r$, write $f \in c^{\infty}$

**Theorem (Abel's Theorem):** assume $f(x) = \sum_{n=0}^{\infty} a_n x^n$ for $|x| < R$, and $\sum_{n=0}^{\infty} a_n R^n$ converges, then $\lim_{x\to \mathrm{R}^-} F(x) = \sum_{n=0}^{\infty} a_n R^n$

**Remark:** if $R(f)$ is the radius of convergence for $f(x) = \sum_{n=0}^{\infty} a_n x^n$, then $R(f) = R(f') = R\left(\int f\,\mathrm{d}x\right)$

**Remark:** everything works for $\sum_{n=0}^{\infty} a_n (x-c)^n$ with any $c \in \mathrm{R}$

## 6. Taylor series

**Theorem:** Assume $f$ is $c^{\infty}$ on $(c-R, c+R)$, if $f(x) = \sum_{n=0}^{\infty} a_n (x-c)^n$ for $|x-c| < r$, then $\forall k \in Z^+$, $a_k = \frac{f^{(k)}(c)}{k!}$, where $f^{(0)}(c) = f(c)$

**Remark:** a power series representation (if exists) for $f(x)$ is unique and must be $\sum_{n=0}^{\infty} \frac{f^{(n)}(c)}{n!}(x-c)^n$

**Definition:** Assume $f^{(k)}(c)$ exists for all $k \in N$. The taylor series of $f$ about $x = c$ is $\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$, if $c = 0$, it is called the Maclaurin series

Assume $f^{(k)}(c)$ exists for all $k \le n \in N$, the nth degree Taylor polynomial for $f$ about $x = c$ is $P_{n,c}(x) =$

**Theorem (Taylor series test)**: Assume $f$ is $c^{\infty}$ on $(c-R, c+R)$, let $M_n(r) = \sup\left\{\left|f^{(n)}(x)\right| : |x-c| < R\right\}$, if $\frac{M_n(r)r^n}{n!} = 0$, then $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$ for all $|x-c| < r$

**Theorem (Taylor's Theorem)**: Assume $f^{n+1}$ exists on $(c-R, c+R)$, $\forall x \in (c-R, c+R)$, $\exists t = t(x)$ such that $f(x) - P_{n,c}(x) = \frac{f^{n+1}(t)}{(n+1)!}(x-c)^{n+1}$

# First order differential equations

2019年7月23日 14:27

**Definition:** A first order differential equation is an equation relating $y = f(x)$, $\frac{\mathrm{d}y}{\mathrm{d}x}$ and $x$

## 1. Separation of variables

**Definition:** A separable first order differential equation is one in the form of $\frac{\mathrm{d}y}{\mathrm{d}t} = g\big(y(t)\big)f(t)$ for some continuous functions. (LGE is a special case where $g\big(y(t)\big) = \left(1 - \frac{y(t)}{L}\right)y(t)$ and $f(t) = k$)

To derive a formula for $y(t)$:

$$\frac{\mathrm{d}y}{\mathrm{d}t} = g\big(y(t)\big)f(t) \Rightarrow \frac{\mathrm{d}y}{g\big(y(t)\big)} = f(t)\,\mathrm{d}t$$

Integrate both sides

$$\int \frac{\mathrm{d}y}{g\big(y(t)\big)} = \int f(t)\,\mathrm{d}t$$

$$\int \frac{\mathrm{d}y}{dt}\frac{1}{g\big(y(t)\big)}dt = \int f(t)\,\mathrm{d}t$$

$$G(y(t)) = \int f(t)\,\mathrm{d}t$$

$$y(t) = G^{-1}\left(\int f(t)\,\mathrm{d}t\right)$$

In fact, this works until $t_1$ where $g\left(y(\,t_1)\right)$ is first zero, if $g(\,y_0) \neq 0$ and $t_1 > 0$, $G^{-1}$ exists and $\int \frac{\mathrm{d}y}{g(y(t))}$ will be increasing or decreasing until $g(y) = 0$

### a. Easy case:

$\frac{\mathrm{d}y}{dt} = ky(t); y(0) = y_0 \Rightarrow y = y_0\mathrm{e}^{kt}$

### b. Logistic Growth Equation

$$\frac{\mathrm{d}y}{\mathrm{d}t} = k\left(1 - \frac{y(t)}{L}\right)y(t); y(0) = y_0$$

It has two trivial solutions $y(t) = 0$ and $y(t) = L$, corresponding to initial conditions $y_0 = L$ and $y_0 = 0$

$$\int \frac{\mathrm{d}y}{\left(1 - \frac{y}{L}\right)y} = \int k\,\mathrm{d}t$$

$$\int \frac{L\,\mathrm{d}y}{(L-y)y} = kt + C$$

$$\int \left(\frac{1}{L-y} + \frac{1}{y}\right)\mathrm{d}y = kt + C$$

$$\ln|y| - \ln|L-y| = kt + C$$

$$\ln\left|\frac{y}{L-y}\right| = kt + C$$

$$\frac{y}{L-y} = C_1\mathrm{e}^{kt}$$

$y = \frac{C_1 L\mathrm{e}^{kt}}{1 + C_1\mathrm{e}^{kt}}$ until first time $y(t) \notin (0, \mathrm{L})$ so $\frac{y}{L-y} > 0$

$\Leftrightarrow y = \frac{L}{\frac{L-y_0}{y_0}\mathrm{e}^{-kt} + 1} \in (0, \mathrm{L})$, for all $t \geq 0$

**Remark:** The presence of $y^2$ in separable equations makes $y(x) \to \infty$ at some finite $x$. If $g(y) \le k(1+|y|)$, then the solution of $\frac{\mathrm{d}y}{\mathrm{d}t} = g\big(y(t)\big)f(t)$ will not have a $y(x) \to \infty$

# 2. First order linear differential equations

$\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$, where $p(x)$ and $q(x)$ are continuous functions

Note: if $p(x) = 0$, it is a separable equation

$p(x) \neq 0$, consider multiplying both sides by $\mathrm{e}^{\mu(x)} > 0$

$$\mathrm{e}^{\mu(x)}\left[\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y\right] = \mathrm{e}^{\mu(x)}q(x)$$

If we choose $\mu(x)$ such that $\mathrm{e}^{\mu(x)}\left[\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y\right] = \frac{\mathrm{d}(\mathrm{e}^{\mu(x)}y)}{\mathrm{d}x}$, call $\mu(x)$ the integrating factor

Then, the LDE can be rewritten as $\frac{\mathrm{d}(\mathrm{e}^{\mu(x)}y)}{\mathrm{d}x} = \mathrm{e}^{\mu(x)}q(x)$

Integrate both sides, $\mathrm{e}^{\mu(x)}y = \int \mathrm{e}^{\mu(x)}q(x)\,\mathrm{d}x$

To find $\mu(x)$, solve $\mathrm{e}^{\mu(x)}\left[\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y\right] = \frac{\mathrm{d}(\mathrm{e}^{\mu(x)}y)}{\mathrm{d}x}$

$$\mathrm{e}^{\mu(x)}\left[\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y\right] = \mathrm{e}^{\mu(x)}[\frac{\mathrm{d}y}{\mathrm{d}x} + \mu'(x)y]$$

$$\Rightarrow p(x)y = \mu'(x)y$$

$$\Rightarrow \mu(x) = \int p(x)\,\mathrm{d}x$$

Note that adding constant to $\mu(x)$ does not affect $y$

**Theorem:** $y$ solves a linear differential equation $\frac{\mathrm{d}y}{\mathrm{d}x} + p(x)y = q(x)$ if and only if $y = \mathrm{e}^{-\mu(x)}\int \mathrm{e}^{\mu(x)}q(x)\,\mathrm{d}x$ where $\mu(x) = \int p(x)\,\mathrm{d}x$

# Vectors and geometry

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Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

- A vector is a quantity with both magnitude and direction indicated by arrows
- Magnitude $|\vec{a}|$ is the length of the vector $\vec{a}$.
- Two vectors are the same if they have the same direction and magnitude
- Addition: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.
- Scalar multiplication $c\vec{a}=\vec{a}+\vec{a}+\cdots+\vec{a}$.
- Zero vector $\vec{0}$: the only vector of magnitude 0, has no direction.
- The vector from $(0,0,0)$ to $(a,b,c)$ is denoted as $< a,b,c >$.
- Unit vectors:
    - $\vec{i} =< 1,0,0 >$.
    - $\vec{j} =< 0,1,0 >$.
    - $\vec{k} =< 0,0,1 >$.

Dot product

- Geometric definition: $\vec{a}\cdot\vec{b}=|\vec{a}|\left|\vec{b}\right|\cos\theta$.
- Algebraic: $\vec{a} =< a_1,a_2,a_3 >$, $\vec{b} =< b_1,b_2,b_3 >$, then $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$.
- Remark: $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$.

Cross product

- Geometric: $\vec{a}\times\vec{b}=|\vec{a}|\left|\vec{b}\right|\sin\theta$.
    - Direction of $\vec{a}\times\vec{b}$ is normal to both $\vec{a}$ and $\vec{b}$.
- Algebraic: $\vec{a}\times\vec{b}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.
- Remark: $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}\parallel\vec{b}$.

Triple product: $\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)=\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$.

Equations for lines in $\mathbb{R}^3$

- A line is determined by a point $(x_0,y_0,z_0)$ on the line and a vector $\vec{v} =< a,b,c >$ in the direction of the line
- Parametric equation: $\begin{cases} x=x_0+at \\ y=y_0+bt \\ z=z_0+ct \end{cases}$.
- 2 linear equation when $a,b,c\neq 0$, $\frac{x-x_0}{a}=\frac{y-y_0}{b}=\frac{z-z_0}{c}$.

Equation for a plane:

- $\vec{N}\cdot< x-x_0,y-y_0,z-z_0 >=0$.
- $a(x-x_0)+b(y-y_0)+c(z-z_0)=0$ or equivalently $d=ax+by+cz$.

Equations and surfaces

- Planes are solutions to linear equations
- For quadratic equations in 2 variables ($x^2,y^2,xy,x,y,c$), we get circles, ellipses, parabolas, hyperbolas
- A quadratic surface in $\mathbb{R}^3$ is given by an equation which is a linear combination of $x^2,y^2,z^2,xy,yz,xz,x,y,z,c$.

- If the equation only involves 2 of the 3 variables, it is a cylinder
- To sketch/understand surfaces, we use the curves obtained by planes parallel to coordinate planes
  - Contour curves: setting $z = c$ constant.
  - Trace curves: $x = c$ or $y = c$ constant.

Functions of 2 and 3 variables

- A function of 2 variables with domain $D \subset \mathbb{R}^2$ is a rule $f$ which assigns to each point $(x, y) \in D$, a $f(x, y) \in \mathbb{R}$, write $f: D \to \mathbb{R}$
- Often the domain is implicit
- For functions of 3 variables, we can only draw the contour/level surfaces

# Partial Derivatives

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Continuity and limits

- For $\lim_{(x,y)\to(a,b)} f(x,y)$, there are infinite number of directions which $(x,y)$ can approach $(a,b)$ along, we need them all to be the same
- For limits to the origin, the easiest way is setting $x = tx_0$, $y = ty_0$.

Partial derivatives

- For a function $f(x,y)$, we can treat $x$ as a variable and $y$ as a constant or vice versa
- $\frac{\partial f}{\partial x} = f_x$ is the derivative of $f$ with respect to $x$.
- $\frac{\partial f}{\partial y} = f_y$ is the derivative of $f$ with respect to $y$.
- In terms of limits, $f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0+\Delta x, y_0) - f(x_0, y_0)}{\Delta x}$.

Higher derivatives

- $f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$.
- $f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$.
- Theorem: partial derivatives commute $f_{xy} = f_{yx}$.
- $f_{xx}, f_{yy}$ tells the concavity in $xz, yz$ plane.
- $f_{xy}$ tells how $f_y$ changes as we change $x$.

Implicit differentiation

- For any 3 variable function $f(x,y,z)$, we can implicitly define $z$ as a function of $x, y$.
- $z$ is dependent on $x, y$, and we can calculate $z_x, z_y$ directly.

Linear approximation

- Consider $l_1: z = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0)$ and $l_2: z = f(x_0, y_0) + f_y(x_0, y_0)(y - y_0)$.
- They lie in the tangent plane
- Then $f(x,y) \approx f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$.

Chain rule

- Given $z = f(x,y)$, $x = g(t)$, $y = h(t)$, we have $z_t = z_x x_t + z_y y_t$.
- Similarly, if $z = f(g(s,t), h(s,t))$, then $\begin{pmatrix} z_s \\ z_t \end{pmatrix} = \begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix} \begin{pmatrix} z_x \\ z_y \end{pmatrix}$.
  - $\begin{pmatrix} x_s & y_s \\ x_t & y_t \end{pmatrix}$ is the Jacobian matrix
- In polar coordinates $x = r\cos\theta$, $y = r\sin\theta$, $z = f(x,y)$.
  - $z_r = z_x \cos\theta + z_y \sin\theta$.
  - $z_\theta = z_x(-r\sin\theta) + z_y(r\cos\theta)$.

Directional derivative

- Let $\vec{u}$ be the directional vector, $D_{\vec{u}} f(x_0, y_0)$ =rate of change at $(x_0, y_0)$, as we move in the direction $\vec{u}$ at unit speed, $|\vec{u}| = 1$.
- $D_{\vec{u}} f = \frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt} = f_x(x_0, y_0)a + f_y(x_0, y_0)b = \langle f_x, f_y \rangle \cdot \vec{u}$.
- $\nabla f = \langle f_x, f_y \rangle$ is the gradient of $f$, it is a vector field.
  - $D_{\vec{u}} f = \nabla f \cdot \vec{u}$.
- If $\vec{u}$ is tangent to a contour line, then $D_{\vec{u}} f = 0 \Rightarrow \nabla f \cdot \vec{u} = 0$, $\nabla f \perp$contour.
- $D_{\vec{u}} f$ is greatest when $\vec{u}$ is in the direction of $\nabla f$.
  - $\nabla f$ points to the direction in which $f$ increases the fastest.
- If $F(x,y,z)$ is a function of 3 variables, then $\nabla F$ is a vector field in $\mathbb{R}^3$, properties hold.

- Tangent plane: $z = z_0 - \frac{F_x}{F_z}(x - x_0) - \frac{F_y}{F_z}(y - y_0)$.

Classification of critical points

- For $f: D \to \mathbb{R}$, if $D$ is closed and bounded, $f(x, y)$ will achieve its global max/min at either a critical point or on the boundary.
- A point $(x_0, y_0)$ is critical if $\nabla f(x_0, y_0) = 0$.
- Discriminant (determinant of Hessian matrix)
  - $D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{vmatrix} = f_{xx}f_{yy} - f_{xy}^2$.
- Classification:
  - $D(x_0, y_0) > 0, f_{xx} > 0$, local min.
  - $D(x_0, y_0) > 0, f_{xx} < 0$, local max.
  - $D(x_0, y_0) = 0$ not a critical point (inconclusive).
  - $D(x_0, y_0) < 0$, saddle point.

Lagrange multiplier

- Max/min of $f(x, y)$ restricted to boundary curve occurs when the contour curve is tangent to the boundary curve.
- Look for points $(x_0, y_0)$ on the boundary curve $g(x, y)$ where $\nabla f(x_0, y_0) = \lambda \nabla g(x_0, y_0)$.
  - $\lambda$ is the Lagrange multiplier.
  - This means $f_x = \lambda g_x$, $f_y = \lambda g_y$, $g = 0$.

# Multiple integrals

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Definition: $\iint_D f(x,y)dxdy = \lim_{N\to\infty, M\to\infty} \sum_{i=1}^{M} \sum_{j=1}^{N} f\left(x_i, y_j\right)\Delta x_i \Delta y_j$.

Average value of $f$ in $D = \frac{1}{Area(D)} \iint_D f(x,y)dA$

Properties

- FTC still apply
- Linearity: $\iint_D Af(x,y) + Bg(x,y)dxdy = A\iint_D f(x,y)dxdy + B\iint_D g(x,y)dxdy$.

Theorem:

- If $D = [a,b]\times[c,d]$, $\iint_D f(x,y)dxdy = \int_c^d \int_a^b f(x,y)dx\,dy$.
- Fubini Theorem: $\iint_D f(x,y)dxdy = \int_c^d \int_a^b f(x,y)dx\,dy = \int_a^b \int_c^d f(x,y)dy\,dx$.

$D$ is vertically sliceable if it is of the form $D = \{(x,y) : g_1(x) \le y \le g_2(x), a \le x \le b\}$.

- Then $\iint_D f(x,y)dxdy = \int_c^d \int_{g_1(x)}^{g_2(x)} f(x,y)dy\,dx$.

$D$ is horizontally sliceable if it is of the form $D = \{(x,y) : g_1(y) \le x \le g_2(y), a \le y \le b\}$.

- Then $\iint_D f(x,y)dxdy = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x,y)dx\,dy$.

Sometimes in a region that is both vertically and horizontally sliceable, an integral is possible to do in only one way

If $f(x,y)$ is odd in $x$, $f(-x,y) = -f(x,y)$, and $R$ is symmetric under reflection about $y - axis$, then $\iint_R f(x,y)dxdy = 0$

Integration in polar coordinates

- $x = r\cos\theta$, $y = r\sin\theta$, $\theta = \arctan\frac{y}{x}$.
- Let $R = \{(r,\theta) : a \le r \le b, \alpha \le \theta \le \beta\}$, $\Delta r = \frac{b-a}{N}$, $\Delta\theta = \frac{\beta-\alpha}{M}$.
- Then $\iint_R f(r,\theta)dA = \lim_{N\to\infty, M\to\infty} \sum_{i=1}^{M} \sum_{j=1}^{N} f\left(r_j, \theta_i\right)\Delta r_j \Delta\theta_i = \int_a^b \int_\alpha^\beta f(r,\theta)rd\theta\,dr$.
- Radially sliceable region: $R = \{(r,\theta) : g_r(\theta) \le r \le g_2(\theta), \alpha \le \theta \le \beta\}$.
    - Then $\iint_R f(r,\theta)dA = \int_\alpha^\beta \int_{g_1(\theta)}^{g_2(\theta)} f(r,\theta)rdr\,d\theta$.

Applications

- Mass
    - Metal object of shape $R$, suppose it is made of a metal of density $\rho$,
        - then $m(R) = \rho Area(R)$
    - Suppose $\rho = \rho(x,y)$.
        - Mass= $\iint_R \rho(x,y)dxdy$.
- Center of mass
    - $(\overline{x}, \overline{y}) = \left(\frac{\iint_R x\rho(x,y)dxdy}{\iint_R \rho(x,y)dxdy}, \frac{\iint_R y\rho(x,y)dxdy}{\iint_R \rho(x,y)dxdy}\right)$.
- Surface area
    - $Area\left(P_{ij}\right) = \left|\vec{a}\times\vec{b}\right| = \sqrt{1 + f_x^2 + f_y^2}\Delta x\Delta y$.
    - Total area $S(A) = \iint_R \sqrt{1 + f_x^2 + f_y^2}dxdy$.
    - $S(A) \ge Area(R)$.
    - $z = f(x,y) + C$ has the same surface area as $f(x,y)$.

Triple integral

- $\iiint_E F(x,y,z)dV$.
- $Volume(E) = \iiint_E dV$.
- Type I: solid between two graphs $z = u_1(x,y)$, $z = u_2(x,y)$, $(x,y) \in R$.
  - $E = \{(x,y,z) : (x,y) \in R, u_1 \le z \le u_2\}$.
  - $\iiint_E FdV = \iint_R \int_{u_1}^{u_2} Fdz\,dxdy$.
- Type II: solid between two graphs $x = u_1(y,z)$, $x = u_2(y,z)$, $(y,z) \in R$.
  - $E = \{(x,y,z) : (y,z) \in R, u_1 \le x \le u_2\}$.
  - $\iiint_E FdV = \iint_R \int_{u_1}^{u_2} Fdx\,dzdy$.
- Type III: solid between two graphs $y = u_1(x,z)$, $y = u_2(x,z)$, $(x,z) \in R$.
  - $E = \{(x,y,z) : (x,z) \in R, u_1 \le y \le u_2\}$.
  - $\iiint_E FdV = \iint_R \int_{u_1}^{u_2} Fdy\,dxdz$.

Cylindrical coordinates

- Let $x = r\cos\theta$, $y = r\sin\theta$, then $(r,\theta,z)$ forms the cylindrical coordinates.
- Let $E = \{(x,y,z) : (x,y) \in R, g_1 \le z \le g_2\}$, $R = \{(r,\theta) : h_1 \le r \le h_2, \theta \in [\alpha,\beta]\}$.
- Then $\iiint_E FdV = \iint_R \int_{g_1}^{g_2} Fdz\,dA = \int_\alpha^\beta \int_{h_1}^{h_2} \int_{g_1}^{g_2} Frdz\,dr\,d\theta$.

Spherical coordinates

- $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$, $\theta \in [0,2\pi]$, $\phi \in [0,\pi]$.
  - $\phi$ measured from positive $z - axis$.
- $\Delta V = \rho^2 \sin\phi\Delta\rho\Delta\phi\Delta\theta$.
- $\iiint_E FdV = \iiint_E F\rho^2\sin\phi\,d\rho d\phi d\theta$.

# Vectors and curves

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Vectors in $\mathbb{R}^2$ and $\mathbb{R}^3$

- A vector is a quantity with both magnitude and direction indicated by arrows
- Magnitude $|\vec{a}|$ is the length of the vector $\vec{a}$.
- Two vectors are the same if they have the same direction and magnitude
- Addition: $\vec{a}+\vec{b}=\vec{b}+\vec{a}$.
- Scalar multiplication $c\vec{a}=\vec{a}+\vec{a}+\cdots+\vec{a}$.
- Zero vector $\vec{0}$: the only vector of magnitude 0, has no direction.
- The vector from $(0,0,0)$ to $(a,b,c)$ is denoted as $< a,b,c >$.
- Unit vectors:
  - $\vec{i} =< 1,0,0 >$.
  - $\vec{j} =< 0,1,0 >$.
  - $\vec{k} =< 0,0,1 >$.

Dot product

- Geometric definition: $\vec{a}\cdot\vec{b}=|\vec{a}|\left|\vec{b}\right|\cos\theta$.
- Algebraic: $\vec{a} =< a_1,a_2,a_3 >$, $\vec{b} =< b_1,b_2,b_3 >$, then $\vec{a}\cdot\vec{b}=a_1b_1+a_2b_2+a_3b_3$.
- Remark: $\vec{a}\cdot\vec{b}=0 \Leftrightarrow \vec{a}\perp\vec{b}$.

Cross product

- Geometric: $\vec{a}\times\vec{b}=|\vec{a}|\left|\vec{b}\right|\sin\theta$.
  - Direction of $\vec{a}\times\vec{b}$ is normal to both $\vec{a}$ and $\vec{b}$.
- Algebraic: $\vec{a}\times\vec{b}=\begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$.
- Remark: $\vec{a}\times\vec{b}=0 \Leftrightarrow \vec{a}\parallel\vec{b}$.

Curves

- Define $r$: $\{(x(t),y(t))\in\mathbb{R}^2 : r(t) =< x(t),y(t) >\}$.

Derivatives

- $r'(t)=\frac{d}{dt}r(t)=\lim_{h\to 0}\frac{r(t_0+h)-r(t_0)}{h}$.
- Rules:
  - $\frac{d}{dt}(a\cdot b)=a'\cdot b+a\cdot b'$.
  - $\frac{d}{dt}(a\times b)=a'\times b+a\times b'$.
  - $\frac{d}{dt}\Big(a\big(s(t)\big)\Big)=a'\big(s(t)\big)s'(t)$, $s(t)$ is a scalar.
- Derivative of $r(t)$ is tangent to $r(t)$, $r'(t)\cdot r(t)=0$
  - Unit tangent $T=\frac{r'(t)}{|r'(t)|}$.
- Arclength is related to the magnitude of the local velocity vector: $\frac{ds}{dt}=\left|\frac{dr}{dt}\right|$.
  - $s(T)=\int_{T_0}^{T}\left|r'(t)\right|dt+s(T_0)$.

For 3D inputs

- Position $r(t) =< x(t),y(t),z(t) >$.
- Velocity $r'(t) =< x',y',z' >$.
- Acceleration $r''(t) =< x'',y'',z'' >$.
- Speed $|r'(t)|=\sqrt{(x')^2+\left(y'\right)^2+(z')^2}$.

- Distance travelled $s(T) - S(T_0) = \int_{T_0}^{T} |r'(t)| dt$.

Parametrization methods

- Polar coordinates
- Cartesian coordinates
- Arclength

Curvature

- $\rho$ is the radius of curvature.
  - $\rho = \left|\frac{ds}{d\theta}\right|$.
  - Center of curvature: $p + \rho N$.
- $k = \frac{1}{\rho}$ is the curvature and is a measure of how tight the curve turns.
  - $k = \left|\frac{ds}{d\theta}\right|^{-1} = \frac{|r' \times r''|}{|r'|^3}$.
  - When $k$ is max, $a \perp v$ iff $v$ is constant.
  - When $k = 0$, $a \parallel v$.
  - If $r \parallel a$, then $r \times v$ is constant, $a = v'T + kvN$.

Unit tangent and normal

- $T = \frac{r'}{|r'|} = \frac{dr}{ds}$.
- $N = \frac{T'}{|T'|}$, it is in the direction of $r' \times r''$.
  - $\frac{dT}{ds} = N(s)k(s)$.

Frenet Frame

- Binormal vector $B = T \times N$ is orthogonal to both $T$ and $N$.
- $\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$.
- Torsion: $\tau(s) = -B' \cdot N = \frac{r' \times r'' \cdot r'''}{|r' \times r''|^2}$.
  - $\tau > 0$, rotation is counter clockwise.

Path integral

- A measure of work done on a particle moving along a curve $\gamma$ inside a scalar force field $f(x, y, z)$.
- $\int_\gamma f(x, y, z) ds = \int_a^b f(r(t)) |r'(t)| dt$.
- In general, if $\gamma_1$ and $\gamma_2$ are reversed, $\gamma_1 = -\gamma_2$,
  - then $-\int_{\gamma_1} f(r(t)) |r'(t)| dt = \int_{\gamma_2} f(r(t)) |r'(t)| dt$.
  - But it does not affect the integration with respect to arclength
  - Need to ensure $a \le t \le b$ and the curve is positvely oriented.

Vector fields

- Velocity field and force field
- V field $v(x, y, z) = < v_x, v_y, v_z >$.
  - E.g. $v = < y, x >$.

Gradients

- $\nabla = < \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} >$.
- Potential function: a vector field is said to be conservative if there exists a scalar and a continuous function $\phi$ such that $v = \nabla \phi$ or $F = \nabla \phi$.

Irrotational flow (curls)

- Curl describe the rotation of a vector field

- They also help check if a vector field is conservative
- $curl\, F = \nabla \times F$.
- If $\nabla \times F = 0$, then the vector field is conservative

Some important operations

- $grad\, f = \nabla f =< f_x, f_y, f_z >$.
- $div\, F = \nabla \cdot F = \frac{\partial}{\partial x}F_1 + \frac{\partial}{\partial y}F_2 + \frac{\partial}{\partial z}F_3$. The rate of which fluid is exiting a volume
- $curl\, F = \nabla \times F = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_1 & F_2 & F_3 \end{vmatrix}$.

Streamlines

- It maps out trajectories of massless particles in a vector field
- $r' \times v(r(t)) = 0$.
- This gives a family of curves that are instantaneously tangent to the vector field, so the vector field can be defined as: $v = \nabla \times \psi$, where $\psi$ is the stream function (velocity potential).

Line integrals in vector fields

- We want the work done on a particle travelling inside a vector field
- $W = \int_\gamma F(t) \cdot T(t) ds = \int_\gamma F \cdot dr = \int_a^b F(r(t)) \cdot r'(t) dt$.
- If a vector field is conservative, then $\int_\gamma F \cdot dr = \phi(r(b)) - \phi(r(a))$.

Path independence

- $F$ is conservative if there exists a scalar and continuous potential function such that $F = \nabla\phi$.
- $F$ if conservative if the curl of the vector field is zero, $\nabla \times F = 0$.
- For conservative fields, $\int_{\gamma_1} F \cdot dr = \int_{\gamma_2} F \cdot dr = \phi(p_1) - \phi(p_0)$ for any path from $p_0$ to $p_1$.

Summary for a continuous vector field in $\mathbb{R}^2$ or $\mathbb{R}^3$.

- $F = \nabla\phi$ if $F$ is conservative.
- $\int_\gamma F \cdot dr = 0$ for closed curves.
- The integral is path independent for curves that start and end at the same point.
- If $F$ is continuous and differentiable, then $F$ is conservative if and only if $\nabla \times F = 0$.

Green's theorem

- The line integral of $F(x, y)$ around a simple closed curve is the same as the double integral of $\nabla \times$ F with the boundary.
- Define $\partial\Omega$ to be the boundary.
- Orientation:
  - Counter clockwise is positive.
  - Clockwise is negative
- $\int_{\partial\Omega} F \cdot dr = \iint_\Omega \nabla \times F dA$.
  - $F_x, F_y$ need to be continuous and differentiable.
  - $\int_{\partial\Omega} F \cdot dr > 0$ if $F$ on average is along the direction of $dr$.
  - $\int_{\partial\Omega} F \cdot dr < 0$ if $F$ on average is against the direction of $dr$.
  - A counter clockwise rotation within $\Omega$ and on $\partial\Omega$ is when $\nabla \times F > 0$.
- If $\nabla \times F = 1$, we have $\int_{\partial\Omega} F \cdot dr = Area(\Omega)$.
  - Need $\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} = 1$.
- It surrounds vector fields that are not continuous/differentiable at every point with the surface $\Omega$.
- Suppose we have a region $\Omega_1$ with a hole $\Omega_2$ in it, $\partial\Omega_1$ is positively oriented and $\partial\Omega_2$ is negative oriented. Then $\iint_\Omega \nabla \times F dA = \iint_{\Omega_1} \nabla \times F dA + \iint_{\Omega_2} \nabla \times F dA$.
  - $\iint_\Omega \nabla \times F dA = \int_{\partial\Omega_1} F \cdot dr + \int_{\partial\Omega_2} F \cdot dr = \int_{\partial\Omega} F \cdot dr$.

Divergence theorem

- 2D divergence theorem is to diverge what Green's theorem is to curl
- The flux $F$ through a boundary curve $\partial\Omega$ is the same as the differentiable integral of $\nabla \cdot F$ over all $\Omega$.
- 2D: $\int_{\partial\Omega} F \cdot n ds = \iint_{\Omega} \nabla \cdot F dA$.
- 3D: $\iint_{S} F \cdot n d\Sigma = \iiint_{V} \nabla \cdot F dV$.

# Surface integrals and theorems

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Parametrized surfaces

- Build a function for the surface: root finding method to find $x, y, z$ at the surface.
- Parametrize the surface such that each point is described by two parameters $u, v$, and get $(u, v) \in \mathbb{R}^2$, $r(u, v) =< x(u, v), y(u, v), z(u, v) >\in \mathbb{R}^3$.
- Parametrized plane: $r(u, v) =< u, v, -\frac{A}{C}u - \frac{B}{C}v - \frac{D}{C} >$.

Tangent plane: $n \cdot< x - x_0, y - y_0, z - z_0 > = 0$.

- Given $r(u, v) =< x(u, v), y(u, v), z(u, v) >$.
- $T_u =< x_u, y_u, z_u >$.
- $T_v =< x_v, y_v, z_v >$.
- $n = T_u \times T_v$.

A surface $\Omega$ is smooth if it has a smooth parametrization $r(u, v)$ such that $x, y, z$ are smooth functions and $T_u \times T_v \neq 0$ for any $u, v$.

Surface area

- To find the surface area of a complex surface, construct a tangent plane at $r\left(u_0, v_0\right)$ such that $r_u = T_u, r_v = T_v$.
- The surface area is $\iint_{\mathrm{D}} \left|r_u \times r_v\right| dA$, where $D$ is the parametrized region.
- If we isolate a small region, we can see that the surface can be linearly approximated.
    - $P = r\left(u_0, v_0\right) + r_u \Delta u + r_v \Delta v$, when $\Delta u, \Delta v$ are small.
- The area of the cell is equivalent to the magnitude of the vector that is orthogonal to the plane

A useful parametrization (surface of revolution)

- $r(u, v) =< f(v)\cos u, f(v)\sin u, v >$.
- This ensures a rectangular parameterization domain.

Surface integral

- Surface integral of a scalar function: $\iint_{\Omega} f\left(x, y, z\right) d\Omega = \iint_{\mathrm{D}} f\left(r(u, v)\right)\left|r_u \times r_v\right| dA$.
- Surface integral of a continuous vector field. To find the flux of $F$ through a surface $\Omega$.
    - Outward normal: $n = \frac{r_u \times r_v}{|r_u \times r_v|}$.
    - $\iint_{\Omega} F \cdot n d\Omega = \iint_{\mathrm{D}} F\left(r(u, v)\right) \cdot \left(r_u \times r_v\right) d\mathrm{A}$.
    - For a continuously differentiable and smooth vector field, we can apply divergence theorem: $\iint_{\Omega} F \cdot n d\Omega = \iiint_{V} \nabla \cdot F dV$.

Stokes' theorem

- It relates the surface integral of the curl of a vector field with the line integral of that same vector field around the boundary of the surface integral
- For each small piece $\int_{\partial\Omega_i} F \cdot dr = (\nabla \times F) \cdot n d\Omega_i$.
- $\int_{\partial\Omega} F \cdot dr = \iint_{\Omega} (\nabla \times F) \cdot n d\Omega$.
- Must make sure that $n$ is oriented positively with counter clockwise rotation and negatively with clockwise rotation.
- There are thus two ways to calculate the surface integral of complex shapes
    - Project the surface to the plane the boundary curve $\partial\Omega$ creates.
    - Cur the hemisphere into sectors instead of the plane that's bounded within the boundary curve.
- If there is no bounding curve, for a closed surface, $\int_{\partial\Omega} F \cdot dr = \iint_{\Omega} (\nabla \times F) \cdot n d\Omega = 0$.