Vectors and geometry

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1. Vectors:

Column vector: $\begin{pmatrix} a \\ b \\ c \end{pmatrix}$

Row vector: (a, b, c)

Addition: $\vec{v} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$ $\vec{w} = \begin{pmatrix} a_2 \\ b_2 \\ c_2 \end{pmatrix}$ $\vec{v} = \begin{pmatrix} a_1 + a_2 \\ b_1 + b_2 \\ c_1 + c_2 \end{pmatrix}$

Scalar product: $\vec{v} = \begin{pmatrix} a_1 \\ b_1 \\ c_1 \end{pmatrix}$, $c\vec{v} = \begin{pmatrix} ca_1 \\ cb_1 \\ cc_1 \end{pmatrix}$

Zero vector: $\vec{0}$ $0\vec{v} = \vec{0}$

A vector space is a set of vectors with 2 operations satisfying properties: $F(R, R) = \{f(x)\}$

2. Geometry

Length of $\vec{a} = \begin{pmatrix} a \\ b \end{pmatrix}$

a. $|\vec{a}| = \sqrt{a^2 + b^2}$

b. $|\vec{a}| \geq 0$ $|\vec{a}| = 0$ if and only if $\vec{a} = \vec{0}$

c. $|s\vec{a}| = |s||\vec{a}|$

d. $|\vec{a} + \vec{b}| \leq |\vec{a}| + |\vec{b}|$

Unit vector

$\vec{u}$ is a unit vector if and only if $|\vec{u}| = 1$

If $\vec{a}$ is a non-zero vector, then $\frac{\vec{a}}{|\vec{a}|}$ is a unit vector

Distance

Distance of $(\vec{a}, \vec{b})$ is $|\vec{b} - \vec{a}|$

Dot product

$\vec{a} \cdot \vec{b} = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n = |\vec{a}| |\vec{b}| \cos\theta$

If $\vec{a} \cdot \vec{b} = 0$ and $\vec{a}, \vec{b}$ are non-zero vectors, then $\vec{a} \perp \vec{b}$

$\vec{a} \cdot \vec{a} = |\vec{a}|^2$

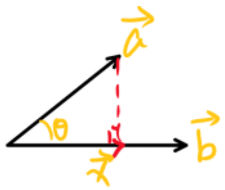
$\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$

$\vec{a} \cdot (\vec{b} + \vec{c}) = \vec{a} \cdot \vec{b} + \vec{a} \cdot \vec{c}$

$s \cdot \vec{a} \cdot \vec{b} = s(\vec{a} \cdot \vec{b})$

$\vec{0} \cdot \vec{a} = 0$

Projection

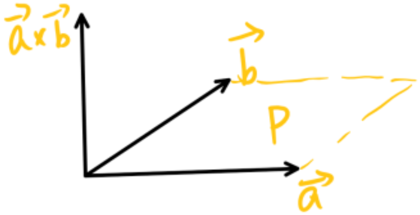


$\vec{x}$ is the projection of $\vec{a}$ onto $\vec{b}$

$\vec{x} = \frac{\vec{a} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b}$

Cross product

$$\vec{a} \times \vec{b} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$ where i, j, k are the unit vectors in R^3



$\vec{a} \times \vec{b} \perp \vec{a}$ and $\vec{a} \times \vec{b} \perp \vec{b}$

$\left|\vec{a} \times \vec{b}\right| = A(P)$

The direction of $\vec{a} \times \vec{b}$ satisfies right hand rule.

$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$

$\vec{a} \times \left(\vec{b} + \vec{c}\right) = \vec{a} \times \vec{b} + \vec{a} \times \vec{c}$

$s \cdot \vec{a} \times \vec{b} = s\left(\vec{a} \times \vec{b}\right)$

$\vec{a} \times \vec{a} = \vec{0}$

$$\vec{c} \cdot (\vec{a} \times \vec{b}) = \det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix}$$

3. Lines and planes

a. Lines in R^2

Equation form l: $ax + by + c = 0, \vec{n} = (a, b)$ is the normal vector

Parametric form l: $\overrightarrow{OP} + s\vec{v}$, P is a point on the line, $\vec{v}$ is the direction vector

Parallel lines: $\overrightarrow{v_1} = c\overrightarrow{v_2}$ or $\overrightarrow{v_1} \cdot \overrightarrow{n_2} = 0$

b. Planes in R^3

Equation form: $ax + by + cz = d, \vec{n} = (a, b, c)$ is the normal vector

Parametric form: $\overrightarrow{OP} + s\vec{v} + t\vec{w}$, P is a point on the line, $\vec{v}, \vec{w}$ are two direction vectors on the plane. $\vec{v} \neq k\vec{w}$, $s, t, k \in R$

c. Lines in R^3

Parametric form: $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \\ z_0 \end{pmatrix} + t\begin{pmatrix} u \\ v \\ w \end{pmatrix}$ where $\vec{d} = \begin{pmatrix} u \\ v \\ w \end{pmatrix}$ is the direction vector and $t \in R$

d. Distances:

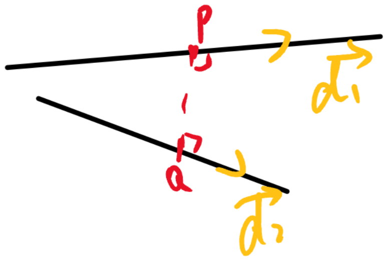
i. Point to line (2D): $d = \frac{|ax_0 + by_0 + c|}{\sqrt{a^2 + b^2}}$

ii. Point to plane (3D): $d = \frac{\left|\overrightarrow{OP} \cdot \vec{n}\right|}{|\vec{n}|}$

iii. Distance between skew line

Common normal vector $\vec{n} = \overrightarrow{v_1} \times \overrightarrow{v_2}$

$$d = \frac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{|\vec{n}|}$$



Linear system

2019年7月26日 9:40

1. Systems of linear equations

A system of linear equations contains m equations of n variables, the highest order is 1
A solution is $(x_1, x_1, \dots, x_n)$ that satisfies all equations
Any system of linear equations has 0,1, ∞ solutions

2. Linear independence

Definition (Linear relationship): a relation between vectors $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ is $a_1\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots + a_n\overrightarrow{v_n} = \vec{0}$, if all a_i are 0, then it is a trivial relation
Definition (Linear independence): $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ are linearly dependent if there is a non-trivial relation among them. Otherwise, they are independent
i.e. $a_i = 0$ is the only solution for $\overrightarrow{v_1} + a_2\overrightarrow{v_2} + \cdots + a_n\overrightarrow{v_n} = \vec{0}$ if $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_n}$ are linearly independent
If some $\overrightarrow{v_i} = \vec{0}$, then $\overrightarrow{v_1}, \overrightarrow{v_2}, \dots, \overrightarrow{v_i}, \dots, \overrightarrow{v_n}$ are dependent
$n + 1$ vectors in R^n are dependent
A basis for R^n is a set of n linearly independent vectors in R^n
If 3 vectors in R^3 are dependent, then they lie on the same plane and $\begin{vmatrix} \overrightarrow{v_1} \\ \overrightarrow{v_2} \\ \overrightarrow{v_3} \end{vmatrix} = 0$

3. Solving a linear equation by Gaussian elimination

Matrix of coefficients:
$\begin{pmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{pmatrix}$ is an augmented matrix of the system $\begin{cases} a_1x + b_1y + c_1z = d_1 \\ a_2x + b_2y + c_2z = d_2 \\ a_3x + b_3y + c_3z = d_3 \end{cases}$

Use Gaussian elimination to make the matrix in the form of $\begin{pmatrix} 1 & 0 & 0 & x_0 \\ 0 & 1 & 0 & y_0 \\ 0 & 0 & 1 & z_0 \end{pmatrix}$ if the system has one solution (x_0, y_0, z_0)
If one row is all zeros, then the system has infinite solutions
If one row has the pivot (the first 1) in the last column, then the system has no solution

4. Homogeneous system

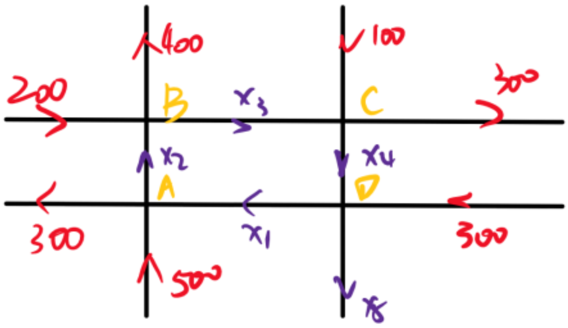
$\begin{pmatrix} a_1 & b_1 & c_1 & 0 \\ a_2 & b_2 & c_2 & 0 \\ a_3 & b_3 & c_3 & 0 \end{pmatrix}$
Solutions are lines/planes which go through the origin
i.e. $\vec{x} = a\vec{v} + b\vec{w} + c\vec{u}$ which is a span of $(\vec{v}, \vec{w}, \vec{u})$
The solution of a homogeneous system is always a span
Properties of spans:

a. $\vec{0}$ lies in span
b. If $\vec{v}, \vec{w}$ lie in a span, then$\vec{v} + \vec{w}$ lies in the span
c. If $\vec{v}$ lies in a span, then $c\vec{v}$ lies in the span

5. Applications

a. Traffic network

n intesections $\Rightarrow n$ equations
For each section total in=total out
E.g.



$$\begin{cases} 500 + x_1 = 300 + x_2 \\ 200 + x_2 = 400 + x_3 \\ x_3 + 100 = 300 + x_4 \\ 300 + x_4 = x_1 + x_5 \end{cases} \Rightarrow \begin{cases} x_1 = 200 + \mathrm{t} \\ x_2 = 400 + \mathrm{t} \\ x_3 = 200 + \mathrm{t} \\ x_4 = \mathrm{t} \\ x_5 = 100 \end{cases}$$

If there are n blocks, then there are n free variables

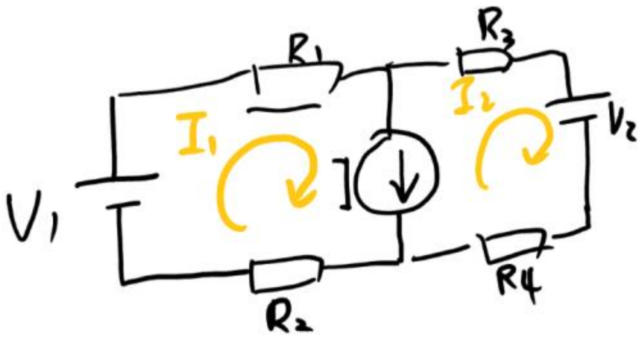
b. Resistor networks

i. Method 1 is like traffic network, for each node, current in = current out

ii. Method 2: Loop current

Voltage drop around a loop is 0

Current source may contribute to the current difference, set its voltage be E



$$\begin{cases} I_1R_1 + I_1R_2 + E - V_1 = 0 \\ I_2R_4 + I_2R_3 - E + V_2 = 0 \\ I_1 - I_2 = I \end{cases}$$

Matrices and determinants

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1. Matrices

A matrix is a table of numbers

$A_{m\times n}$ is a matrix of m rows and n columns

The rank of a matrix is the number of pivots in its row reduced echelon form

a. Matrix linear combination

i. A matrix $A_{m\times n}$ can be written as a row of column vectors
$A_{m\times n} = (\vec{a_1}\ \vec{a_2}\ \dots\ \vec{a_n})$ $\vec{a_n}$ is a vector in R^m

ii. A matrix $A_{m\times n}$ multiplying a column vector from the left, if defined, is equal to a linear combination of its column vectors

$$A\vec{x} = (\vec{a_1}\ \vec{a_2}\ \dots\ \vec{a_n})\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = x_1\vec{a_1} + x_2\vec{a_2} + \cdots + x_n\vec{a_n}$$

iii. Matrix B multiplying A from the left if defined, can be expressed as $BA = B(\vec{a_1}\ \vec{a_2}\ \dots\ \vec{a_n}) = (B\vec{a_1}\ B\vec{a_2}\ \dots\ B\vec{a_n})$

b. Linear transformation:

Definition: an $m \times n$ matrix $A_{m\times n}$ multiplying a vector in R^n from the left if defined, transform the vector to R^m, $A_{m\times n}\overrightarrow{x_{n\times 1}} = \overrightarrow{x_{m\times 1}}$. $A_{m\times n}$ defines a transformation $R^n \to R^m$

Every linear map $T: R^n \to R^m$ is a matrix transformation for a unique matrix $A_{m\times n}$

Special case: $A_{n\times n}$:$R^n \to R^n$

A transformation $T: R^n \to R^m$ is linear if and only if $T(\alpha\vec{x_1} + \beta\vec{x_2}) = \alpha T(\vec{x_1}) + \beta T(\vec{x_2})$

Linear transformation preserves properties of the original graph

i. Transpose:

If $A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix}$, then $A^{\mathrm{T}} = \begin{pmatrix} 2 & 1 \\ 5 & 3 \end{pmatrix}$

$(AB)^{\mathrm{T}} = B^{\mathrm{T}}A^{\mathrm{T}}$

ii. Inverse transformation

Let $T: R^n \to R^m$ be a linear transformation with $A_{m\times n}$, if there is a pivot in every row and every column of rref(A) (only possible if m=n, det(A)≠0), we can define the inverse transformation T^{-1}, which associates to every $\vec{y}$, the unique $\vec{x}$, $T(\vec{x}) = \vec{y}$ and $T^{-1}(\vec{y}) = \vec{x}$

To find the inverse matrix, use the identity matrices

E.g.

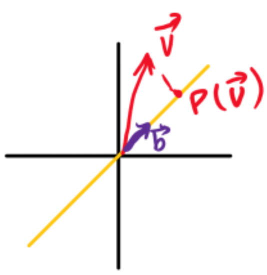
$$A = \begin{pmatrix} 2 & 5 \\ 1 & 3 \end{pmatrix},\ A^{-1} = \begin{pmatrix} 2 & 5 & 1 & 0 \\ 1 & 3 & 0 & 1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 3 & -5 \\ 0 & 1 & -1 & 2 \end{pmatrix}$$

The right side of the last matrix is the inverse of A

$(AB)^{-1} = B^{-1}A^{-1}$

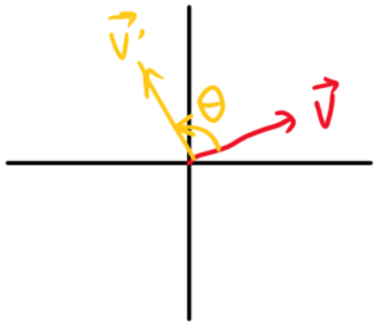
iii. Projection

$P = \left(proj\begin{pmatrix} 1 \\ 0 \end{pmatrix}\ \ proj\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right)$ where $proj\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}$, $\vec{b}$ is the direction vector of the projection line



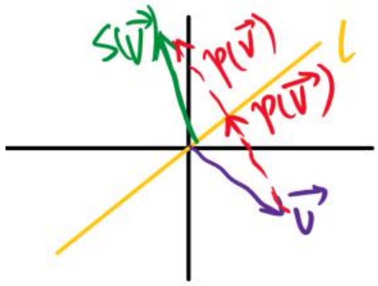
iv. Rotation

$R = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$, it rotates a graph about origin counterclockwise by an angle θ



v. **Reflection**

If P is the projection of $\vec{v}$ onto a line L through the origin, S is the reflection of $\vec{v}$ across L, then $S(\vec{v}) = 2P(\vec{v}) - \vec{v}$, the matrix for S is $S = 2P - I$, where I is the identity matrix



vi. **Composition of matrix transformations**

$R^p \xrightarrow{B} R^n \xrightarrow{A} R^m$, $A \circ B$ is the composition of A and B, $[A \circ B]\vec{x} = A(B\vec{x})$

Every composition of transformation is another transformation because the composition of linear transformation is linear

Definition: $A \circ B = AB$ if A, B exists and can be multiplied

The (i, j) entry of AB is the dot product of ith row of A and jth column of B

c. Dynamic Systems

Definition: $R^n \xrightarrow{A} R^n$ outputs vectors of the same type as the input systems, A is a $n \times n$ square matrix. We can then iterate A and get a dynamic system. Call vectors in R^n the state vectors. A describe change of the state of the systems.

Example: Fibonacci's rabbits

Start with 1 pair, after 1 month, each pair of rabbits produce one pair of rabbits every month.

We have two types of rabbits:

Let $j_n = \#\ of\ pairs\ of\ juveniles$; $a_n = \#\ of\ pairs\ of\ adults$;

$\begin{pmatrix} j_n \\ a_n \end{pmatrix}$ is the state vector after n months.

$j_{n+1} = a_n, a_{n+1} = a_n + j_n \Rightarrow \begin{pmatrix} j_{n+1} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j_n \\ a_n \end{pmatrix}$

$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is the transition matrix of the dynamic system

$\begin{pmatrix} j_1 \\ a_1 \end{pmatrix}$ is the initial vector, $\begin{pmatrix} j_n \\ a_n \end{pmatrix} = A^n \begin{pmatrix} j_1 \\ a_1 \end{pmatrix}$ is the solution

d. Markov process/Random walk

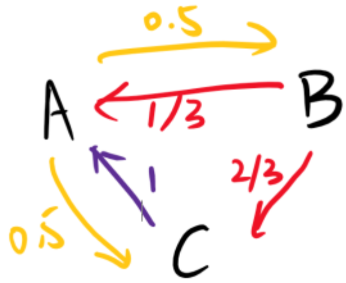
Definition: it is a dynamic system with a transition matrix such that all columns add up to 1

It may/ may not have a equilibrium state depending on the initial condition.

Ways to think of random walks:

i. Many users, the state describes the number of users on each node
ii. One user, the state describes the likelihood of the user being on each node

Example (has equilibrium state): Internet webpages



$$\begin{cases} a_{n+1} = \frac{1}{3}b_n + c_n \\ b_{n+1} = \frac{1}{2}a_n \\ c_{n+1} = \frac{1}{2}a_n + \frac{2}{3}b_n \end{cases} \Rightarrow \begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{3} & 1 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \end{pmatrix} \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$$

$\begin{pmatrix} 0 & \frac{1}{3} & 1 \\ \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{3} & 0 \end{pmatrix}$ is the transition matrix, its columns all add to 1, because the total number of users is constant

It has an equilibrium state, where $\begin{pmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{pmatrix} = \begin{pmatrix} a_n \\ b_n \\ c_n \end{pmatrix}$ by definition of limits

$\Rightarrow$ the proportion in equilibrium state is $6:3:5$

Example (does not have equilibrium state):

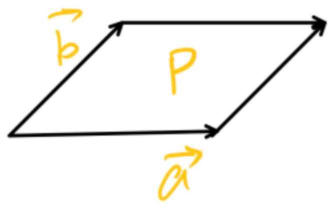
$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the transition matrix, even thougth the equilibrium is unique $eq = t\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, if we start with $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$, it will oscillate between these two states and never reaches the equilibrium state.

2. Determinant

$\det(A)$ is a number, A must be a $n \times n$ matrix

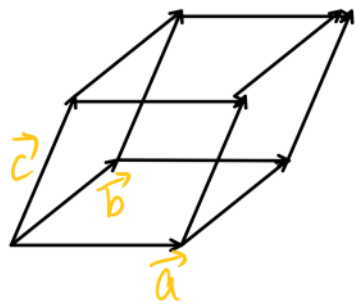
Geometry:

In R^2



$$\det\begin{pmatrix} \vec{a} \\ \vec{b} \end{pmatrix} = \pm A(P)$$

In R^3



$$\det\begin{pmatrix} \vec{a} \\ \vec{b} \\ \vec{c} \end{pmatrix} = \pm V(P)$$

If $\det\begin{pmatrix}\vec{a}\\ \vec{b}\\ \vec{c}\end{pmatrix} = 0$, $\vec{a}$,$\vec{b}$,$\vec{c}$ lies in a plane ($\vec{c} = s\vec{a} + t\vec{b}$)

Determinant operations:

$$\begin{vmatrix}a & b\\ c & d\end{vmatrix} = ad - bc$$

$$\begin{vmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{vmatrix} = a_{11}\begin{vmatrix}a_{22} & a_{23}\\ a_{32} & a_{33}\end{vmatrix} - a_{12}\begin{vmatrix}a_{21} & a_{23}\\ a_{31} & a_{33}\end{vmatrix} + a_{13}\begin{vmatrix}a_{21} & a_{22}\\ a_{31} & a_{32}\end{vmatrix}$$

In general $\begin{vmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{vmatrix} = a_{11}\det(A_{11}) - a_{12}\det(A_{12}) + \cdots + (-1)^{n-1}a_{1n}\det(A_{1n})$

$= \sum_{j=1}^{n}(-1)^{j-1}a_{1j}\det(A_{1j})$

This is the Laplace expansion of $\det(A)$ along the first row

We can use the expansion recursively to reduce the rank of the determinant to 2

A_{1j} is the matrix of $\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ deleting the first row and jth column.

The **Laplace expansion** can be used along any row $\det(A) = \sum_{j=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij})$ and any column, $\det(A) = \sum_{i=1}^{n}(-1)^{i+j}a_{ij}\det(A_{ij})$

A_{ij} is the matrix of $\begin{pmatrix}a_{11} & \cdots & a_{1n}\\ \vdots & \ddots & \vdots\\ a_{n1} & \cdots & a_{nn}\end{pmatrix}$ deleting the ith row and jth column.

The determinant of a **lower/upper triangle matrix** is the product of the diagonal terms

Example: $\begin{vmatrix}a_{11} & 0 & 0\\ a_{21} & a_{22} & 0\\ a_{31} & a_{32} & a_{33}\end{vmatrix} = \begin{vmatrix}a_{11} & a_{12} & a_{13}\\ 0 & a_{22} & a_{23}\\ 0 & 0 & a_{33}\end{vmatrix} = a_{11}a_{22}a_{33}$

Determinant & row operations

i. Replace operations (e.g. a_{1i}=a_{1i}-a_{2i}) do not affect the determinant
ii. Rescale a row, rescale the determinant by the same factor
iii. Row swap changes sign of the determinant

Properties:

$\det(A^{\mathrm{T}}) = \det(A)$

$\det(AB) = \det(A)\det(B)$

$\det(A) \neq 0$ if and only if A is invertible

For 2×2 matrix $A = \begin{pmatrix}a & \mathrm{b}\\ c & d\end{pmatrix}$, $A^{-1} = \frac{1}{ad-bc}\begin{pmatrix}d & -\mathrm{b}\\ -c & a\end{pmatrix}$

If there is a row/column with only 0, the determinant is 0

An $n \times n$ determinant has $n!$ Terms

Complex numbers

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1. basics

$C = \mathbb{R}^2$

To write $\begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2$ as a complex number: $x + yi$

Addition in $C \Leftrightarrow$ Addition in $\mathbb{R}^2$

multiplication in C, use distribution law and $i^2 = -1$

Properties: let $z = x + yi$ be a complex number

$|z| = \sqrt{x^2 + y^2}$ is the modulus

$|zw| = |z||w|$ if w is defined as z

$\bar{z} = x - yi$ is the complex conjugate of z

$\overline{zw} = \bar{z}\bar{w}$

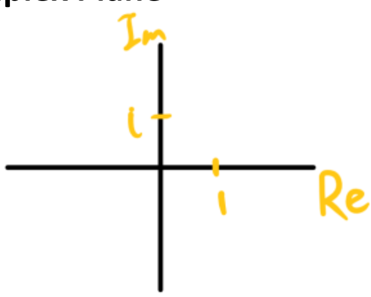
$|z|^2 = z\bar{z}$

$\frac{a + bi}{c + di} = \frac{(a + bi)(\mathrm{c} - \mathrm{di})}{(c + di)(c - di)} = \frac{(a + bi)(\mathrm{c} - \mathrm{di})}{c^2 + d^2}$

Theorem: Every polynomial with complex coefficient factors completely using complex numbers

Theorem: All of matrix algebra can be done with C

Complex Plane



2. Polar form of complex numbers

Complex numbers of modulus 1 ($|z| = 1$) lies on the unit circle, we can define $z = \cos\theta + i\sin\theta$, where θ is the angle of z with the positive real axis.

θ is called the argument of z, it is not unique (but we can make it unique by restricting the domain of θ)

Claim that multiplication by $\cos\theta + i\sin\theta$ is equivalent to rotation by angle θ

Proof: let $z = a + bi$, $(\cos\theta + i\sin\theta)z = (a\cos\theta - b\sin\theta) + i(a\sin\theta + b\cos\theta)$

$$= \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix}$$

Polar form of an arbitrary z

$z = |z|(\cos\theta + i\sin\theta)$

Let $r = |z|$, $z = r(\cos\theta + i\sin\theta)$ is the polar form

Multiplication by z is equivalent to rotation by angle θ and scale by a factor of r

3. Complex exponentials

Let $s, t \in \mathbb{R}$, $\mathrm{e}^{s+it} = \mathrm{e}^s\mathrm{e}^{it}$ is a complex exponential

Define $\mathrm{e}^{it} = \cos t + i\sin t$

Proof: let $\varphi(t) = \cos t + i\sin t$, then $\varphi'(t) = -\sin t + i\cos t = i\varphi(t)$, this is the property e^{it} should have

Another way to justify is by Taylor series

$z = r(\cos\theta + i\sin\theta) = r\mathrm{e}^{i\theta}$

Eigen-analysis

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1. Eigenvalues and eigenvectors

a. Real eigenvalues and eigenvectors

Recall Fibonacci's rabbit problem (with offspring changed from 1 to 2)

$$\begin{pmatrix} j_{n+1} \\ a_{n+1} \end{pmatrix} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j_n \\ a_n \end{pmatrix}$$

Let $\overrightarrow{x_n} = \begin{pmatrix} j_n \\ a_n \end{pmatrix}$, $\overrightarrow{x_{n+1}} = \begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \overrightarrow{x_n}$, initial state $\overrightarrow{x_0} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

If we plot $\overrightarrow{x_n}$ on a coordinate, the vector seems to get closer to a line $j = a$, which is important for this dynamic system.

The property of this line: if a state vector $\overrightarrow{x_n}$ is on the line, then $\overrightarrow{x_{n+1}}$ is also on the line

i.e. $\overrightarrow{x_{n+1}} = \lambda \overrightarrow{x_n}$ for some $\lambda \in \mathbb{R}$

$\Rightarrow A\overrightarrow{x_n} = \lambda \overrightarrow{x_n} \Rightarrow A\vec{x} = \lambda \vec{x}$ which means applying a transition matrix A is equivalent to multiplying a scalar λ

$$\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} j_n \\ a_n \end{pmatrix} = \lambda \begin{pmatrix} j_n \\ a_n \end{pmatrix} \Rightarrow \left(\begin{pmatrix} 0 & 2 \\ 1 & 1 \end{pmatrix} - \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} \right) \begin{pmatrix} j_n \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$\Rightarrow \begin{pmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{pmatrix} \begin{pmatrix} j_n \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, this homogeneous system shall have a non-trivial solution (we need the coefficient matrix be not invertible)

i.e. $\det \begin{pmatrix} -\lambda & 2 \\ 1 & 1-\lambda \end{pmatrix} = 0 \Rightarrow \lambda = 2$ or $\lambda = -1$

If $\lambda = 2$ $\begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} j_n \\ a_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, solution: $\begin{pmatrix} j_n \\ a_n \end{pmatrix} = a \begin{pmatrix} 1 \\ 1 \end{pmatrix}$, the line is spanned by the vector $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ (has slope 1). $\begin{pmatrix} j \\ a \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector correspoding to the eigenvalue $\lambda = 2$

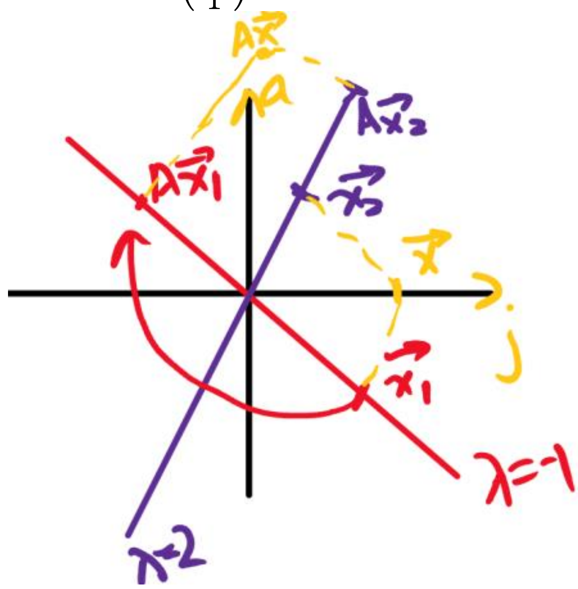
If $\vec{x}$ is a non-zero vector and λ is a scalar such that $A\vec{x} = \lambda \vec{x}$, then λ is an eigenvalue of A and $\vec{x}$ is an eigenvector of A. λ and $\vec{x}$ do not depend on the initial vector

$\lim_{n\to\infty} \frac{a_n}{j_n} = 1 =$ the slope of the eigenvector , is the limiting proportion from slope of eigenvectors

$\lim_{n\to\infty} \frac{j_{n+1}}{j_n} = \lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 2$ is the limiting growth rate=eigenvalue

To complete the problem, we still have $\lambda = -1$

Eigenvector is $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$, the eigen space is spanned by $\begin{pmatrix} -2 \\ 1 \end{pmatrix}$



Key fact: $\begin{pmatrix}-2\\1\end{pmatrix}$ and $\begin{pmatrix}1\\1\end{pmatrix}$ forms a basis of $\mathbb{R}^2$, consisting of eigenvectors of matrix $A = \begin{pmatrix}0 & 2\\1 & 1\end{pmatrix}$

The initial condition $\overrightarrow{x_0} = \begin{pmatrix}1\\0\end{pmatrix}$ can be expressed by $c_1\begin{pmatrix}1\\1\end{pmatrix} + c_2\begin{pmatrix}-2\\1\end{pmatrix}$

$\Rightarrow c_1 = \frac{1}{3}, c_2 = \frac{1}{3}$

We can then find $\begin{pmatrix}j_n\\a_n\end{pmatrix} = A^n\begin{pmatrix}j_0\\a_0\end{pmatrix} = \frac{1}{3}(2)^n\begin{pmatrix}1\\1\end{pmatrix} + \frac{1}{3}(-1)^n\begin{pmatrix}-2\\1\end{pmatrix}$

Now, the dynamic system is solved completely

b. Complex eigenvalues and eigenvectors

In general, complex eigenvalues are associated with rotational behavior in dynamic systems.

Example: rotation by $\frac{\pi}{2}$ counter-clockwise

$A = \begin{pmatrix}0 & -1\\1 & 0\end{pmatrix} \Rightarrow \lambda = \pm i$

When $\lambda = i, \vec{v} = \begin{pmatrix}i\\1\end{pmatrix}$; when $\lambda = -i, \vec{v} = \begin{pmatrix}-i\\1\end{pmatrix}$

Summary:

Characteristic equation of a matrix A: $\det(\lambda I - A) = 0$, λ is the eigenvalue

To find eigenvectors: $A\vec{v} = \lambda\vec{v}$, $\vec{v}$ is the eigenvector

The general solution is $\overrightarrow{x_n} = A^n\overrightarrow{x_0} = c_1\lambda_1^n\overrightarrow{v_1} + c_2\lambda_2^n\overrightarrow{v_2} + \cdots + c_k\lambda_k^n\overrightarrow{v_k}$, where c_i can be solved by initial condition

This works because there exists a basis of the state space consisting of eigenvectors for the transition matrix

Note: if one eigenvalue $\lambda_1 > |\lambda_i|$, for all $i \neq 1$, then the long-term behavior is determined by λ_1 and its eigenspace, it is the limiting growth rate

Note: random walk where the column of A sum to 1 (A is a stochastic matrix), then 1 is always an eigenvalue of A

Theorem: if in addition to being a stochastic matrix, all entries of A are positive, then

i. The multiplicity of 1 as an eigenvalue of A is 1 (up to scaling a unique fixed vector/equilibrium)

ii. All other eigenvalues satisfies $|\lambda_i| < 1$, no matter what the initial condition is, the system will converge to the equilibrium

iii. If all $|\lambda_i| = 1$, and λ_i are complex numbers, then the system will not converge, instead, it is periodic

iv. All eigenvectors corresponding to eigenvalues $|\lambda_i| \neq 1$ have the property that their components sum to 0. (e.g. in $\mathbb{R}^2$, $x + y = 0$)

2. Diagonalization

P and D are two matrix

If P: the columns are eigenvectors, D has the corresponding eigenvalues on diagonal then $AP = PD$

i.e. $P = (\overrightarrow{v_1}\,\overrightarrow{v_2} \ldots \overrightarrow{v_n})$, $D = \begin{pmatrix}\lambda_1 & 0 & \ldots & 0\\ 0 & \lambda_2 & 0 & \vdots\\ \vdots & 0 & \ddots & 0\\ 0 & \ldots & 0 & \lambda_n\end{pmatrix}$

Then $APP^{-1} = PDP^{-1} \Rightarrow A = PDP^{-1}$

$$\lim_{n\to\infty} A^n = (PDP^{-1})^n = P\begin{pmatrix}\lim_{n\to\infty}\lambda_1^n & 0 & \ldots & 0\\ 0 & \lim_{n\to\infty}\lambda_2^n & 0 & \vdots\\ \vdots & 0 & \ddots & 0\\ 0 & \ldots & 0 & \lim_{n\to\infty}\lambda_k^n\end{pmatrix}P^{-1}$$

Theorem: A is a $k \times k$ matrix with entries from $\mathbb{R}$, then $\mathbb{R}^k$ admits a basis consisting of eigenvectors for A(A is diagonalizable)

If the characteristic polynomial of $A(\det(\lambda I - A) = 0)$ has to factor into linear factors (this problem can be solved by passing to C) and for eigenvalues of A, the algebraic multiplicity has to equal the number of independent eigenvectors (geometric multiplicity)

Vector differential equations/Continuous dynamic systems

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1. Discrete model vs. continuous model

Continuous models are better model with shorter time intervals

Example: $2m^3$ chemical spill in upper lake, a beach resort is in the lower lake, 0.008 parts/billion is safe for swimming, predict the concentration of this pollutant in the lower lake.

River flow rate: $3km^3/day$, volume of lakes: $V_{upper} = 50km^3$, $V_{lower} = 100km^3$

Let x be the amount of pollutant in the upper lake in m^3, y be the amount of pollutant in the lower lake in m^3, $\begin{pmatrix} x \\ y \end{pmatrix} \in R^2$ is the state vector with initial condition $\begin{pmatrix} 2 \\ 0 \end{pmatrix}$

Discrete model with time interval 1 day:

Concentration: $\frac{x_n}{50}\ m^3/km^3$

$$x_{n+1} = x_n - \frac{3}{50}x_n$$

$$y_{n+1} = y_n + \frac{3}{50}x_n - \frac{3}{100}y_n$$

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} \frac{47}{50} & 0 \\ \frac{3}{50} & \frac{97}{100} \end{pmatrix} \begin{pmatrix} x_n \\ y_n \end{pmatrix}$$

Continuous model take h as small time interval:

$$x(t+h) = x(t) - h \cdot \frac{3}{50}x(t)$$

$$y(t+h) = y(t) + h \cdot \frac{3}{50}x(t) - h \cdot \frac{3}{100}y(t)$$

In terms of change over a small time interval:

$$\frac{x(t+h) - x(t)}{h} = -\frac{3}{50}x(t)$$

$$\frac{y(t+h) - y(t)}{h} = \frac{3}{50}x(t) - \frac{3}{100}y(t)$$

Take limits of both sides and we get a system of differential equations:

$$\begin{cases} x'(t) = -\frac{3}{50}x(t) \\ y'(t) = \frac{3}{50}x(t) - \frac{3}{100}y(t) \end{cases}$$

$$\Rightarrow \begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = \begin{pmatrix} -\frac{3}{50} & 0 \\ \frac{3}{50} & -\frac{3}{100} \end{pmatrix} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

Let $A = \begin{pmatrix} -\frac{3}{50} & 0 \\ \frac{3}{50} & -\frac{3}{100} \end{pmatrix}$, its eigenvalues are $\lambda_1 = -\frac{3}{100}$, $\lambda_2 = -\frac{3}{50}$, corresponding eigenvectors are $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$

The general solution is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{-\frac{3}{100}t} \begin{pmatrix} 0 \\ 1 \end{pmatrix} + c_2 e^{-\frac{3}{50}t} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$

c_1, c_2 depends on the initial conditions, $c_1 = -\frac{3}{25}$, $c_2 = \frac{6}{25}$

The general solution for a 2-variable differential equation is $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{\lambda_1 t}\overrightarrow{v_1} + c_2 e^{\lambda_2 t}\overrightarrow{v_2}$, where λ_1, λ_2 are the eigenvalues of the transition matrix A in $\begin{pmatrix} x'(t) \\ y'(t) \end{pmatrix} = A\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$, and $\overrightarrow{v_1}, \overrightarrow{v_2}$ are the corresponding eigenvectors. c_1, c_2 depends on the initial conditions.

Differential equations:

$\overrightarrow{x(t)}$ is a state vector in $\mathbb{R}^n$ depending on time t

$\overrightarrow{x'(t)} = A\overrightarrow{x(t)}$, A is a constant $n \times n$ matrix

If A admits a basis $\vec{v_1}, \vec{v_2}, \dots, \vec{v_n}$ of $\mathbb{R}^n$ of eigenvectors for A with corresponding eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. The general solution is : $\overrightarrow{x(t)} = c_1 e^{\lambda_1 t}\vec{v_1} + c_2 e^{\lambda_2 t}\vec{v_2} + \cdots + c_n e^{\lambda_n t}\vec{v_n}$, $c_1, c_2, \dots, c_n$ are coefficient determined by initial condition $\overrightarrow{x(0)} = c_1\vec{v_1} + c_2\vec{v_2} + \cdots + c_n\vec{v_n}$

2. Differential equations with complex eigenvalues

Example: $A = \begin{pmatrix} -3 & 2 \\ -1 & -1 \end{pmatrix}$, initial condition: $\overrightarrow{y(0)} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}$

eigenvalues: $\lambda_1 = -2 + i, \lambda_2 = -2 - i$, corresponding eigenvectors: $\vec{v_1} = \begin{pmatrix} 1-i \\ 1 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 1+i \\ 1 \end{pmatrix}$, $c_1 = 1, c_2 = 1$

$$\overrightarrow{y(t)} = e^{(-2+i)t}\begin{pmatrix} 1-i \\ 1 \end{pmatrix} + e^{(-2-i)t}\begin{pmatrix} 1+i \\ 1 \end{pmatrix} = \begin{pmatrix} 2e^{-2t}\cos t \\ 2e^{-2t}(\cos t + \sin t) \end{pmatrix}$$

It spirals to the origin

Remarks:

complex eigenvalues: $\lambda_1 = a + bi, \lambda_2 = a - bi, a, b \in \mathrm{R}, b \neq 0$

Complex eigenvectors: $\vec{u} \pm i\vec{v}$

General solutions: $\overrightarrow{x(t)} = c_1 e^{(a+bi)t}\vec{v_1} + c_2 e^{(a-bi)t}\vec{v_2}$

$= C_1 Re\left[e^{(a+bi)t}(\vec{u} + i\vec{v})\right] + C_2 Im\left[e^{(a+bi)t}(\vec{u} + i\vec{v})\right]$

Note: $e^{(a+bi)t} = e^{at}(\cos bt + \sin bt)$

If $a = 0$, i.e. eigenvalues are purely imaginary, then the system is purely periodic with $T = \frac{2\pi}{b}$.

If $a > 0$, the system rotates away from the origin with rate e^{at}

If $a < 0$, the system rotates towards the origin with rate e^{at}

3. Real application

a. Damping oscillators

Start with non-damping

$F = -ky, F = my'', y'' = \frac{F}{m} = -\frac{k}{m}y$

Take $y_1 = y$ and $y_2 = y'$, $\begin{cases} y_1'(t) = y_2(t) \\ y_2'(t) = y'' = -\frac{k}{m}y_1' \end{cases}$

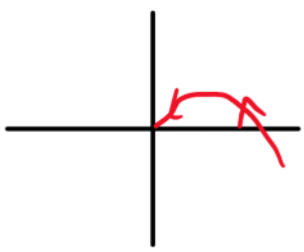
$$\begin{pmatrix} y_1'(t) \\ y_2'(t) \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -\dfrac{k}{m} & 0 \end{pmatrix}\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

Add damping

$y'' = -\frac{k}{m}y - cy'$ $A = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & -c \end{pmatrix}$

$$\Rightarrow \lambda = -\frac{c}{2} \pm \sqrt{\left(\frac{c}{2}\right)^2 - \frac{k}{m}}$$

If $\left(\frac{c}{2}\right)^2 - \frac{k}{m} \geq 0$, eigenvalues are real negative numbers



If $\left(\frac{c}{2}\right)^2 - \frac{k}{m} < 0$, eigenvalues are complex numbers

$e^{-\frac{c}{2}t}e^{\pm\sqrt{\left(\frac{c}{2}\right)^2-\frac{k}{m}}t}$ have no eigen spaces, the state vectors spiral to the origin

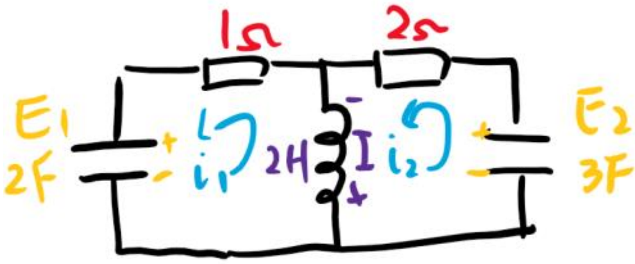
If $c = 0$, state vectors spiral/rotate in an ellipse

b. L-C-R circuits

Capacitors: voltage source with variable voltage $\frac{dV}{dt} = -\frac{i(t)}{C}$

Inductors: current source with time dependent current $\frac{di}{dt} = -\frac{V(t)}{L}$

Example:



Unknown: loop currents (i_1,i_2) & voltage drops across inductors (e)

Known: voltages at capacitors (E_1,E_2) & current at inductors (I)

Loop 1: $e + 1i_1 + E_1 = 0$

Loop 2: $-e + 2i_2 - E_2 = 0$

Current at inductor : $I = i_2 - i_1$

$$\begin{pmatrix} -E_1 \\ E_2 \\ I \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 2 & -1 \\ -1 & 1 & 0 \end{pmatrix}\begin{pmatrix} i_1 \\ i_2 \\ e \end{pmatrix}$$

Differential equation:

$$\begin{pmatrix} \frac{dE_1}{dt} \\ \frac{dE_2}{dt} \\ \frac{dI}{dt} \end{pmatrix} = \frac{1}{18}\begin{pmatrix} -3 & 3 & -6 \\ 2 & -2 & 2 \\ 6 & 3 & -6 \end{pmatrix}\begin{pmatrix} E_1 \\ E_2 \\ I \end{pmatrix}$$

$$\lambda_1 = -0.2, \overrightarrow{v_1} = \begin{pmatrix} 0.3 \\ -0.9 \\ -0.4 \end{pmatrix}$$

$$\lambda_2 = -0.2 + 0.3i, \overrightarrow{v_2} = \begin{pmatrix} -0.1 + 0.6i \\ 0.3 + 0.1i \\ 0.7 \end{pmatrix}$$

The general solution:

$$\begin{pmatrix} E_1 \\ E_2 \\ I \end{pmatrix}(t)$$

$$= c_1 e^{-0.2t}\begin{pmatrix} 0.3 \\ -0.9 \\ -0.4 \end{pmatrix} + c_2 e^{-0.2t}\left[\cos 0.3t \begin{pmatrix} 0.1 \\ 0.3 \\ 0.7 \end{pmatrix} - \sin 0.3t \begin{pmatrix} 0.6 \\ 0.1 \\ 0 \end{pmatrix}\right]$$

$$+ c_3 e^{-0.2t}\left[\cos 0.3t \begin{pmatrix} 0.6 \\ 0.1 \\ 0 \end{pmatrix} - \sin 0.3t \begin{pmatrix} 0.1 \\ 0.3 \\ 0.7 \end{pmatrix}\right]$$

If $c_2, c_3 = 0$, then no osillation, exponential decay

If $c_1 = 0$, then we always stay in the span $\left\{\begin{pmatrix}0.1\\0.3\\0.7\end{pmatrix}, \begin{pmatrix}0.6\\0.1\\0\end{pmatrix}\right\}$ inside state space of R^3

The graph in general will be a helix with demolishing radius