Basics

October 9, 2020 9:07 PM

Arithmetic in $\mathbb{C}$:

- $(a+bi)\pm(c+di)=(a\pm c)+(b\pm di)$
- $(a+bi)(c+di)=(ac-bd)+(ad+bc)i$
- $\dfrac{a+bi}{c+di}=\dfrac{(ac+bd)+(bc-ad)i}{c^2+d^2}$
- $z=a+bi$
  - $Re(z)=a, Im(z)=b$
  - $|z|=\sqrt{(a^2+b^2)}$
  - $\bar{z}=a-bi$
    - $Z=z+z, zz=zz, (z/z)=z/z$
    - $|\bar{z}|=|z|$
    - $Re(z)=\dfrac{1}{2}(\bar{z}+z)$
    - $Im(z)=\dfrac{1}{2i}(\bar{z}+z)$
    - $\bar{z}z=|z|^2$
    - $\frac{z_1}{z_2}=\frac{z_1\overline{z_2}}{|z_2|^2}, \frac{1}{z}=\frac{\bar{z}}{|z|^2}$

Polar forms

- $r\, cis\theta=r(\cos\theta+i\sin\theta)$
- $z_1=r_1\, cis\theta\,, z_2=r_2\, cis\theta$, then $z_1z_2=r_1r_2cis(\theta_1+\theta_2)$
- De Moivre's formula: $(cis\theta)^n=cis(n\theta)$
- The m-th roots of $z=rcis\theta$ is $\zeta=r^{\frac{1}{m}}cis\left(\frac{\theta+2k\pi}{m}\right)$
- Can write $r\, cis\theta=re^{i\theta}$
  - $z^{\frac{1}{m}}=r^{\frac{1}{m}}e^{\frac{\theta+2k\pi}{m}}$

Planar sets

- Examples
  - $|z-z_0|=\rho$, circle center at z_0, radius ρ
  - $|z-z_0|<\rho$, open disk
  - $|z|<1$, unit disk
- Open sets: A subset $S\subset\mathbb{C}$ is an open set if $\forall z_0\in S, \exists\rho>0: |z-z_0|<\rho$ lies in S
  - $1<|z|<2$ is open
  - $1\le|z|\le 2$ is not open
- Connected set: a set for which every pair of points in the set has some polygonal path (several straight lines) in the set that joins them
  - A domain is an open set
- A cut plane is defined as $\mathbb{C}\backslash(-\infty,0]$

Analytic functions

October 9, 2020 9:22 PM

Complex Functions

- Let $S \subset \mathbb{C}$, a function f with domain S is a mapping from S to $\mathbb{C}$, i.e. $\forall z \in S$, there is a unique $f(z) \in \mathbb{C}$

Limits

- $w_0 = \lim_{z \to z_0} f(z)$ means $\forall \epsilon > 0, \exists \delta > 0$ such that $|f(z) - w_0| < \epsilon$ whenever $0 < |z - z_0| < \delta$

Continuity: f is continuous at z_0 if

- f is defined in $\{z: |z - z_0| < \delta\}$ for some $\delta > 0$
- $\lim_{z \to z_0} f(z) = f(z_0)$
- $f = u(x, y) + iv(x, y)$ is continuous at $z_0 = x_0 + iy_0$ if and only if $u(x, y)$ and $v(x, y)$ are continuous at (x_0, y_0)

Differentiation

- Write $\Delta z = z - z_0, f'(z_0) = \frac{df}{dz}(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0}$
- $(f + g)' = f' + g', (fg)' = f'g + fg'$, quotient rule, chain rule holds
- $\frac{d}{dz} z^n = nz^{n-1}$ for $n = 0, + -1, + -2 \ldots$
- Differentiability and analyticity
  - f is differentiable at z_0 if $f'(z_0)$ exists
  - f is analytic at z_0 if $f'(z)$ exists for all z in some open disk centered at z_0
  - f is analytic in a domain D if $f'(z)$ exists for all z in D
- To show a function is not differentiable, find the limit from two directions, $\Delta z = \Delta x$ and $\Delta z = \Delta y$, show that they are not equal

Cauchy-Riemann equations

- $f = u(x, y) + iv(x, y)$ is analytic if and only if $u_x = v_y$ and $v_x = -u_y$
- $f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$
- If $f(z)$ is analytic in D and if $f'(z) = 0$ everywhere in D, then $f(z)$ is constant in D

Harmonic functions

- $\phi(x, y)$ a real valued function is harmonic if $\phi_{xx}, \phi_{yy}, \phi_{xy}, \phi_{yx}$ are continuous and $\phi_{xx} + \phi_{yy} = 0$
- Cauchy-Riemann equations gives an easy way to find such functions
- If $f = u + iv$ is analytic, then u, v are harmonic
  - If u is harmonic and $u + iv$ is analytic on D, then v is a harmonic conjugate of u
  - Level curves of u, v always intersect at right angle when $f'(z) = 0$
- If u and v are harmonic, then
  - $u + v$ is harmonic
  - uv is harmonic if and only if u and v are harmonic conjugate

Elementary functions

October 9, 2020 9:22 PM

Polynomials: $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_1 z + a_0$, $a_j \in \mathbb{C}$ and $a_n \neq 0$

- If all a_j are real and if z_0 is a zero, then $\overline{z_0}$ is also a zero
- Every non-constant polynomial with complex coefficients has at least one zero in $\mathbb{C}$
  - A polynomial of degree n has exactly n zeros counted according to multiplicity
- If z_1 is a zero pf $p(z)$, then $p(z) = (z - z_1)q(z)$, with $deg(q) = n - 1$, and we can continue to factor q
- Taylor form of polynomial: $p(z) = \Sigma_{k=0}^{n} \frac{p^{(k)}(z_0)}{k!} (z - z_0)^k$

Rational functions: $R(z) = \frac{p(z)}{q(z)} = a \frac{(z-z_1)(z-z_2)\ldots(z-z_n)}{(\zeta-\zeta_1)(\zeta-\zeta_2)\ldots(\zeta-\zeta_m)}$

- $z_1, z_2, \ldots, z_n$ are zeros of R and $\zeta_1, \zeta_2, \ldots, \zeta_m$ are poles of R

Exponential functions: $f(z) = e^z = e^{x+iy} = e^x(\cos y + i \sin y)$

- $e^{z_1} e^{z_2} = e^{z_1 + z_2}, \frac{e^{z_1}}{e^{z_2}} = e^{z_1 - z_2}$
- $\frac{d}{dz} e^z = e^z$, e^z is entire
- $\forall z \in \mathbb{C}, e^z \neq 0$, $Range(e^z) = \mathbb{C} \backslash \{0\}$
- $\forall k \in \mathbb{Z}, z \in \mathbb{C}, e^z = e^{z + 2k\pi i}$
  - If $e^{z_1} = e^{z_2}$, then $z_2 = z_1 + 2k\pi i$

Trig and hyperbolic trig function

- $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$
- Define $z \in \mathbb{C}$, $\cos z = \frac{e^{iz} + e^{-iz}}{2}, \sin z = \frac{e^{iz} - e^{-iz}}{2i}$
  - Trig identity holds and $\frac{d}{dz} \cos z = -\sin z, \frac{d}{dz} \sin z = \cos z, T = 2\pi$
  - The range can become all complex numbers
- Define $\sinh z = \frac{e^z - e^{-z}}{2}, \cosh z = \frac{e^z + e^{-z}}{2}$
  - $\cosh iz = \cos z, \sinh iz = i \sin z$
  - $\frac{d}{dz} \cosh z = -\sinh z, \frac{d}{dz} \sinh z = \cosh z$

Logarithm functions

- $\text{Log}\, z = \log|z| + i\, Arg(z)$ is the principal branch of the log, where $Arg(z) \in [-\frac{\pi}{2}, \frac{\pi}{2}]$ is the principal argument of z
  - We can have $\log z = \log|z| + i\left(Arg(z) + 2k\pi\right)$to have all the branches
  - On the cut plane, each branch of Log z is analytic
- $\log z_1 + \log z_2$ holds if we choose branches correctly
- $f(z) = \log g(z)$, then $f'(z) = \frac{g'(z)}{g(z)}$
- $Arg\ z$ is harmonic in the domain
- $Log|z|$ is harmonic in the entire plane except the origin
- $f(z) = Log(g(z))$ is analytic at z provided that $g(z)$ satisfies
  - $|g(z)| > 0$
  - $-\pi < Arg\ w < \pi$
- We take $\theta = -\pi$ to be the cut, we have $L_{-\pi} := Log|z| + i\, Arg\ z$ to be the principle branch
  - If we take τ to be the cut, we have the domain $(\tau, \tau + 2\pi]$
  - $L_\tau = Log|z| + i(Arg(z) + \pi + \tau)$
  - $L_0 := Log|z| + i\,(Arg(z) + \pi)$ flips the domain
  - E.g. $Log(-z) + i\pi = L_0(z)$ is analytic

General powers

- $z^{\alpha} = e^{\alpha \log z} = e^{\alpha\,(\mathrm{Log}|z| + i\,(Arg(z) + 2k\pi))}$
- $i^{\frac{1}{n}} = e^{\left(\frac{\pi}{2n} + \frac{2k\pi}{n}\right)i}$, but only n distinct values
- Properties
  - When α is a positive integer $z^n = e^{n \log z} = z \cdot z \cdots z$
  - z^{α} has infinitely many values if and only if α is not a rational real number
  - In previous two cases, every branch of z^{α} is analytic for $z \neq 0$, $\frac{d}{dz} z^{\alpha} = \frac{\alpha}{z} z^{\alpha}$
  - $z^{\alpha_1} z^{\alpha_2} = z^{\alpha_1 + \alpha_2}$ for suitable choices of branches

Inverse trig functions

- Since we know the trig functions if we want to find $\mathrm{w} = \arcsin z$, we just need to solve the function $\sin w = z$
- $\arcsin z = -i \log\left(iz + \left(1 - z^2\right)^{\frac{1}{2}}\right)$
- $\operatorname{arcsinh} z = \log\left(z + \left(1 + z^2\right)^{\frac{1}{2}}\right)$

Riemann surfaces

- Suppose we have a many-valued function 1 to n, we can produce a 1-1 function by taking the domain of z values to be n copies of the cut plane suitably glued together.
- This works even if the function is 1 to ∞, such as $Log\ z$

Complex integration

October 9, 2020 9:22 PM

$\gamma \subset \mathbb{C}$ is a smooth arc if it is the range of some $z = z(t) a \leq t \leq b$ such that

- $z'(t)$ exists and is continuous for $t \in [a, b]$
  - $z'(t) = x'(t) + i\, y'(t)$ is the tangent vector of $z(t)$
- $z'(t) \neq 0$ for any $t \in [a, b]$
- $z(t)$ is one-to-one on $[a, b]$, or for a smooth closed curve, $z(t)$ is one-to-one on $[a, b)$ but $z(a) = z(b)$ and $z'(a) = z'(b)$

Contour: directed piece-wise smooth curve (arc or closed curve) $\Gamma = \gamma_1 + \gamma_2 + .. \gamma_n$

Parameterization by arc length

Suppose $z(t) = x(t) + iy(t), t \in [a, b]$, then arclength $s(t) = \int_a^t \sqrt{x'(u)^2 + y'(u)^2}\, du$

And $s'(t) = \sqrt{x'(u)^2 + y'(u)^2} = |z'(t)|$

Total length of $\gamma = l(\gamma) = \int_a^b s'(t) dt = \int_a^b |z'(t)| dt$

Partition γ is called p_n, let $\Delta z_k = z_k - z_{k-1}$, then Riemann sum $s(P_n) = \Sigma_{k=1}^{n} f(c_k) \Delta z_k = \int_\gamma f(z) dz = \lim_{n \to \infty} s(P_n)$ where $mesh(P_n)$=arclength of the longest bit of γ between any z_{k-1} and z_k

If f is continuous on γ, then $\int_\gamma f(z) dz = \int_a^b f(z(t)) z'(t) dt$

If $f = u + iv$, then $\Delta z = \Delta x + i \Delta y$, $\int_\gamma f(z) dz = \int_\gamma (udx - vdy) + i \int_\gamma (vdx + udy)$

If $\Gamma = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, then $\int_\Gamma f(z) dz = \int_{\gamma_1} f(z) dz + \int_{\gamma_2} f(z) dz + \cdots + \int_{\gamma_n} f(z) dz$

Properties of integrals

- $\displaystyle\int_\gamma \alpha f(z) + \beta g(z) dz = \alpha \int_\gamma f(z) dz + \beta \int_\gamma g(z) dz$
- If $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$, then $\int_\Gamma f(z) dz = \int_{\Gamma_1} f(z) dz + \int_{\Gamma_2} f(z) dz + \cdots + \int_{\Gamma_n} f(z) dz$
- $\left| \int_\Gamma f(z) dz \right| \leq M l(\Gamma)$ where $M = max|f(z)|$ on Γ and $l(\Gamma)$ is the arclength of Γ

Suppose f is continuous in a domain D and F is analytic in D and $F'(z) = f(z)\ \forall z \in D$, then for a contour Γ in D from α to β, $\int_\Gamma f(z) dz = F(\beta) - F(\alpha)$

Let f be continuous in a domain D. Then the following a equivalent

- $\exists F$ such that $F'(z) = f(z) \forall z \in D$
- $\int_\Gamma f(z) dz = 0$ for all closed contour Γ in D
- $\int_{\Gamma_1} f(z) dz = \int_{\Gamma_2} f(z) dz$, if Γ_1 and Γ_2 are contours with the same initial and terminal points

Cauchy Integral theorem

- A simply connected domain D is one domain such that every simple (no self-intersecting) closed contour in D has every point inside it and in D
  - $D = \{z: |z| < 1\}$ is simply connected
  - $D = \{z: 0 < |z| < 1\}$ is not simply connected
  - Cut plane is simply connected
- If f is analytic in a simply connected domain D and Γ is a simple closed contour in D, then $\int_\Gamma f(z) dz = 0$
  - f has an antiderivative F in D and integrals of f from α to β are path independent
  - Green's theorem: $\int_\Gamma (Pdx + Qdy) = \int \int_R (\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}) dxdy$

- Extensions
  - Assume f is analytic inside and on a simple closed contour Γ, then $\int_\Gamma f(z)dz = 0$
  - Γ_1 can be continuously deformed in D to Γ_2 but not to Γ_3. Suppose f is analytic in D, then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$
  - Suppose f is analytic in a domain M which is not simply connected (such a domain is called multiply connected). Let $\gamma = \gamma_1 + \gamma_2 + \gamma_3$ (γ_i are all boundaries of M). Suppose f is analytic on M and γ, then $\int_\gamma f(z)dz = 0$
    - Convention: orient boundary contours such that M lies to our left as we traverse the boundary

If a function is analytic in D, then it has an antiderivative in D

Cauchy Integral Formula

- Suppose f is analytic inside and on a positively-oriented closed contour Γ. Let z_0 be a point inside Γ. Then $f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)}dz$
  - When $f = 1$, $\int_\Gamma \frac{dz}{z-z_0} = 2\pi i$ doesn't matter if Γ is a general contour
  - If f is analytic inside and on Γ, then the values of the integral on Γ determines f everywhere in Γ
  - If z_0 lies outside Γ, then $\int_\Gamma \frac{f(z)}{z-z_0}dz = 0$, given that $f(z)$ is analytic inside and on Γ
- Consequences of Cauchy Integral Formula
  - An analytic function has derivatives of all orders
    - $f^{(n)}(z) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta-z)^{n+1}}d\zeta$
    - If $f = u + iv$ is analytic, then all partial derivatives of u, v must exist
  - If f is continuous in a domain D and $\int_\Gamma f(z)dz = 0$ for every closed contour Γ in D, then f is analytic
  - If we want to know $G(c) = \int \frac{g(z)}{z-c}dz$, then
    - $G'(c) = 2\pi i g'(c)$and $G''(c) = 2\pi i g''(c)$
- Cauchy integral formula holds on a multiply-connected domain M provided we integrate over the complete boundary of M

Cauchy estimate: let f be analytic inside and on $C_R = \{z: |z - z_0| = R\}$. If $|f(z)| \le M, \forall z \in C_R$, then $|f^{(n)}(z_0)| \le \frac{n!M}{R^n}$ $(n = 0,1,2, \ldots)$

- Liouville's theorem: If f is entire and bounded, then f is constant

Series representations for analytic functions

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A sequence is a list of complex numbers $c_0, c_1, c_2, \dots$
We say $\{c_n\}$ converges to c and write $\lim_{n\to\infty} c_n = c$ if $\forall\epsilon > 0, \exists N$ such that $n > N, |c - c_n| < \epsilon$

An infinite series or series is an infinite sum $\Sigma_{j=0}^{\infty} c_j$. The nth partial sum $S_n = \Sigma_{j=0}^{n} c_j$

- The series converges and has sum $s = \lim\limits_{n\to\infty} c_n$ if the limit exists and is finite
- Otherwise it diverges

Basic facts and examples

- Geometric series: $\Sigma_{j=0}^{\infty} c_j = 1 + c + c^2 + \cdots + c^n + \cdots$
    - If $|c| < 1$, $\lim\limits_{n\to\infty} S_n = \frac{1}{1-c}$, converges
    - If $|c| > 1$, $S_n = \frac{c^{n+1}-1}{c-1}$, diverges
    - If $|c| = 1$ and $c = 1$, $S_n = n$, diverges
    - If $|c| = 1$ and $c \neq 1$, c^{n+1} oscillates and does not approach a limit, so it diverges
        - E.g. $(c = i)$
- P-series: $\Sigma_{j=0}^{\infty} \frac{1}{j^p}$
    - If $p = 1$, diverges
    - If $p \in (0,1)$, diverges
    - If $p \in (1, \infty)$, converges
- If $\Sigma_{j=0}^{\infty} c_j$ converges, then $\lim\limits_{n\to\infty} c_n = 0$, if the limit is not zero then the series diverges
    - Reverse is false
- Comparison test: if $\left|c_j\right| \leq M_j$, and $\Sigma_{j=0}^{\infty} M_j$ converges, then $\Sigma_{j=0}^{\infty} \left|c_j\right|$ converges and $\Sigma_{j=0}^{\infty} c_j$ converges
    - When $\Sigma_{j=0}^{\infty} \left|c_j\right|$ converges, we say $\Sigma_{j=0}^{\infty} c_j$ is absolutely convergent
- Ratio test: suppose $l = \lim_{j\to\infty} \left|\frac{c_{j+1}}{c_j}\right|$ exists
    - If $L < 1$, then Σc_j converges absolutely
    - If $L > 1$, then Σc_j diverges
    - If $L = 1$, cannot conclude

Sequences and series of functions

- The sequence $F_0(z), F_1(z), \dots$ converges uniformly to $F(z)$ on the set $T \subset \mathbb{C}$, if $\forall\epsilon > 0, \exists N$ such that $n > N \Rightarrow |F(z) - F_n(z)| < \epsilon$ for all $z \in T$
- The series $\Sigma_{j=0}^{\infty} f_j(z)$ converges uniformly to $F(z)$ on T, if the sequence $F_n(z) = \Sigma_{j=0}^{n} f_j(z)$ converges uniformly to $F(z)$ on T
    - $\Sigma_{j=0}^{\infty} z^j$ converges uniformly to $\frac{1}{1-z}$ on $|z| \leq r < 1$, but not on $|z| < 1$

Power series: $\Sigma_{n=0}^{\infty} a_n \left(z - z_0\right)^n$

- If $\Sigma_{n=0}^{\infty} a_n z^n$ converges for some value $z = z_1$, then it converges absolutely for all z with $|z| < |z_1|$.
    - If it diverges for some value $z = z_1$, then it diverges for $|z| > |z_1|$.
- Let R=radius of the largest circle within which $\Sigma_{n=0}^{\infty} a_n z^n$ converges, R is called the radius of convergence of $\Sigma_{n=0}^{\infty} a_n z^n$.
    - If $\Sigma_{n=0}^{\infty} a_n z^n$ converges for all $z \in \mathbb{C}$, then $R = \infty$
    - If $\Sigma_{n=0}^{\infty} a_n z^n$ converges only for $z = 0$, then $R = 0$
- If $\Sigma_{n=0}^{\infty} a_n z^n$ converges in $|z| < R$, then converges in every closed disk $|z| \leq R' < R$
- If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$ exists, then $R = \frac{1}{L}$

- If $\lim_{n\to\infty}\sqrt[n]{|a_n|} = l$ exists, then $R = \frac{1}{l}$
  - This may not always exist, we introduce limsup to fix this problem
    - $s = \text{limsup } s_n$ is the smallest number such that $\forall\epsilon > 0$, $\exists N$ such that $\forall n > N$, $s_n < s + \epsilon$
      - If $s_n \to \infty$, then $\text{limsup}_{n\to\infty} s_n = \infty$, if $s_n \to -\infty$, then $\text{limsup}_{n\to\infty} s_n = -\infty$
    - If $\text{limsup}_{n\to\infty}\sqrt[n]{|a_n|} = l$ exists, then $R = \frac{1}{l}$

Uniform convergence

- If f_n are continuous on $T \subset \mathbb{C}$ and f_n converges uniformly to f on T, then f is continuous on T
- If f_n are continuous on $T \subset \mathbb{C}$, f_n converges to f uniformly and Γ is a contour in T, then $\lim_{n\to\infty}\int_\Gamma f_n(z)dz = \int_\Gamma f(z)dz$
- If f_n are analytic in a domain D and if f_n converges uniformly to f in D then f is analytic in D
- Let $\Sigma_{n=0}^{\infty} a_n z^n$ have radius of convergence R and define $f(z) = \Sigma_{n=0}^{\infty} a_n z^n$ for $|z| < R$, then f is analytic in $|z| < R$

The derivative of $f(z) = \Sigma a_n z^n$ can be computed term by term, the radius of convergence are the same

- Actually $a_m = \frac{f^{(m)}(z_0)}{m!}$, and $f(z) = \Sigma\frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$

Convergent power series are analytic functions and analytic functions can be represented by a power series in some disk

Taylor series

- Suppose f is analytic in a domain D, let $z_0 \in D$, C be a circle of radius r with r<distance from z_0 to boundary of D, then $f(z) = \frac{1}{2\pi i}\int_C \frac{f(\zeta)}{\zeta - z}d\zeta$, for z inside C and by writing $\frac{1}{\zeta - z} = \frac{1}{\zeta - z_0}\Sigma\left(\frac{z - z_0}{\zeta - z_0}\right)^n$, we can deduce that $f(z) = \Sigma\frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$
- $e^z = \Sigma\frac{1}{n!}z^n, R = \infty$
- $\sin z = \Sigma\frac{(-1)^n z^{2n+1}}{(2n+1)!}, R = \infty$
- $\cos z = \Sigma\frac{(-1)^n z^{2n}}{(2n)!}, R = \infty$
- $Log z = \Sigma\frac{(-1)^n}{n}(z - 1)^n, R = 1$
- $Log(1 - z) = \Sigma - \frac{z^n}{n}$
- $\frac{1}{z} = \frac{d}{dz}\, Log\, z = \Sigma\,(-1)^n(z - 1)^n, R = 1$

Laurent series (usually used for functions with singularities)

- $\Sigma_{n=-\infty}^{\infty} a_n(z - z_0)^n$
  - z_0 is called center
  - The function is undefined at z_0 if negative powers
- Radius of convergence is an annulus ($R_1 < |z - z_0| < R_2$)
  - $\Sigma_{n=0}^{\infty} a_n(z - z_0)^n$ converges for $|z - z_0| < R_2$
  - $\Sigma_{n=-\infty}^{-1} a_n(z - z_0)^n$ converges for $|z - z_0| > R_1$
  - Series may or may not converge on the boundaries
  - Possibly $R_1 = 0$ (with only finitely many negative terms)
- $e^{\frac{1}{z}} = \Sigma_{n=0}^{\infty}\frac{1}{n!}\left(\frac{1}{z}\right)^n$ converges for $0 < |z| < \infty$ ($z_0 = 0$)
- $f(z) = \frac{z^2}{z-1} = \frac{z}{1-\frac{1}{z}} = z + 1 + \frac{1}{z} + \cdots = \Sigma_{n=-\infty}^{1} z^n$ for $1 < |z| < \infty$ ($z_0 = 0$)
- $f(z) = \frac{z^2}{z-1} = \frac{((z-1)+1)^2}{z-1} = \frac{1}{z-1} + 2 + z - 1$ for $0 < |z - 1| < \infty$ ($z_0 = 1$)

- General properties
  - A Laurent series defines a (single valued) analytic function in its annulus of convergence
  - Given f analytic (and single valued) in the annulus $R_1 < |z - z_0| < R_2$, we can always expand f in a Laurent series
  - The differentiation formula is no longer valid, but the integral is still valid with $a_n = \frac{1}{2\pi i}\int \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta, n \in \mathbb{Z}$
  - Size of annulus determined by location of singularities (annulus goes up to the nearest singularities)
    - E.g. $f(z) = \frac{1}{z^2 - z - 2} = \frac{1}{(z-2)(z+1)}$ different Laurent series represents f in three regions, $|z| < 1$, $1 < |z| < 2$ with $f(z) = \left(-\frac{1}{6}\right)\Sigma\left(\frac{z}{2}\right)^n - \frac{1}{3z}\Sigma\left(-\frac{1}{z}\right)^n$, $2 < |z| < \infty$
- Purpose of Laurent series
  - Expansion at a singularity
  - Expansion between singularities
  - Classification of singularities

Isolated singularities:

- An isolated singularity of f is a point z_0 such that f is not analytic at z_0, but f is analytic in some punctured disk $0 < |z - z_0| < \delta$
  - $\frac{1}{\sin z}$ has isolated singularities at $n\pi$
  - $Log\ z$ has non-isolated singularities at each z_0 in $(-\infty, 0]$
- Singularity categories
  - Removable singularity: Laurent series has no negative powers ($\Sigma_{n=0}^{\infty} a_n (z - z_0)^n$)
    - $f(z) \to a_0\ as\ z \to z_0$
    - If f is analytic on $0 < |z - z_0| < \delta$ and bounded, then z_0 is a removable singularity
    - Can remove the singularity and get an analytic function by defining a piecewise function
    - It means that $f(z)$ has a limit point at z_0
  - Pole of order m: finite number of negative powers ($\Sigma_{n=-m}^{\infty} a_n (z - z_0)^n$)
    - $|f(z)| \to \infty$ if $z_n \to z_0$
    - E.g. $f(z) = \frac{\sin z}{z^4}$ has a pole $z_0 = 0$ of order 3
    - Poles of order 1 are called simple poles ($f(z) = \frac{\sin z}{z^2}$ $z_0 = 0$ has order 1)
    - The same terminology used for zeros: and $a_m \neq 0$ has a zero of order m at z_0, if $\Sigma_{n=m}^{\infty} a_n (z - z_0)^n$, where $m > 0$
      - $m = 1, z_0$ is a simple zero
  - Essential singularity: infinite number of negative powers ($\Sigma_{n=-\infty}^{\infty} a_n (z - z_0)^n$)
    - Casorati – Weierstrass theorem: $\forall b \in \mathbb{C}$, exists a sequence $z_n \to z_0$ such that $f(z_n) \to b$, also, $\exists z_n$ such that $|f(z_n)| \to z_0$
    - Picard's theorem: if f has an essential singularity at z_0, then in every disk $0 < |z - z_0| < \delta$, $f(z)$ assumes every complex value with possibly one exception
      - E.g.$f(z) = e^{\frac{1}{z}}$ has an essential singularity at z_0, but $e^{\frac{1}{z}}$ can never achieve 0

Residue theory

November 20, 2020 8:28 AM

Residue: Let $f(z) = \Sigma_{n=-\infty}^{\infty} a_n (z - z_0)^n$ be the Laurent series of f in the neighborhood of an isolated singularity z_0 of f. Then the residue of f at z_0 is a_{-1}, written as $a_{-1} = Res(f; z_0)$

- $Res(f; z_0) = a_{-1} = \frac{1}{2\pi i} \int_C f(z) dz$
- If f has a simple pole at z_0, then $Res(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z)$
- If $f(z) = \frac{P(z)}{Q(z)}$ with $P(z_0) \neq 0$, and Q has a simple zero at z_0 ($Q(z_0) = 0$ and $Q'(z_0) \neq 0$), then $Res(f; z_0) = \frac{P(z_0)}{Q'(z_0)}$
- Note: residue is always a finite number

Residue at a higher order pole

- Suppose f has a pole of order $m \geq 1$ at z_0,
  $f(z) = \frac{a_{-m}}{(z - z_0)^m} + \cdots + \frac{a_{-1}}{z - z_0} + a_0 + a_1(z - z_0) + \cdots$,
  then $(z - z_0)^m = a_{-m} + \cdots + a_{-1}(z - z_0)^{m-1} + \cdots$.,
  And $Res(f; z_0) = a_{-1} = \lim_{z \to z_0} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right)$

Residue theorem:

- Let f be analytic inside and on a simple closed contour C except for a finite number of isolated singularities $z_1, z_2, \ldots z_k$ inside C, then $\int_C f(z) dz = 2\pi i \Sigma_{j=1}^{k} Res(f; z_j)$

Trigonometric Integrals: $\int_0^{2\pi} f(\cos\theta, \sin\theta) d\theta$

- Change of variables $z = e^{i\theta}$
- E.g. $I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = 4\pi Res\left(\frac{1}{bz^2 + 2az + b}; z_1 \right) = \frac{4\pi}{2bz_1 + 2a} = \frac{2\pi}{\sqrt{a^2 - b^2}}$ (with $|b| < a$)
- Usually $\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_{-\pi}^{\pi} \frac{d\theta}{a + b\cos\theta} = 2\int_0^{\pi} \frac{d\theta}{a + b\cos\theta}$

Improper integrals on $(-\infty, \infty)$

- $\int_{-\infty}^{\infty} f(x) dx$ with $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials, with
  - $deg\ Q \geq deg P + 2$, $f(x) = O\left(\frac{1}{x^2}\right)$, so integral converges
  - Q has no zeros on real axis
- $\int_{-\infty}^{\infty} f(x) dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) dx$
- Let S_R be a semi circle in the upper half plane, $C_R = [-R, R] + S_R$,
  then $\int_{C_R} f(z) dz = 2\pi i \Sigma_{z_j \in C_R} Res(f; z_j) = \int_{-R}^{R} f(x) dx + \int_{S_R} f(z) dz$
  And as $R \to \infty$, $\left| \int_{S_R} f(z) dz \right| \leq O\left(\frac{1}{R^2}\right) \pi R = O\left(\frac{1}{R}\right) \to 0$
  This gives that $\int_{-\infty}^{\infty} f(x) dx = 2\pi i \Sigma_{z_j \in C_R} Res(f; z_j)$
  Also, if choose lower half plane $\int_{-\infty}^{\infty} f(x) dx = -2\pi i \Sigma_{LHP} Res(f; z_j)$
- E.g. $\int_{-\infty}^{\infty} \frac{x^2}{x^4 + a^4} dx = \frac{\sqrt{2}\pi}{4a}$
- The formula also gives that $\Sigma Res(f; z_j) = \Sigma_{LHP} Res(f; z_j) + \Sigma_{UHP} Res(f; z_j) = 0$

Principle value of $\int_{-\infty}^{\infty} f(x) dx$ is $p.v. \int_{-\infty}^{\infty} f(x) dx = \lim_{R \to \infty} \int_{-R}^{R} f(x) dx$ if the limit exists

- Note: $\int_{-\infty}^{\infty} f(x) dx$ is defined to be $\lim_{R \to \infty} \int_0^R f(x) dx + \lim_{R' \to \infty} \int_{-R'}^{0} f(x) dx$ when both limits exist

- Principle value can exist even if neither of the limit exists
- E.g. $p.v. \int_{-\infty}^{\infty} x dx = 0$

Improper integrals of the form $\int_{-\infty}^{\infty} f(x)g(x)dx$ where $f(x)$ is a rational and $g(x) = \sin ax$ or $g(x) = \cos ax$

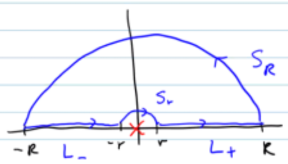
- We can consider $\int_{-\infty}^{\infty} f(x)e^{iax}dx$, and this gives the cosine and sine integrals by real and imaginary parts
- Assume: $deg\ Q \geq degP + 1$ and Q has no zeros on real axis
- Taking the upper half plane, we have $\int_{-R}^{R} f(x)e^{iax}dx = \int_{C_R} f(z)e^{iaz}dz - \int_{S_R} f(z)e^{iaz}dz$,
    - Jordan's lemma: if $deg\ Q \geq degP + 1, a > 0$, then $\int_{S_R} f(z)e^{-iaz}dz \to 0$ as $R \to \infty$
- So $\int_{-\infty}^{\infty} f(x)e^{iax}dx = 2\pi i \Sigma_{z_j \in C_R} Res\left(f(z)e^{iaz}; z_j\right)$
- If f is even, then $\int_0^{\infty} f(x) \cos ax\, dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)e^{iax}dx$
- If f is odd, then $\int_0^{\infty} f(x) \sin ax\, dx = \frac{1}{2}\int_{-\infty}^{\infty} f(x)e^{iax}dx$

Indented contours: $\int_{-\infty}^{\infty} f(x)dx$ when f has a singularity on the real axis

- Example: $I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$ (We'll see this converges.)

  Let $C = L_- + S_r + L_+ + S_R$

  & consider $\oint_C \frac{e^{iz}}{z} dz$.

  By Cauchy Int Thm, $\oint_C \frac{e^{iz}}{z} dz = 0$

  so $\text{Im} \oint_C \frac{e^{iz}}{z} dz = 0$

  As $R \to \infty$ & $r \to 0$, $\text{Im} \int_{L_- + L_+} \frac{e^{iz}}{z} dz \to p.v. \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$

  By Jordan's Lemma, $\int_{S_R} \frac{e^{iz}}{z} dz \to 0$ as $R \to \infty$.

  $\therefore\ p.v. \int_{-\infty}^{\infty} \frac{\sin x}{x} dx + \lim_{r \to 0} \text{Im} \int_{S_r} \frac{e^{iz}}{z} dz + 0 = 0$ (from S_R)

  i.e., $p.v. \int_{-\infty}^{\infty} \frac{\sin x}{x} dx = -\lim_{r \to 0} \text{Im} \int_{S_r} \frac{e^{iz}}{z} dz$

  Note: $e^{iz} = \cos z + i \sin z$, that's why we take the imaginary part

  Then $\int_{S_r} \frac{e^{iz}}{z} dz \to -i\pi$ as $r \to 0$, we have $\int_{-\infty}^{\infty} \frac{\sin x}{x} dx = \pi$
- Actually, $\lim_{(r\to 0)} \int_{S_r} \frac{e^{iz}}{z} dz = 2\pi i Res\left(\frac{e^{iz}}{z}; 0\right) \cdot l$, where l is the fraction that we go through on the little circle
    - In our example, we go through half circle, so $l = \frac{1}{2}$
- If f has a simple pole at $z = c$ and T_r is the circular arc of $T_r: z = c + re^{i\theta}, (\theta_1 \leq \theta \leq \theta_2)$
    - Then $\lim_{(r\to 0)} \int_{T_r} f(z)dz = i(\theta_2 - \theta_1)Res(f; c)$
    - If clockwise/lower half circle, then we have $\lim_{(r\to 0)} \int_{T_r} f(z)dz = -i\pi Res(f; c)$
    - If there is $\sin ax$ or $\cos ax$, take e^{iax}, then consider separate cases, if pole z_0 is on the real axis, then use $\pi i Res(z_0)$, otherwise, use $2\pi i Res(z_0)$

Mandelbrot set

November 30, 2020 8:34 AM

Given an entire function $f(z)$ and a point $z_0 \in \mathbb{C}$

- The orbit of z_0 is $\{z_0, z_1, z_2, \ldots\}$ where $z_i = f(z_{i-1})$, denote $f^2(z) = f(f(z)) = (f \circ f)(z)$
- z_0 is a fixed point of f if $f(z_0) = z_0$, its orbit is just $\{z_0\}$
    - Behavior near a fixed point z_0: $f(z) = f(z_0) + f'(z_0)(z - z_0) + \cdots$
      So $f(z) - z_0 = f'(z_0)(z - z_0), |f(z) - z_0| = |f'(z_0)||z - z_0|$
      If $|f'(z_0)| < 1$, then $f(z)$ is closer to z_0 than z
      If $|f'(z_0)| > 1$, then f(z) is further to z_0 than z
    - Classification of fixed points:
        - Attracting: $|f'(z_0)| < 1$
        - Repelling: $|f'(z_0)| > 1$
        - Indifferent (neutral): $|f'(z_0)| = 1$
    - E.g. $f(z) = z^2$ has fixed points $z = 0$ and $z = 1$
        - $f'(0) = 0, z = 0$ is attracting
        - $f'(1) = 2, z = 1$ is repelling
        - Orbit of $z = re^{i\theta}$ is $\{z, z^2, z^4, z^8, \ldots\}$
            - $z_n \to 0$ if $r < 1$, and $z_n \to \infty$ if $r > 1$
        - Domain of attraction of 0 is $|z| < 1$, and ∞ is $|z| > 1$
- A periodic point z_0 (period k) is attracting if $|(f^k)'(z_0)| < 1$ and repelling if $|(f^k)'(z_0)| > 1$
- The Julia set $J(f)$ is the closure of the set of all repelling fixed points of f
    - E.g. for $f(z) = z^2$, has a dense set of periodic points on $|z| = 1$,
      $|(\mathrm{f}^{\mathrm{k}})'(z_0)| = \left|\frac{d}{dz} z^{2^k}\right| > 1, J(f) = \{z: |z| = 1\}$

Mandelbrot set: let $f_c(z) = z^2 + c$ for c a complex number

- $M = \{c \in \mathbb{C}:\ f_c^n(0) < \infty\ as\ n \to \infty\}$ i.e. the orbit $\{0, c, c^2 + c, \left(c^2 + c\right)^2 + c, \ldots\}$ does not go to infinity
- Theorem:
    - M is connected
    - $c \in M$ if and only if J_C is connected
    - For c near 0, the Hausdorff dimension of J_C is $1 + \frac{|c|^2}{4 \log 2} + o(|c|^2)$
    - For real $c \in \left[-\frac{3}{4}, \frac{1}{4}\right]$, J_C is simple closed curve
    - M is a subset of $\{z: |z| \leq 2\}$

Brief review

January 10, 2022 11:17 AM

A domain in $\mathbb{C}$ is an open path-connected set

For f defined on a neighborhood of z_0, $f'(z_0) = \lim_{z \to z_0} \frac{f(z)-f(z_0)}{z-z_0}$ if the limit exists

- f is analytic on D if $f'(z)$ exists for all $z \in D$.

Analyticity ⇒ Cauchy-Riemann equation: $f = u + iv$ has $u_x = v_y, u_y = -v_x$.

- If the partial derivatives u_x, u_y are continuous, then CR equation means analytic.
- If $f = u + iv$ is analytic on D, then u, v are harmonic ($u_{xx} + u_{yy} = v_{xx} + v_{yy} = 0$)
- All level curves of u, v intersect in right angles

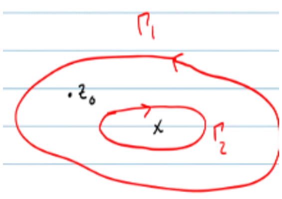
Note: $f(z) = Log(z) = Log(r) + i\theta$, with $z = re^{i\theta}$, $\theta \in (-\pi, \pi]$.

- The principal branch has a branch cut at $\theta = \pi$ or $\{(x, y) : x \leq 0, y = 0\}$.

Contour integral

- Contour: directed piecewise smooth curve Γ, $z = z(t)$ for $t \in [a, b]$.
- Integral: $\int_\Gamma f(z)dz = \int_a^b f(z(t))z'(t)dt$.
  - It obeys $\left|\int_\Gamma f(z)dz\right| \leq \max_{z\in\Gamma}|f(z)| \cdot length(\gamma)$.
- Theorem: suppose f is continuous on D and there is an analytic function F such that $F' = f$ on D, then $\int_\Gamma f(z)dz = F(\beta) - F(\alpha)$.
  - Independent of the contour
  - Zero for closed contour (F must be analytic)

Cauchy integral theorem

- Simply connected domain: a connected set with subset enclosed by every simple closed contour is contained in the domain
  - Any closed curve can be deformed to a point without leaving the domain
  - There is no hole
- If f is analytic on a simply connected domain D and Γ is a closed contour in D, then $\int_\Gamma f(z)dz = 0$.
- Corollary: if D is a domain and Γ_1 can be continuously deformed into Γ_2 and f is analytic on D, then $\int_{\Gamma_1} f(z)dz = \int_{\Gamma_2} f(z)dz$.
- Cauchy integral formula: if f is analytic inside and on a positively oriented (counter clockwise) closed contour and if z_0 is inside Γ, then $f(z_0) = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{z-z_0}dz$.
  - f has derivatives of all orders and $f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz$.
  - Also works on a multiply-connected domain $f(z_0) = \frac{1}{2\pi i}\int_{\Gamma_1+\Gamma_2} \frac{f(z)}{z-z_0}dz$.
    - 

Taylor series

- If f is analytic on D and $z_0 \in D$, then f has a Taylor series $f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n$
  - $a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{(z-z_0)^{n+1}}dz = \frac{f^{(n)}(z_0)}{n!}$.
- It converges at least in the largest disk in D centered at z_0

Laurent series

- If f is analytic on D and $z_0 \in D$, then f has a Laurent series $f(z) = \sum_{n=-\infty}^{\infty} a_n\left(z - z_0\right)^n$
  - $a_n = \frac{1}{2\pi i}\int_\Gamma \frac{f(z)}{\left(z-z_0\right)^{n+1}}dz.$
- Typically, the inner circle deforms to the isolated singularity z_0

Residue

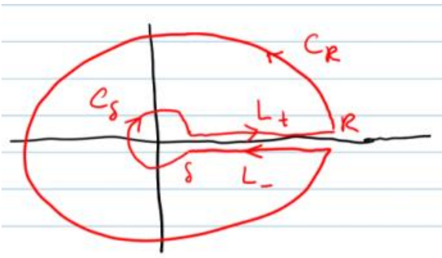
- If z_0 is an isolated singularity, then $Res\left(f; z_0\right) = a_{-1} = \frac{1}{2\pi i}\int_\Gamma f(z)dz.$
  - If $f(z) = \frac{P(z)}{Q(z)}$ with $P\left(z_0\right) \neq 0$, and Q has a simple zero at z_0 ($Q(z_0) = 0$ and $Q'\left(z_0\right) \neq 0$), then $Res(f; z_0) = \frac{P(z_0)}{Q\prime(z_0)}$
  - Suppose f has a pole of order $m \geq 1$ at z_0, $Res(f; z_0) = a_{-1} = \lim_{z \to z_0} \frac{1}{(m-1)!}\frac{d^{m-1}}{dz^{m-1}}\left(\left(z - z_0\right)^m f(z)\right)$

- Residue theorem: Let f be analytic inside and on a simple closed contour C except for a finite number of singularities $z_1, \ldots, z_k$ inside C, then $\int_C f(z)dz = 2\pi i \sum_{j=1}^{k} Res\left(f; z_j\right)$.

Residue theory cont.

January 7, 2022 9:44 PM

Keyhole contour

- 

Integration of rational functions $\int_0^\infty f(x)x^a dx$.

- a not an integer
- f a rational function with no poles on positive x-axis
- Choose $C = L_+ + L_- + C_R + C_\delta$.
- Require:
  - $f(z)z^a = O\left(\frac{1}{R^p}\right)$, for $p > 1$ as $R \to \infty$, so that $\left|2\pi R f(z)z^a\right| \to 0$.
  - $f(z)z^a = O\left(\frac{1}{\delta^q}\right)$, for $q < 1$ as $\delta \to 0$.
- Choose branch of z^a with cut along $[0, \infty)$.
- $\int_{C_R} f(z)z^a dz = \int_{C_\delta} f(z)z^a dz = 0$ with the $M \cdot l$ bound.
- $\int_{L_-} f(z)z^a dz = \int_R^\delta f(x)x^a e^{2\pi i a} dz = -e^{2\pi i a} \int_\delta^R f(x)x^a dx = -e^{2\pi i a} \int_{L_+} f(z)z^a dz.$
- Finally, $\left(1 - e^{2\pi i a}\right) \int_0^\infty f(x)x^a dx = 2\pi i \sum Res\left(f(z)z^a; z_j\right)$.
  - z_j are poles of f.

Argument principle

- Def: A function f on a domain D is meromorphic if at every $z \in D$ either f is anlytic or has a pole (no branch cuts or singularities)
  - A function f on D is holomorphic if it is differentiable at every $z \in D$ (no poles).
- Theorem (Argument principle): Let C be a simple closed positively-oriented contour. Let f be analytic and non-zero on C and meromorphic inside C. Then $\frac{1}{2\pi i} \int_C \frac{f'(z)}{f(z)} dz = N_0(f) - N_p(f)$.
  - $N_0(f)$ is the number of zeros of f in C with multiplicity
  - $N_p(f)$ is the number of poles of f in C with multiplicity
- Reason for the name
  - For $\frac{d}{dz} \log f(z) = \frac{d}{dz} \mathrm{Log}\left|f(z)\right| + \frac{d}{dz} i \arg f(z) = \frac{f'(z)}{f(z)}$.
  - $\arg z$ may not be globally well-defined for all $z \in C$.
  - Solution: break C into small pieces, small enough that $\arg f$ is well-defined on each piece.
    - On this piece, $\int_C \frac{f'(z)}{f(z)} dz = \mathrm{Log}\left|f(z)\right|\Big|_{z_1}^{z_2} + i\,(change\ in \arg z\ \ from\ z_1\ to\ z_2)$.
  - Sum over pieces, if C is closed, $\mathrm{Log}\left|f(z)\right|$ cancels.
  - Then $N_0(f) - N_p(f) = \frac{1}{2\pi} \Delta_C(\arg f)$.
    - $\Delta_C(\arg f)$ is the total change in $\arg f$ over C.

Rouche's theorem

- Let f and h be analytic inside and on a simple closed contour C, and suppose $|h(z)| < \left|f(z)\right|$ for each $z \in C$. Then f and $f + h$ have the same number of zeros inside C with multiplicity
- Corollary: zeros of non-constant functions are isolated

Open mapping theorem

- If f is analytic on D and not constant on D, then its range $f(D) = \left\{f(z) : z \in D\right\}$ is an open set

Analytic continuation

January 21, 2022 11:03 AM

Consider the power series, $Log(1+z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \frac{z^4}{4} + \cdots$, which converges for $|z| < 1$.

- Is any information loss in representing a function in the cut plane by a power series in the unit disk?
- In fact, a power series in a disk for a function contains all information about f, including its domain of analyticity, singularities, other branches etc.

E.g. $f(z) = \frac{1}{z}$, $z_0 = \epsilon > 0$.

- Compute power series $\frac{1}{z} = \sum_{n=0}^{\infty}(-1)^n \frac{(z-\epsilon)^n}{\epsilon^{n+1}}$ converges for $|z-\epsilon| < \epsilon$.
- This power series for $|z-\epsilon| < \epsilon$ determines $f(z), \forall z \neq 0$.
- We can choose z_1 close to 2ϵ , and determine $f^{(n)}(z_1)$ from the power series, and form a new Taylor series at z_1.
    - $f_1(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_1)}{n!}(z-z_1)^n$ has radius of convergence $R_1 = |z_1|$.
- By repeating this procedure, can produce $f(z)$, $\forall z \neq 0$.

Terminology

- Power series of f at z_0 is the element of f at z_0
- The sequence of centers $z_0, z_1, \ldots$ is called a chain of centers
- The process of going from one element to another is analytic continuation

Fraction combinatorics

- $\binom{c}{n} = \frac{c(c-1)\ldots(c-n+1)}{n!}$.
- e.g. $\binom{\frac{1}{2}}{n} = \frac{\frac{1}{2}\cdot\left(-\frac{1}{2}\right)\cdot\ldots\cdot\left(\frac{1}{2}-n+1\right)}{n!}$.

As we move around the analytic continuation, we may change the branch

Conclusions and theorems

- Suppose f is analytic in a domain D and $f = 0$ on some arc $l \subset D$ or even just a sequence of points $z_n \to z_0$. Then $f(z) = 0$ on D.
- Suppose f_1, f_2 are analytic on D and $f_1 = f_2$ on $l \subset D$, then $f_1 = f_2$ on D.
- Principle of permanence: suppose $f_1, \ldots, f_n$ are analytic on D and $P(x_1, \ldots, x_n)$ is a polynomial in n variables. Then if $P(f_1(z), \ldots, f_n(z)) = 0$ for z on some arc $l \subset D$, then $P(f_1, \ldots, f_n) = 0$.
    - $\cos^2 x + \sin^2 x = 1$ for all $x \in \mathbb{R}$, then $\cos^2 z + \sin^2 z = 1$ for all $z \in \mathbb{C}$.
- Monodromy theorem: suppose analytic continuation of an element produces elements at all points of a simply-connected domain D. Then these elements determine a single valued analytic function on D.
- Analytic continuation can sometimes be done by summation
- If an element at z_0 has a finite radius of convergence R, then there must be a singularity of the function on the circle $|z - z_0| = R$.
    - Singularity: there cannot be an element at this point
- There are functions analytic in a disk which cannot be analytically continued beyond the disk.
    - The disk boundary is a natural boundary.
    - $z + z^2 + z^4 + z^8 + \cdots = \sum_{n=1}^{\infty} z^{2^n}$ has natural boundary $|z| = 1$.
- Riemann surfaces: suppose we have a many-valued function 1 to n. We can produce a one-to-one function by taking n copies of the cut plane suitably glued together
    - Analytic continuation via a chain of centers going twice around the origin yields a single-valued function on the Riemann surface
    - This also works if the function is 1 to ∞ such as $\log z$.

Conformal mapping

January 10, 2022 11:03 AM

Applications

- Fluid dynamics (2D problem):
  - Find the streamlines (level curves)
  - Solution: $u(x,y) = c$ where $\nabla^2 u = u_{xx} + u_{yy} = 0$ outside the wing.
  - Method: find f mapping exterior of the wing to exterior of a disk with f analytic
    Then solve the problem on the disk and use f to pull back streamlines to D
- Heat conduction
  - Find the steady state temperature in the strip with given boundary conditions
  - Method: find analytic f mapping strip to upper half plane, solve the equation and use f to map back to the strip
- Electric potential
  - Same as heat conduction

Mapping

- $w = f(z) = u(x,y) + iv(x,y)$.
- e.g.
  - $D = \left\{z = re^{i\theta}: 0 < r < \infty, 0 < \theta < \frac{\pi}{2}\right\}, w = z^2$.
  - Then $w = r^2 e^{2i\theta}, |w| = r^2 \in (0,\infty), \arg w = 2\theta \in (0,\pi)$.
- Properties
  - If ϕ is harmonic on D', then $\phi \circ f$ should be harmonic on D
    - $\phi \circ f$ is called pull back
  - $f: D \to D'$ should be a bijection (one to one and onto)
  - Similarity: small figures in z plane maps to roughly similar figures in the w-plane
  - Boundary behavior: if f maps ∂D bijectively onto $\partial D'$, then f maps D bijectively to D'.

Local vs. global invertible

- Def: f is locally invertible at z_0 if there is a neighborhood of z_0 on which f has an inverse.
  - e.g. $w = z^2$ is not globally invertible but it is locally invertible except at $z = 0$.
- If f is analytic at z_0 and $f'(z_0) \neq 0$, then there is an open disk D centered at z_0 such that f is one to one and onto $f(D)$.

Conformal mapping

- A mapping is conformal at z_0 if it preserves angles at z_0 (both magnitude and sign)
- Let f be anlytic at z_0. Then f is conformal at z_0 if and only if $f'(z_0) \neq 0$.
- Remarks
  - $r = |f'(z_0)|$ is a magnification factor
  - Let $w = f(z) = z^2$. Then $f'(z) = 2z$, so f is conformal except at $z = 0$. At $z = 0$, angles are doubled.
    - Since f' has simple zero at $z = 0$.
  - Small figures are rotated and uniformly magnified where f is conformal
  - If $f: D \to D' = f(D)$ is bijective and conformal, then $f^{-1}: D' \to D$ is bijective and conformal, since by inverse function theorem $\frac{d}{dw} f^{-1}(w) = \frac{1}{f'(z)} \neq 0$.
    - Also, if $f: D \to D'$ and $g: D \to D'$ are both bijective and conformal, then so is $g \circ f$ since $(g \circ f)'(z) = g'(f(z))f'(z) \neq 0$.
    - The set of bijective conformal maps $f: D \to D$ forms a group (with identity $z \to z$)

Riemann mapping theorem:

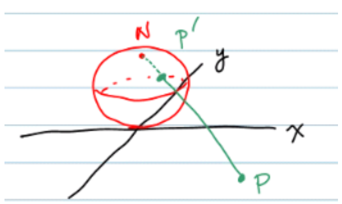
- Main problem: given 2 domains (possibly unbounded) with boundaries C, C', find a conformal bijective function $f: D \to D'$.

- Notation: $u = open\ unit\ disk = \{z \in \mathbb{C} : |z| < 1\}$.
- Theorem: let D be a simply connected domain which is not the entire plane. Then there exists a conformal bijection $f: D \to u$.
    - In fact, for any fixed $z_0 \in D$, there is a unique such f with $f(z_0) = 0$ and $f'(z_0) > 0$.
- Drawback: don't know what f is, only that it exists
- The uniqueness entails 3 real degrees of freedom
    - Choose $z_0 = x_0 + iy_0$.
    - Rotation of the disk to ensure $f'(z_0) > 0$ (in general $f'(z_0) = re^{i\phi}$)
- Corollary: if D_1, D_2 are any 2 simply-connected domains (both not $\mathbb{C}$), then there is a conformal bijection $f: D_1 \to D_2$.

Mobius transformations (fractional linear transformations)

- Maps of the form $w = \frac{az+b}{cz+d}$ where $a, b, c, d \neq 0$ and $ad - bc \neq 0$.
    - If $ad - bc = 0$, then $\frac{b}{a} = \frac{d}{c}$ and $w = \frac{a}{c}\frac{z+\frac{a}{b}}{z+\frac{d}{c}} = \frac{a}{c}$ constant.
- Derivative: $\frac{dw}{dz} = \frac{ad-bc}{(cz+d)^2}$.
    - Is never zero
    - So the map is conformal everywhere except at its unique pole $z = -\frac{d}{c}$.
- Basic Mobius transformations
    - Rotation by ϕ: $w = f_1(z) = e^{i\phi}z$ $(a = e^{i\phi}, b = c = 0, d = 1)$
    - Magnification by r: $w = f_2(z) = rz$ $(a = r > 0, b = c = 0, d = 1)$
    - Translation by b: $w = f_3(z) = z + b$ $(a = 1, c = 0, d = 1)$
    - Affine transform: $w = f(z) = az + b = f_3 \circ f_2 \circ f_1(z)$, $a = re^{i\phi} \neq 0$.
        - Such a linear f maps lines to lines and circles to circles.
    - Inversion map: $w = f(z) = \frac{1}{z}$ $(a = 0, b = 1, c = 1, d = 0)$.
        - $re^{i\theta} \to \frac{1}{r}e^{-i\theta}$.
        - Inversion in unit circle and reflection in real axis.
- Let S be the set of circles and straight lines in the plane
    - A line is a circle with radius ∞.
    - $w = \frac{1}{z}$ maps S to S.
        - A circle passes origin gets inverted to a line
        - An element of S has equation $az\bar{z} - \alpha z - \bar{\alpha}\bar{z} + d = 0$ with $\alpha\bar{\alpha} > ad$ $(a, d \in \mathbb{R}$, $\alpha \in \mathbb{C})$
            - $a = 0$: a line.
            - $a \neq 0$: a circle $\left(x - \frac{b}{a}\right)^2 + \left(y - \frac{c}{a}\right)^2 = \frac{b^2}{a^2} + \frac{c^2}{a^2} - \frac{d}{a}$.
    - Any Mobius transformation $w = \frac{az+b}{cz+d}$ maps S to S.
- A line in $\mathbb{C}$ is a "circle" that passes throuh ∞. In this case all elements of S (formally lines and circles) can be thought of as "circles"
    - Mobius transformations map "circles" to "circles"
- $w = e^{i\theta}\frac{z-z_0}{z-\bar{z}_0}$ maps the upper half plane to the unit disk with 3 degrees of freedom.
    - Choose $z_0 = a + bi \in UHP$ for which $w = 0$.
    - Choose an angle θ for rotation of disk

Point at infinity

- Also called: Riemann sphere, stereographic projection, Alexandroff one-point compactification
- Geometric version 1
    - 

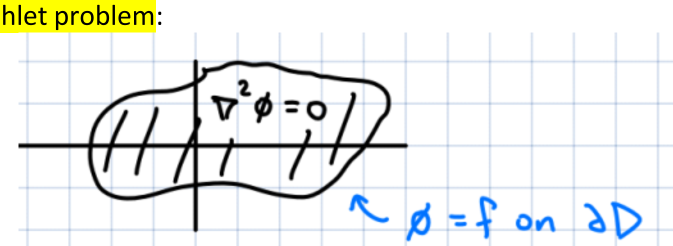
- Get a 1-1 mapping between sphere without the north pole and $\mathbb{C}$.
- Gives a bijection from sphere to $\mathbb{C} \cup \{\infty\}$.

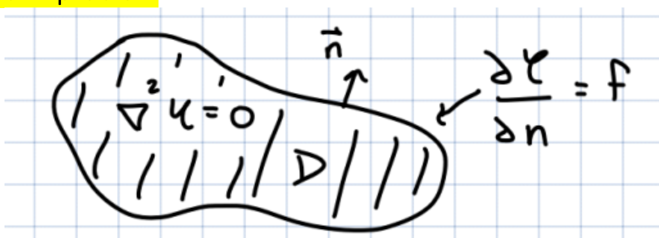
- Geometric version 2
  - Similar to 1, but have center of sphere at origin
  - North sphere outside the unit disk
  - South sphere inside the unit disk
- Analytic version
  - Consider $\mathbb{C} \cup \{\infty\}$, neighborhood of ∞ is $\infty \cup F^C$ for any closed bounded $F \subset \mathbb{C}$.
    - $\infty \cup F^C$ is open.
  - Compliment of F on Riemann sphere is an open neighborhood of N.
  - The map $z \to \frac{1}{z}$ is a continuous map on $\mathbb{C} \cup \{\infty\}$.
    - $0 \to \infty$ and $\infty \to 0$.
  - On $\mathbb{R}$, F^C is disconnected, so we need both $\pm\infty$.
  - On, $\mathbb{C}$, open disks around ∞ are connected, there is only one point at ∞.

Schwarz-Christoffel transformation

- Transforms UHP to a polygon
- Let P be a polygon in the w-plane, with vertices $w_0, w_1, \ldots, w_n$ and exterior angles $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$ with $\alpha_i \in (-1,1)$. Then there are complex constants A, B and real ordered numbers $x_1, \ldots, x_n$ (2 of which can be arbitrary) such that $w = f(z)$ maps UHP conformally one to one onto P, where $f(z) = A\int_{z_0}^{z} (\zeta - x_1)^{-\alpha_1} \ldots (\zeta - x_n)^{-\alpha_n} d\zeta + B$.
  - A is a magification and rotation, B is a translation.
    - $A = 1, B = 0$ gives a polygon similar to P (same angles $\alpha_1\pi, \alpha_2\pi, \ldots \alpha_n\pi$).
  - Note: sum of exterior angles $\sum_{i=1}^{n} \alpha_i\pi = 2\pi$.
  - Tangent vectors are not rotated except at x_i, where they are rotated by $\alpha_i\pi$.
  - Freedom to choose x_1, x_2 doesn't extend to $x_3, \ldots, x_n$, which are needed to set scale of sides

Applications to boundary value problems

- Dirichlet problem:
  - 

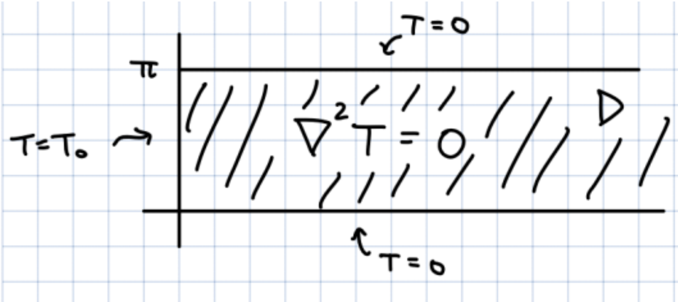
  - Find ϕ defined $\overline{D}$ such that $\nabla^2\phi = 0$ inside D, $\phi = f$ on ∂D.
  - Solution is unique if D is bounded
  - Solution exists for nice D and f.
  - Poisson formula
    - $D = \{z : |z| < R\}$.
    - $\phi(re^{i\theta}) = \frac{R^2 - r^2}{2\pi} \int_0^{2\pi} \frac{f(Re^{it})}{R^2 - 2Rr\cos(\theta - t) + r^2} dt$.
  - For unbounded domains, the solution to the Dirichlet problem may not be unique, unless some condition at ∞ is imposed (e.g. solution remain bounded)
    - e.g. $D = UHP, \nabla^2\phi = 0, \phi = 0$ at $y = 0$, has solution $\phi(x, y) = \lambda y$ for any $\lambda \in \mathbb{R}$.
- Neumann problem
  - 

  - Find ϕ defined $\overline{D}$ such that $\nabla^2\phi = 0$ inside D, $\frac{\partial\phi}{\partial n} = f$ on ∂D.

- With $\phi = \phi(x,y)$, $\frac{\partial\phi}{\partial n} = (\phi_x, \phi_y) \cdot (n_1, n_2)$.

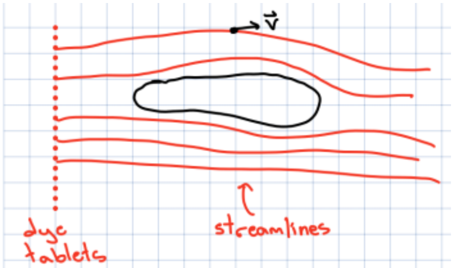
- Theorem: if D is bounded, then the solution to the Neumann problem is unique, up to an additive constant
- Note: if $f: D \to D'$ is analytic (conformal), $\phi: \overline{D'} \to \mathbb{R}$ is harmonic with $\frac{\partial\phi}{\partial n}$ on $\partial D'$, then $\psi = \phi \circ f$ is harmonic on D with $\frac{\partial\psi}{\partial n} = 0$ on ∂D.

Heat conduction example

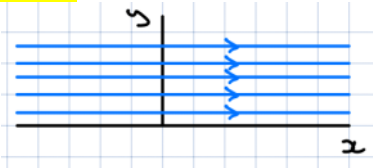
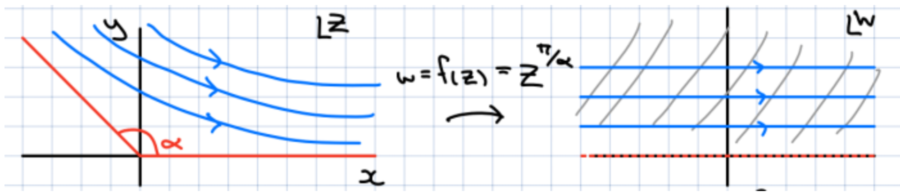
- 

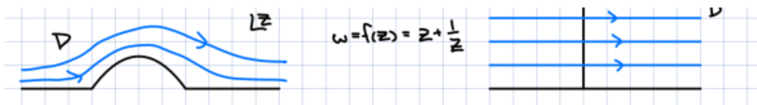
- $\nabla^2 T = 0$ in $D = \{z: 0 < Im(z) < \pi\}$ with indicated boundary conditions.
- Step 1: map strip to UHP
  - $w = f(z) = \cosh z$.
- Step 2: solve the problem in UHP
  - Note: the argument function $\arg z$ is harmonic in UHP since $\arg z = Im(\log z)$.
  - $\phi(w) = \frac{T_0}{\pi}(-Arg(w+1) + Arg(w-1))$.
  - Since $\phi(w) = Im\left(\frac{T_0}{\pi}(-Log(w+1) + Log(w-1))\right)$, ϕ is harmonic in UHP and obeys the boundary conditions.
  - Write $w = u + iv$, $\phi(u,v) = \frac{T_0}{\pi} Arg(u^2 + v^2 - 1 + 2iv)$.
- Step 3: pull back to strip by $T = \phi \circ f$.
  - Let $z = x + iy$, $f(z) = \cosh z = \cosh x \cos y + i \sinh x \sin y$.
  - So $u = \cosh x \cos y$, $v = \sinh x \sin y$.
  - Note: $u^2 + v^2 - 1 = \sinh^2 x - \sin^2 y$.
  - $T(x,y) = \phi\left(u(x,y), v(x,y)\right) = \frac{2T_0}{\pi} \arctan\left(\frac{\sin y}{\sinh x}\right)$.
- Isotherms are $T(x,y) = const$. Given by $\sin y = C \sinh x$.

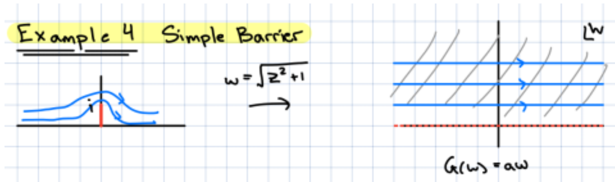
2D fluid flow

- 

- Problem: determine $v(x,y)$ and streamlines
- Notation: $v = (v_1, v_2) = v_1 + iv_2$.
- Assume velocity satisfies $v(x,y) \to (a, 0)$ as $x \to -\infty$, $a > 0$.
- Assumptions
  - v is independent of time (steady state), v is smooth.
  - Flow is irrotational (curl free, $\frac{\partial v_1}{\partial y} = \frac{\partial v_2}{\partial x}$) and incompressible (divergence free $\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} = 0$).
  - There are streamlines along boundary of the obstacle
- Define $f(z) = v_1 - iv_2 = \bar{v}$ where $z = x + iy$.
- Then f is analytic since $\frac{\partial v_1}{\partial x} = \frac{\partial(-v_2)}{\partial y}$ and $\frac{\partial v_1}{\partial y} = -\frac{\partial(-v_2)}{\partial x}$ i.e. obeys CR equation.
- Let $F(z) = \int_{z_0}^{z} f(\zeta) d\zeta$ (complex potential) independent of path for a simply connected domain

- If we know F, we can
  - Obtain v: $v = v_1 + iv_2 = \overline{f(z)} = \overline{F'(z)}$.
  - Streamlines: write $F(z) = \psi(x,y) + i\phi(x,y)$, ψ, ϕ are harmonic, streamlines are the level curves $\phi(x,y) = const.$
    - v is orthogonal to $\nabla\phi$.
- Parallel flow
  - 
  - Velocity $v = (a, 0), a > 0$ is constant everywhere
  - $F'(z) = \bar{v} = a$, so $F(z) = az = a(x + iy)$.
  - $\phi(x,y) = ay$, streamlines are $y = const.$
  - Any flows can be mapped to the parallel flow
- Flow around corner
  - 

  - Potential in the UHP is $G(w) = aw$.
  - Potential in z-plane (pull back) is $F(z) = G(f(z)) = az^{\frac{\pi}{\alpha}}$.
  - $\bar{v} = \overline{F'(z)} = \frac{a\pi}{\alpha}\frac{z^{\overline{\frac{\pi}{\alpha}}}}{z}$.
  - $\phi(x,y) = 2axy$, streamlines are $xy = const.$
- Cylinder obstacle
  - 

  - $F(z) = G(f(z)) = a\left(z + \frac{1}{z}\right)$.
  - $\phi(x,y) = a\left(y - \frac{y}{x^2+y^2}\right)$, streamlines are $y - \frac{y}{x^2+y^2} = const.$
- Simple barrier
  - 

  - $F(z) = G(f(z)) = a\sqrt{z^2 + 1}$.
  - Streamlines are $Im\left(a\sqrt{z^2 + 1}\right) = const.$
- This has genuine application to aircraft design (Joukowsky transformation)

Asymptotic evaluation of integrals

January 10, 2022 11:04 AM

Examples

- Stirling's formula: $n! \sim n^n e^{-n}\sqrt{2\pi n}$ as $n \to \infty$.
  - In more detail, $n! = n^n e^{-n}\sqrt{2\pi n}\left(1 + \frac{1}{12n} + \frac{1}{288n^2} + O\left(\frac{1}{n^3}\right)\right)$.
- Prime number theorem: let $\pi(x)$ be the number of primes less than or equal to x ($x > 0$), then $\pi(x) \sim \frac{x}{\log x}$ as $x \to \infty$.

O-Notations

- $A_R = \{z = re^{i\theta}, \alpha \le \theta \le \beta, r > R\}$.
- $f(z) = O(g(z))$ means $\exists R, M$ such that $|f(z)| \le M|g(z)|$ for all $z \in A_R$.
  - i.e. $\left|\frac{f(z)}{g(z)}\right| \le M$, for all $z \in A_R$.
- $f(z) = o(g(z))$ means $\forall \epsilon > 0$, $\exists R$ such that $|f(z)| \le \epsilon|g(z)|$ for all $z \in A_R$.
  - i.e. $\lim_{z\to\infty} \frac{f(z)}{g(z)} = 0$.
- Examples
  - $f(z) = O(1)$ means $f(z)$ is eventually bounded
  - $f(z) = o(1)$ means $f(z) \to 0$.
  - If $f(z) = o(g(z))$, then $f(z) = O(g(z))$.
  - Take $\alpha = \beta = 0$, then $e^{-x} = o\left(\frac{1}{x^n}\right)$ for all $n \ge 1$.
  - Take $\alpha = \beta = 0$, then $\frac{1}{2x^2+x} = o\left(\frac{1}{x^{2-p}}\right)$ for $p \ge 0$.

Def: We say $f(z) \sim S(z) = a_0 + \frac{a_1}{z} + \frac{a_2}{z^2} + \cdots$, if $f(z) - S_n(z) = o\left(\frac{1}{z^n}\right)$, where $S_n(z) = \sum_{m=0}^{n} \frac{a_m}{z^m}$, for all $n > 0$.

- $S(z)$ may not converge.
- Write $\sim$ instead of $=$ because $S(z)$ may diverge even if f is finite.
- Properties
  - An improved remainder estimate is $f(z) - S_n(z) = O\left(\frac{1}{z^{n+1}}\right)$.
  - Uniqueness: if $f(z) \sim A(z) = \sum_{n=0}^{\infty} \frac{a_n}{z^n}$ and $f(z) \sim B(z) = \sum_{n=0}^{\infty} \frac{b_n}{z^n}$, then $a_n = b_n$ for all $n \ge 0$.
  - Asymptotic expansions can be added or multiplied
  - Two different functions can have the same asymptotic expansion
    - $e^{-x} \sim 0 + \frac{0}{x} + \frac{0}{x^2} + \cdots$, $0 \sim 0 + \frac{0}{x} + \cdots$.
- Example
  - $I(x) = \int_0^\infty \frac{e^{-t}}{t+x} dt \sim \sum_{n=0}^{\infty} \frac{(-1)^n n!}{x^{n+1}}$ for large real $x > 0$.
  - $R_N(x) = I(x) - \sum_{n=0}^{N} \frac{(-1)^n n!}{x^{n+1}} = o\left(\frac{1}{x^{N+1}}\right)$.

Gamma function

- $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} dt$ for $Re(z) > 0$.
- The integral does converge for $Re(z) > 0$.
- $\Gamma(z)$ is analytic for $Re(z) > 0$, and $\Gamma'(z)$ can be computed by differentiating under the integral.
- Recursion, let $Re(z) > 0$, then $\Gamma(z+1) = z\Gamma(z)$.
- Relation to factorial: $\Gamma(n+1) = n!$ For $n \ge 0$.
- $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.
- Analytic continuation.

- $\Gamma(z) = \frac{\Gamma(z+n)}{(z+n-1)\dots(z+1)z}$ with $Re(z) > 0$.
- RHS is analytic for $Re(z+n) > 0$ or $Re(z) > -n$ except for simple poles at $0, -1, \dots, -(n-1)$.
- Since n is arbitrary, we get an analytic continuation of $\Gamma(z)$ to $\mathbb{C} - \{0, -1, -2, \dots, \}$.

Asymptotic equivalence

- Def: two functions $f(z)$ and $g(z)$ are asymptotically equivalent $f(z) \sim g(z)$ if $f(z) = g(z)(1 + o(1))$.
  - i.e. $\lim_{z\to\infty} \frac{f(z)}{g(z)} = 1$ (limit taken in a wedge).
  - A more detailed statement: $\frac{f(z)}{g(z)} \sim 1 + \frac{b_1}{z} + \frac{b_2}{z^2} + \cdots$.
  - But often, just knowing $f(z) \sim g(z)$ is enough, f is hard to compute, g is easy to compute
- g provides an approximation to f with small relative error, but not necessarily small absolute error.
  - $f(z) = \frac{ze^z}{z+1}$, $f(z) \sim e^z$ as $z \to \infty$ on $[0, \infty)$.
    - Absolute error: $|f(z) - e^z| = \left|e^z \cdot \frac{1}{z+1}\right| \to \infty$.
    - Relative error: $\left|\frac{f(Z) - e^z}{f(z)}\right| = \left|\frac{e^z \frac{1}{z+1}}{\frac{ze^z}{z+1}}\right| = \left|\frac{1}{z}\right| \to 0$.
- Common f is $f(z) = \int_\Gamma e^{zh(\zeta)} g(\zeta) d\zeta$, Γ is some contour, $z \to \infty$ in some wedge.
  - $\Gamma(z+1) = \int_0^\infty t^z e^{-t} dt = z^{z+1} \int_0^\infty e^{z(\log u - u)} du$, $h(u) = \log u - u$, $g(u) = 1$.
  - Laplace transform: $\tilde{g}(z) = \int_0^\infty e^{-zt} g(t) dt$, $h(t) = -t$.
  - Fourier transform: $\hat{g}(z) = \int_{-\infty}^\infty e^{-izt} g(t) dt$, $h(t) = -it$.

Laplace transform

- $\tilde{g}(z) = \int_0^\infty e^{-zt} g(t) dt$.
- Expect $\tilde{g}(z)$ for large z to depend only on $g(t)$ for $t \approx 0$.
- If $g(t) = \sum_{n=0}^\infty a_n t^n$ for $|t| < t_0$, then $\tilde{g}(z) \sim \sum_{n=0}^\infty a_n \int_0^\infty e^{-zt} t^n dt$.
  - Let $u = zt$, $\int_0^\infty e^{-zt} t^n dt = \frac{1}{z^{n+1}} \int_0^\infty e^{-u} u^n du = \frac{n!}{z^{n+1}}$.
  - $\tilde{g}(z) \sim \sum_{n=0}^\infty \frac{a_n n!}{z^{n+1}}$.
  - Since $a_n = \frac{g^{(n)}(0)}{n!}$, $\tilde{g}(z) \sim \sum_{n=0}^\infty \frac{g^{(n)}(0)}{z^{n+1}}$.
- Theorem: suppose g is continuous and bounded on $[0, \infty)$ and analytic at $t = 0$ with $g(t) = \sum_{n=0}^\infty a_n t^n$ for $|t| \le t_0$, then $\tilde{g}(z) \sim \sum_{n=0}^\infty \frac{a_n n!}{z^{n+1}} = \frac{a_0}{z} + \frac{a_1}{z^2} + \frac{a_2 2!}{z^3} + \cdots$ as $z \to \infty$ along $[0, \infty)$.
- Watson's lemma: let $\tilde{g}(z) = \int_0^b e^{-zt} g(t) dt$, $b \in (0, \infty]$, where g is locally integrable and $g(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n}$ as $t \to 0$ with $\alpha > -1$, $\beta > 0$.
  - i.e. $g(t) - t^\alpha \sum_{n=0}^N a_n t^{\beta n} = o(t^{\alpha + \beta n})$ as $t \to 0^+$.
  - If $b < \infty$, assume g is bounded ($|g(t)| \le M, \forall t \in [0, b]$).
  - If $b = \infty$, assume $|g(t)| \le Me^{ct}$ for some $c, M > 0$, $\forall t \in [0, \infty)$.
  - Then $\tilde{g}(z) \sim \sum_{n=0}^\infty \frac{a_n \Gamma(\alpha + \beta n + 1)}{z^{\alpha + \beta n + 1}}$ as $z \to \infty$ along $[0, \infty)$.
  - For $\alpha = 0$, $\beta = 1$, matches the previous theorem.
- Improved Watson's lemma: Let $I(z) = \int_0^b e^{-zt} g(t) dt$, $b \in (0, \infty]$, with $g(t) \sim t^\alpha \sum_{n=0}^\infty a_n t^{\beta n}$ as $t \to 0$ with $\alpha > -1$, $\beta > 0$. Assume the integral exists for all sufficiently large z. Then $I(z) \sim \sum_0^\infty a_n \frac{\Gamma(\alpha + \beta n + 1)}{z^{\alpha + \beta n + 1}}$
- e.g. $I(z) = \int_0^\infty \frac{e^{-zt}}{\sqrt{t^2 + 2t}} dt$.
  - $g(t) = \frac{1}{\sqrt{t^2 + 2t}} = \frac{1}{\sqrt{2t}} \frac{1}{\sqrt{1 + \frac{t}{2}}}$.
  - Note: $(1 + x)^p = \sum_{n=0}^\infty \binom{p}{n} x^n$ for any $p \in \mathbb{R}$, $|x| < 1$.

- And $\binom{p}{n} = \frac{p(p-1)\dots(p-n+1)}{n!}$ is the generalized binomial coefficient.

Laplace's method

- Informal: $I(z) = \int_a^b e^{-zh(t)} g(t)dt$, $z \to \infty$ along $[0, \infty)$. Suppose g and h are smooth, a and b may be infinite.
  - Suppose h has a global minimum at $c \in (a, b)$, with $h'(c) = 0, h''(c) > 0$.
  - Then $h(t) = h(c) + \frac{1}{2}h''(c)(t-c)^2 + \cdots$, $e^{-zh(t)} = e^{-zh(c)}e^{-z(h(t)-h(c))}$.
  - Expect $I(z) \sim \int_{c-\epsilon}^{c+\epsilon} e^{-zh(c)}e^{-z[h(t)-h(c)]}g(t)dt = e^{-zh(c)} \int_{c-\epsilon}^{c+\epsilon} e^{-z\frac{1}{2}h''(c)(t-c)^2} g(c)dt$.
  - Actually $I(z) \sim \frac{\sqrt{2\pi}e^{-zh(c)}g(c)}{\sqrt{zh''(c)}}$.
- Formal: let $I(z) = \int_a^b e^{-zh(t)} g(t)dt$. Suppose there exists a unique $c \in (a, b)$ such that $h'(c) = 0$, suppose also that $h''(c) > 0$ and that $h \in C^4$, $g \in C^2$, $g(c) \neq 0$. Then $I(z) = \frac{\sqrt{2\pi}e^{-zh(c)}g(c)}{\sqrt{zh''(c)}}\left[1 + O\left(\frac{1}{z}\right)\right]$.
  - If c is an end point (a or b), instead $I(z) = \frac{1}{2}\frac{\sqrt{2\pi}e^{-zh(c)}g(c)}{\sqrt{zh''(c)}}\left[1 + O\left(\frac{1}{\sqrt{z}}\right)\right]$.
- $I(z) = \int_{-\infty}^{\infty} e^{-z\sinh^2 t}dt = \sqrt{\frac{\pi}{z}}\left[1 + O\left(\frac{1}{z}\right)\right]$.
  - $g(t) = 1$, $h(t) = \sinh^2 t$, $h'(t) = \sinh 2t$, $h''(t) = 2\cosh 2t$.
- Stirling's formula: $\Gamma(z+1) = \int_0^{\infty} t^z e^{-t}dt = z^z e^{-z}\sqrt{2\pi z}\left(1 + O\left(\frac{1}{z}\right)\right)$.
  - Let $t = uz$, $\int_0^{\infty} t^z e^{-t}dt = z^{z+1}\int_0^{\infty} e^{-z(u-\log u)}du$.
  - $g(u) = 1$, $h(u) = u - \log u$, $h'(u) = 1 - \frac{1}{u}$, $h''(u) = \frac{1}{u^2}$.

A useful contour integral

- $I_{\alpha,p}(\nu) = \int_0^{\infty} t^{\alpha-1}e^{i\nu t^p}dt$ where $\nu \in \mathbb{R}$, $\nu \neq 0$, $0 < \alpha < p$.
- $I_{\alpha,p}(\nu) = \frac{\Gamma\left(\frac{\alpha}{p}\right)}{p|\nu|^{\alpha/p}}e^{i\frac{\pi\alpha}{2p}\operatorname{sgn}(\nu)}$.
- Special case: Fresnel integral with $\alpha = 1, p = 2$.
  - $I_{1,2}(\nu) = \int_0^{\infty} e^{i\nu t^2}dt = \frac{\Gamma\left(\frac{1}{2}\right)}{2\sqrt{|\nu|}}e^{\operatorname{sgn}(\nu)\frac{i\pi}{4}}$.
  - For $\nu > 0$, $I_{1,2}(\nu) = \frac{1}{2}\sqrt{\frac{\pi}{\nu}}\frac{1+i}{\sqrt{2}}$, so $\int_0^{\infty}\sin(\nu t^2)\,dt = \int_0^{\infty}\cos(\nu t^2)\,dt = \sqrt{\frac{\pi}{8}}\frac{1}{\sqrt{\nu}}$.

Stationary phase method

- Change the variable names of the previous integral
  - $I_{\lambda,\mu}(z) = \int_a^b t^{\lambda-1}e^{izt^{\mu}}dt = \frac{\Gamma\left(\frac{\lambda}{\mu}\right)}{\mu\,|z|^{\lambda/\mu}}e^{i\frac{\pi\lambda}{2\mu}\operatorname{sgn}(z)}$ for $0 < \lambda < \mu$, $z \in \mathbb{R}$, $z \neq 0$.
- For Fourier type integrals, $I(z) = \int_a^b e^{izh(t)}g(t)dt$ for real-valued g, h.
  - Riemann-Lebesgue Lemma: $\int_{-\pi}^{\pi} f(x)\cos nx\,dx \to 0$ as $n \to \infty$ if f is Riemann integrable
  - Thus $\int_{-\pi}^{\pi} f(x)e^{inx}dx \to 0$ as $n \to \infty$.
  - Idea: adjacent peaks and valleys create cancellation in the integral
  - Cancellations in $\int_a^b e^{izh(t)}g(t)dt$ are least at endpoints due to lack of symmetry and at peaks where $h'(t) = 0$ because $h(t)$ varies more slowly there.
- Endpoint behavior for $h(t) = t$.
  - $\int_a^b e^{izt}g(t)dt \sim \frac{1}{iz}\left(e^{izb}g(b) - e^{iza}g(a)\right)$ using IBP.
- Stationary point
  - Suppose $h'(c) = 0$, $h''(c) \neq 0$, $h'(t) \neq 0$ for all $t \neq c$.
  - Then $\int_{c-\epsilon}^{c+\epsilon} e^{izh(t)}g(t)dt \approx \sqrt{\frac{2\pi}{|h''(c)|}}e^{izh(c)}\frac{1}{\sqrt{z}}e^{i\frac{\pi}{4}\sigma}$, where $\sigma = \operatorname{sgn} h''(c)$.

Stationary phase theorem

- Consider $I(z) = \int_a^b e^{izh(t)} g(t) dt$ with a finite $b > a$ (possibly infinite), assume:
  - In (a, b), h' and g are continuous, $h''(t) > 0$ and $g'(t)$ and $\frac{g(t)}{h'(t)}$ are continuously differentiable with the latter integrable on (a, b).
  - As $t \to a^+$, there are $\mu > \lambda > 0$ such that $h(t) - h(a) \sim c_1 (t-a)^\mu$, $g(t) \sim c_2 (t-a)^\lambda$ and the first is twice differentiable, the second is once differentiable.
  - As $t \to b^-$, $\frac{g(t)}{h'(t)} \to$finite limit which is zero if $b \to \infty$.
- Then $I(z) \sim e^{i\pi \frac{\lambda}{2\mu}} \frac{c_2}{\mu} \Gamma\left(\frac{\lambda}{\mu}\right) \frac{e^{iz\,h(a)}}{(c_1 z)^{\lambda/\mu}}$.
- Special case:
  - Conditions: $a = 0$, $h(t) \sim c_1 t^2$ as $t \to 0^+$ ($c_1 = \frac{h''(0)}{2}$), $g(t) \sim g(0)$ as $t \to 0^+$.
  - $\mu = 2, \lambda = 1$.
  - $I(z) \sim \sqrt{\frac{\pi}{2}} \frac{1}{\sqrt{h''(0)}} e^{\frac{i\pi}{4}} g(0) \frac{1}{\sqrt{z}}$.
  - Half the previous example, the $\frac{1}{2}$ will disappear for an interior point.

Application to Bessel functions $J_n(z)$.

- Def: Let $f(\zeta) = e^{\frac{1}{2}z\left(\zeta - \frac{1}{\zeta}\right)}$ (generating function), $\zeta \in \mathbb{C} - \{0\}$. Fixed $z \in \mathbb{C}$. Since f is analytic on $\mathbb{C} - \{0\}$, it has a Laurent expansion $f(\zeta) = \sum_{-\infty}^{\infty} J_n(z) \zeta^n$ convergent for $z \neq 0$.
- $J_n(z) = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{\zeta^{n+1}} d\zeta$.
- If $C: |z| = 1$, we get $J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin\theta} e^{-in\theta} d\theta$.
  - If $z \in \mathbb{R}$, $|J_n(z)| \leq 1$.
- The power series: $J_n(z) = \left(\frac{z}{2}\right)^n \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{1}{(n+k)!} \frac{1}{2^{2k}} z^{2k}$.
  - Ratio test gives radius of convergence ∞.
  - $J_n(z)$ is entire and has zero of order n at $z = 0$.
  - This is a solution to the Bessel equation $J_n''(z) + \frac{1}{z} J_n'(z) + \left(1 - \frac{n^2}{z^2}\right) J_n(z) = 0$.
- Asymptotic behavior of $J_n(z) = \frac{1}{2\pi} \int_0^{2\pi} e^{iz\sin\theta} e^{-in\theta} d\theta$.
  - $g(\theta) = e^{-in\theta}$, $h(\theta) = \sin\theta$, critical points at $\frac{\pi}{2}, \frac{3\pi}{2}$, need spliting.
  - $J_n(z) = \frac{1}{2\pi} \int_0^{\pi} e^{iz\sin\theta} e^{-in\theta} d\theta + \frac{1}{2\pi} \int_{\pi}^{2\pi} e^{iz\sin\theta} e^{-in\theta} d\theta$.
  - $\sim \frac{2}{\sqrt{2\pi z}} \cos\left(z - \frac{\pi}{4} - \frac{n\pi}{2}\right)$.
  - $-\frac{\pi}{4} - \frac{n\pi}{2}$ is the phase shift
  - This is a damped cosine wave

Method of steepest descent example (Airy function)

- Airy function: $Ai(x) = \frac{1}{\pi} \int_0^{\infty} \cos\left(\frac{1}{3} t^3 + xt\right) dt$.
- Consider the behavior as $x \to \infty$ along $[0, \infty)$.
- Rewrite in exponential form: $Ai(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\left(\frac{1}{3}t^3 + xt\right)} dt$.
- Let $t = \sqrt{x} w$, $Ai(x) = \frac{\sqrt{x}}{2\pi} \int_{-\infty}^{\infty} e^{ix^{\frac{3}{2}}\left(w + \frac{1}{3}w^3\right)} dw$.
- Let $z = \sqrt{x}$, $Ai(z) = \frac{z}{2\pi} \int_{-\infty}^{\infty} e^{iz^3\left(w + \frac{1}{3}w^3\right)} dw$.
- Let $h(w) = w + \frac{1}{3} w^3$, $h'(w) = 1 + w^2$, $h''(w) = 2w$.
  - Critical points: $w = \pm i$, $h'(\pm i) = 0$, $h''(\pm i) = \pm 2i$, $ih(i) = -\frac{2}{3}$.
- Idea: deform the contour from the real axis to a new contour C which passes through a critical point
- Write $h(w) = u(w) + iv(w)$.
  - We want that on C, we have $Im(ih(w)) = u(w) = const = u_0$, C is a level curve of u passing through a critical point. Then $\int_C e^{iz^3 h(w)} dw = e^{iz^3 u_0} \int_C e^{-z^3 v(w)} dw$.

- $iz^3h = iz^3u - z^3v$.
- $u(w) = x - xy^2 + \frac{1}{3}x^3$.
  - Level curves through $i = (0,1)$ is $x\left(1 - y^2 + \frac{1}{3}x^2\right) = 0 = u_0$.
  - $x = 0$ or $3y^2 - x^3 = 3$.
  - Choose the upper branch of the hyperbola (passing through i) as C, because it is the part of steepest descent of $-v(w)$.
  - The $x = 0$ is the path of steepest ascent.
- $v(w) = y - \frac{1}{3}y^3 + x^2y$.
  - Gradient of $-v$ is parallel to level curves of u.

- Deformation from the real axis to C
  - Want to show $\int_{-\infty}^{\infty} e^{iz^3h(w)}dw = \int_C e^{iz^3h(w)}dw$.
  - The integrand is entire, so local deformation are OK by Cauchy integral theorem
  - The only difficulty is at infinity
  - Claim: we can deform the contour to any contour going to ∞ as $\lambda e^{i\theta}$ with $\theta \in \left[0, \frac{\pi}{3}\right)$, similarly on left side with $\theta \in \left(\frac{2\pi}{3}, \pi\right]$.
    - Verification: $\int_{\Gamma_R} e^{z^3ih(w)}dw = \int_0^\beta e^{z^3ih\left(Re^{i\theta}\right)}Rie^{i\theta}d\theta$.
    - $\left|\int_{\Gamma_R} e^{z^3ih(w)}dw\right| \le \int_0^\beta e^{-\frac{1}{3}R^3\sin 3\theta}Rd\theta \le R\int_0^\beta e^{-\frac{1}{3}R^3 m_\beta \theta}d\theta = O\left(\frac{1}{R^2}\right)$.
  - So $Ai\left(z^2\right) = \frac{z}{2\pi}\int_C e^{-z^3v(w)}dw$.
- Apply Laplace's method (asymptotic behavior is dominated by $w = i$ critical point)
  - On the contour C, $-v(w) = ih(w) = i\left(h(i) + h'(i)(w-i) + \frac{h''(i)}{2}(w-i)^2\right)$.
  - $h(i) = \frac{2}{3}i, h'(i) = 0, h''(i) = 2i$.
  - $ih(w) = -\frac{2}{3} - (w-i)^2$.
  - $\int_C e^{-z^3v(w)}dw = \int_C e^{iz^3h(w)}dw \sim e^{-\frac{2}{3}z^3}\int_{C_\epsilon} e^{-z^3(w-i)^2}dw = e^{-\frac{2}{3}z^3}\int_{-\epsilon}^{\epsilon} e^{-z^3t^2}dt$.
  - $\sim e^{-\frac{2}{3}z^3}\int_{-\infty}^{\infty} e^{-z^3t^2}dt = \frac{\sqrt{\pi}}{z^{3/2}}e^{-\frac{2}{3}z^3}$.
  - Put $x = z^2$, $Ai(x) \sim \frac{1}{2\sqrt{\pi}}\frac{1}{x^{1/4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}$ as $x \to \infty$ along $[0, \infty)$.

Steepest descent theorem

- Let $\gamma : (a,b) \to \mathbb{C}$ be a C^1 curve ($a = -\infty$ and/or $b = \infty$ is allowed).
- Let $f(w)$ be continuous along γ and analytic at $w_0 = \gamma(t_0)$, $t_0 \in (a,b)$.
- Let g be a bounded and continuous funciton on γ with $g(t_0) \neq 0$.
- Suppose that for $|z| \ge R$ and $\arg z$ fixed.
  - $\int_\gamma e^{zf(w)}g(w)dw$ converges absolutely.
  - $f'(w_0) = 0, f''(w_0) \neq 0$.
  - $Im(zf(w)) = const$ for w on γ in some neighborhood of w_0.
  - $Re\left(zf(w_0)\right) > Re\left(zf(\gamma(t))\right)$ for all $t \neq t_0$. ($-v(w)$ takes its unique max on γ at the critical point)
- Then $\int_\gamma e^{zf(w)}g(w)dw \sim e^{zf(w)}\frac{\sqrt{2\pi}}{\sqrt{-f''(w_0)}}\frac{1}{\sqrt{z}}g(w_0)$ as $z \to \infty$, $\arg z$ fixed.

- Application to Airy function $Ai(x) = \frac{x}{2\pi}\int_{-\infty}^{\infty} e^{ix^{\frac{3}{2}}\left(w + \frac{1}{3}w^3\right)}dw$.
  - $f(w) = ih(w) = i\left(w + \frac{1}{3}w^3\right)$, $g(w) = 1$, $z = x^{\frac{3}{2}}$.
  - Deform to a contour C such that $Im(ih) = u = const$, $Re(ih) = -v$ is maximal at $w_0 = i$.
  - $f(w_0) = -\frac{2}{3}$, $f''(w_0) = -2$.
  - $A_i(x) \sim \frac{\sqrt{x}}{2\pi}e^{-\frac{2}{3}x^{\frac{3}{2}}}\frac{\sqrt{2\pi}}{\sqrt{2}}\frac{1}{\sqrt{x^{\frac{3}{2}}}} = \frac{1}{2\sqrt{\pi}}\frac{1}{x^{1/4}}e^{-\frac{2}{3}x^{\frac{3}{2}}}$.

- Example: $I(z) = \int_{-\infty}^{\infty} e^{izt}\left(1+t^2\right)^{-z} dt$ as $z \to \infty$ along $[0, \infty)$.
  - Rewrite as $I(z) = \int_{-\infty}^{\infty} \exp\left(z\left(it - \mathrm{Log}\left(1+t^2\right)\right)\right) dt$.
  - Take the branch cut for $\mathrm{Log}\left(1+t^2\right)$ at $t > i$ and $t < -i$.
    - In the cut plane, $\mathrm{Log}\left(1+t^2\right)$ is analytic.
  - Let $f(w) = iw - \mathrm{Log}\left(1+w^2\right)$.
    - $f'(w) = i - \frac{2w}{1+w^2}$, $w_0 = i\left(\sqrt{2}-1\right)$.
    - $f''(w) = \frac{2\left(w^2-1\right)}{\left(1+w^2\right)^2}$, $f''\left(w_0\right) = -\frac{c^2+1}{2c^2}$.
    - $f\left(w_0\right) = -c - \log 2c$ $(c = \sqrt{2} - 1)$.
    - $Re\left(f(w)\right) = -y - \log\left|1+w^2\right| = -y - \frac{1}{2}\log\left(\left(1+x^2-y^2\right) + 4x^2y^2\right)$.
    - $Im\left(f(w)\right) = x - \arctan\frac{2xy}{1+x^2-y^2}$, $Im\left(f\left(w_0\right)\right) = 0$.
  - Path of steepest descent: $Im\left(f\left(w_0\right)\right) = 0$.
  - Substitute into the theorem: $I(z) \sim \frac{2c}{\sqrt{c^2+1}}\sqrt{\frac{\pi}{z}}\, e^{-cz}\frac{1}{(2c)^z}$.

Laplace transform

- Def: for $f: [0, \infty) \to \mathbb{C}$ of exponential order (i.e. $\exists A > 0, D \in \mathbb{R}$ such that $\left|f(t)\right| \leq Ae^{Bt}$ for all $t \geq 0$), its Laplace transform is $\tilde{f}(z) = \int_0^{\infty} e^{-zt} f(t) dt$.
- Facts
  - There exists a unique $\sigma \in [-\infty, \infty)$ such that the integral converges if $Re(z) > \sigma$, diverges if $Re(z) < \sigma$.
  - $\tilde{f}$ is analytic on $Re(z) > \sigma$ and $(z) = -\int_0^{\infty} e^{-zt} t f(t) dt$.
  - If $f(t)$ and $g(t)$ are continuous and $\tilde{f}(z) = \tilde{g}(z)$ for $Re(z) > x_0$ for some x_0, then $f(t) = g(t)$ for all $t \in [0, \infty)$.

Inverse Laplace transform

- Complex inversion formula: suppose $F: \mathbb{C} \to \mathbb{C}$ is analytic except for a finite number of isolated singularities z_j and that F is analytic on $\{z : Re(z) > \sigma\}$. Suppose $\left|F(z)\right| \leq \frac{M}{|z|^{\beta}}$ for all $|z| \geq R$ with $\beta > 0$. For $t \geq 0$, let $f(t) = \sum_j Res\left(e^{zt}F(z); z_j\right)$. Then $\tilde{f}(z) = F(z)$ for $Re(z) > \sigma$.
- Note: decay condition is satisfied if $F(z) = \frac{P(z)}{Q(z)}$, P, Q polynomials, $\deg P \geq \deg Q + 1$.
  - With $Q(z)$ having simple zeros at $z_1, \dots, z_n$, $f(t) = \sum_{j=1}^{n} e^{z_j t}\frac{P\left(z_j\right)}{Q'\left(z_j\right)}$ (special case of Heaviside expansion theorem).
  - The abscissa of convergence is $\sigma\left(f\right) = \max\left\{Re\left(z_1\right), \dots, Re\left(z_n\right)\right\}$.
- Integral form of inverse theorem: suppose the theorem's hypothesis hold, with the rightmost singularities z_j of F on the line $Re(z) = \sigma$, then the abscissa of convergence of F is σ, and $f(t) = \frac{1}{2\pi i}\int_{\alpha - i\infty}^{\alpha + i\infty} e^{zt} F(z) dz$ for $\alpha > \sigma$.

Fourier and inverse Fourier transform

- The Fourier transform of an integrable function $f: \mathbb{R} \to \mathbb{C}$ is $\hat{f}\left(y\right) = \int_{-\infty}^{\infty} e^{-iyt} f(t) dt$.
- The inverse transform is $f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{f}\left(y\right) e^{iyt} dy$.