Introduction & ODEs

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Differential equation is an equation that defines a function implicitly by giving a relationship between a function and its derivatives

ODES: $f(x, y, y', \dots, y^{(n)}) = 0$.
PDES: $f\left(x, y, u(x,y), u_x, u_y, u_{xx}, u_{xy}, u_{yy}\right) = 0$.

Let L be the differential operator.

First order ODEs

- Separable equations: $\frac{dy}{dx} = P(x)Q(y)$.
    - Then $\int \frac{dy}{Q(y)} = \int P(x)dx + C$
- Linear equations: $\frac{dy}{dx} + P(x)y = Q(x)$.
    - General form: $Ly = \left(\frac{d}{dx} + P\right)y = Q(x)$.
    - Recall $\frac{d}{dx}\left(F(x)y\right) = F\frac{dy}{dx} + F'y$.
    - So we can choose F such that $\frac{dF}{dx} = FP(x)$, and get $FLy = F\frac{dy}{dx} + FPy = FQ$.
        - $F = Ae^{\int P(x)dx}$ is the integration factor.
        - Then we have $\frac{d}{dx}\left(e^{\int P(x)dx}y\right) = e^{\int P(x)dx}Q(x)$.
        - Integrating both sides gives the solution

Second order linear ODEs

- Constant coefficient equations: $Ly = ay'' + by' + cy = 0$, (a, b, c are constants)
    - The differential operator is $L \coloneqq aD^2 + bD + c$.
    - Look for y such that $y' = ry$, then we can apply first order techniques.
        - This gives $y = Ae^{rx}$.
    - To solve for the second order equation, guess $y = e^{rx}$, r is a parameter to be determined.
        - Then $Ly = ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$.
        - Since $e^{rx} \neq 0$, we are actually solving $ar^2 + br + c = 0$.
        - $r = \frac{-b}{2a} \pm \frac{\sqrt{b^2-4ac}}{2a}$ with discriminant $\Delta = b^2 - 4ac$.
            - If $\Delta > 0$, two distinct real roots r_1, r_2, general solution $y = Ae^{r_1x} + Be^{r_2x}$.
            - If $\Delta = 0$, double root $r = -\frac{b}{2a}$, general solution $y = (A + Bx)e^{rx}$.
            - If $\Delta < 0$, complex conjugate pair $r_\pm = -\frac{b}{2a} \pm \frac{\sqrt{4ac-b^2}}{2a}i = \lambda \pm i\mu$,
                - general solution $y = Ae^{(\lambda+i\mu)x} + Be^{(\lambda-i\mu)x} = e^{\lambda x}\left((A+B)\cos\mu x + (A-B)i\sin\mu x\right)$
- Cauchy-Euler/Equidimensional equations: $Ly = x^2y'' + axy' + by = 0$.
    - Note: The dimension of x^2y'', xy', y are the same
    - Look for y such that $x\frac{dy}{dx} = ry$, then it is the separable case
        - This gives $y = Ax^r$.
    - To solve for the equation, guess $y = x^r$, r is a parameter
        - Then $Ly = x^2r(r-1)x^{r-2} + axrx^{r-1} + bx^r = (r(r-1) + ar + b)x^r = 0$.
        - We need $r(r-1) + ar + b = r^2 + (a-1)r + b = 0$.
        - $r = -\frac{a-1}{2} \pm \frac{\sqrt{(a-1)^2-4b}}{2}$ with discriminant $\Delta = (a-1)^2 - 4b$.
            - If $\Delta > 0$, two distinct real roots r_1, r_2, general solution $y = Ax^{r_1} + Bx^{r_2}$.
            - If $\Delta = 0$, double root $r = -\frac{a-1}{2}$, general solution $y = (A + B\ln x)x^r$.

- If $\Delta < 0$, complex conjugate pair $r_{\pm} = -\frac{a-1}{2} \pm \frac{\sqrt{4b-(a-1)^2}}{2} i = \lambda \pm i\mu$,
    - general solution $y = C_1 x^{\lambda+i\mu} + C_2 x^{\lambda-i\mu} = x^{\lambda}(A\cos(\mu \ln x) + B\sin(\mu \ln x))$

Series solutions of differential equations

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Taylor series

- Power series $S(x) = a_0 + a_1 x + \cdots + a_n x^n + \cdots = \sum_{n=0}^{\infty} a_n x^n$.
- Suppose we have a function f and know all its derivatives
  - Let $f(x) = a_0 + a_1 x + \cdots a_n x^n$.
  - Then $f'(x) = a_1 + 2a_2 x + \cdots + n a_n x^{n-1} + \cdots$.
  - And $f^{(n)}(x) = n!\, a_n + \frac{(n+1)!}{1!} a_{n+1} x + \cdots$.
  - This gives that $f'(0) = a_1, f''(0) = 2a_2, \ldots, \frac{f^{(n)}(0)}{n!} = a_n$.
  - So $f(x) = f(0) + \frac{f'(0)}{1} x + \cdots + \frac{f^{(n)}(0)}{n!} x^n + \cdots = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$.
- If it is about a point x_0, we have $f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!} (x - x_0)^n$.

We can generate the derivatives of the solution from the ODE

Undetermined coefficients

- E.g. $Ly = y' - 2y = 0$
  - Assume $y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=1}^{\infty} n a_n x^{n-1}$.
  - Then $Ly = a_1 x^0 + a_2 2x + a_3 3x^2 + \cdots - 2(a_0 + a_1 x + a_2 x^2 + \cdots)$.
    - $= (a_1 - 2a_0) + (2a_2 - 2a_1)x + (3a_3 - 2a_2)x^2 + \cdots = 0$.
    - Then $a_1 = 2a_0$, $a_2 = a_1 = 2a_0$, $3a_3 = 2a_2$, $a_n = \frac{2^n a_0}{n!}$.
    - So $y(x) = a_0 \left(1 + 2x + \frac{(2x)^2}{2} + \cdots\right) = a_0 e^{2x}$
  - Another way: $Ly = \sum_{n=0}^{\infty} a_{n+1}(n+1)x^n - 2\sum_{n=0}^{\infty} a_n x^n = 0$
    - Then check the coefficients
- E.g. $Ly = y'' - xy = 0$ (Airy equation)
  - Note: $y'' - c^2 y = 0$ gives exponential solutions
    - $y'' + c^2 y = 0$ gives sin and cos
  - Let $y = \sum_{n=0}^{\infty} a_n x^n, y' = \sum_{n=1}^{\infty} n a_n x^{n-1}, y'' = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2}$.
  - Then $Ly = \sum_{n=-1}^{\infty} (n+3)(n+2) a_{n+3} x^{n+1} - \sum_{n=0}^{\infty} a_n x^{n+1} = 0$.
    - $2 \cdot 1 a_2 = 0$.
    - And $(n+3)(n+2)a_{n+3} = a_n$.
    - So $y(x) = a_0 \left(1 + \frac{x^3}{3\cdot 2} + \frac{x^6}{6\cdot 5\cdot 3\cdot 2} + \cdots\right) + a_1 \left(x + \frac{x^4}{4\cdot 3} + \frac{x^7}{7\cdot 6\cdot 4\cdot 3} + \cdots\right)$

Ordinary points and singular points

- E.g. $Ly = (x-1)y'' + y' = 0$.
  - The solution is $y = C \ln|x-1| + D$.
  - Consider a Taylor series expansion about $x_0 = 0, y = \sum_{n=0}^{\infty} a_n x^n$
    - $Ly = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=2}^{\infty} n(n-1) a_n x^{n-2} + \sum_{n=1}^{\infty} n a_n x^{n-1}$.
    - Then $Ly = \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - \sum_{n=1}^{\infty} n(n+1) a_{n+1} x^{n-1} + \sum_{n=1}^{\infty} n a_n x^{n-1}$.
      - $-1 \cdot 2a_2 + a_1 = 0$.
      - $n(n-1)a_n - n(n+1)a_{n+1} + n a_n = 0$.
      - So $a_{n+1} = \frac{n}{n+1} a_n$.
      - So $y(x) = a_0 + a_1 \left(x + \frac{x^2}{2} + \cdots + \frac{x^n}{n} + \cdots\right)$.
  - Consider the expansion about $x_0 = 1, y = \sum_{n=0}^{\infty} a_n (x-1)^n$.
    - Then $Ly = \sum_{n=2}^{\infty} a_n n(n-1)(x-1)^{n-1} + \sum_{n=1}^{\infty} a_n n(x-1)^{n-1}$.
    - This gives $y = a_0$.
  - $x_0 = 0$ yields two solutions, a_0 and $-a_1 \ln|x-1|$.
    - $x_0 = 0$ is the ordinary point of this ODE
  - $x_0 = 1$ yields only the regular part of the solution $y = a_0$ and does not capture the

singular behavior that occurs as $x_0 = 1$, namely, $-a_1 \ln|x-1|$.
  - When expanding about the ordinary point $x_0 = 0$, the radius of convergence of the series is at least as far as the distance between $x_0 = 0$ and the nearest singular point $x_0 = 1$
- Def: Consider the general second order linear ODE, $P(x)y'' + Q(x)y' + R(x)y = 0$
  Equivalently, we have $y'' + \frac{Q(x)}{P(x)}y' + \frac{R(x)}{P(x)}y = 0$,
  - If $p(x) = \frac{Q(x)}{P(x)} = p_0 + p_1(x-x_0) + \cdots$ and $q(x) = \frac{R(x)}{P(x)} = q_0 + q_1(x-x_0) + \cdots$, (i.e. p(x) and q(x)a are analytic at $x = x_0$), then $x = x_0$ is an ordinary point
    - A Taylor expansion of the form $y = \sum_0^\infty a_n (x-x_0)^n$ will yield two independent solutions
  - If $p(x)$ or $q(x)$ are not analytic about $x = x_0$, then x_0 is a singular point
- If x_0 is an ordinary point, then the radius of convergence of the power series $y = \sum_{n=0}^\infty a_n (x-x_0)^n$ is at least as large as the distance from x_0 to the nearest singular point

Frobenius Series
- E.g. $2xy' + (2x+1)y = 0$
  - Singular point at $x = 0$.
  - Solution: $y = \frac{A}{x^{1/2}} e^{-x}$.
  - We should try a solution of the form $y = x^r \sum_0^\infty a_n x^n$ (Frobenius Series).
    - Let $y = \sum_{n=0}^\infty a_n x^{n+r}$, $y' = \sum_{n=0}^\infty a_n (n+r) x^{n+r-1}$.
    - Then $\sum_{n=0}^\infty 2a_n (n+r) x^{n+r} + \sum_{n=0}^\infty a_n x^{n+r} + \sum_{n=0}^\infty 2a_n x^{n+r+1} = 0$.
      - $a_0(2r+1) = 0$.
      - $a_m(2m+2r+1) + 2a_{m-1} = 0$.
- Around certain classes of singular points x_0, we can assume a Frobenius series $y = \sum_{n=0}^\infty a_n (x-x_0)^{n+r}$.
- If we have a coefficient that vanishes even more radically
  - E.g. $Ly = 2x^2 y' + (2x+1)y = 0$, solution is $y = \frac{A}{x} e^{\frac{1}{2x}}$ (essential singularity at 0).
- Consider the general second order linear ODE $P(x)y'' + Q(x)y' + R(x)y = 0$
  - We can rewrite it as $(x-x_0)^2 y'' + (x-x_0)\left(\frac{(x-x_0)Q(x)}{P(x)}\right)y' + (x-x_0)^2 \frac{R(x)}{P(x)} y = 0$.
  - If $\frac{(x-x_0)Q(x)}{P(x)}$ and $(x-x_0)^2 \frac{R(x)}{P(x)}$ are analytic, then $x = x_0$ is a regular singular point,
    - the Frobenius series $y = \sum_{n=0}^\infty a_n (x-x_0)^{n+r}$ will yield 2 independent solutions to the ODE which satisfies the equation $r(r-1) + p_0 r + q_0 = 0$ (indicial equation)
      - $p_0 = \lim_{x\to x_0} (x-x_0)\frac{Q(x)}{P(x)}$.
      - $q_0 = \lim_{x\to x_0} (x-x_0)^2 \frac{R(x)}{P(x)}$.
      - This gives $(x-x_0)^2 y'' + (x-x_0)p_0 y' + q_0 y = 0$
    - Radius of convergence is the distance from x_0 to the nearest different singular point in the complex plane
  - If $\frac{(x-x_0)Q(x)}{P(x)}$ or $(x-x_0)^2 \frac{R(x)}{P(x)}$ are not analytic, then $x = x_0$ is not regular (irregular singular point).
- E.g. $Ly = 4x^2 y'' - (x^2 + x)y' + y = 0$.
  - $x_0 \neq 0$ are all ordinary points, we can use Taylor expansion directly.
  - $x_0 = 0$ is a regular singular point
    - NOTE: $p_0 = \lim_{x\to 0} x \frac{-(x^2+x)}{4x^2} = -\frac{1}{4}$ and $q_0 = \lim_{x\to 0} x^2 \frac{1}{4x^2} = \frac{1}{4}$.
    - $r(r-1) - \frac{1}{4}r + \frac{1}{4} = 0$ gives $r = \frac{1}{4}, 1$.
    - Then $y_1(x) = a_0 x \left(1 + \frac{x}{7} + \frac{x^2}{77} + \cdots\right)$ and $y_2(x) = a_0 x^{\frac{1}{4}} \left(1 + \frac{x}{4} + \frac{x^2}{32} + \cdots\right)$.
      - The radius of convergence is ∞.
    - We can also plug in the series to calculate r directly.

Radius of convergence

- For a series of numbers $\sum_0^\infty c_n$, the ratio test is $\lim_{n\to\infty}\left|\frac{c_{n+1}}{c_n}\right| = r$,
  - If $r < 1$, converges
  - If $r = 1$, test fails
  - If $r > 1$, diverges
- Then $\lim_{n\to\infty}\left|\frac{a_m x^{m+r}}{a_{m-1}x^{m+r-1}}\right|$ will give the radius of convergence.

Bessel functions

- $Ly = x^2y'' + xy' + (x^2 - \nu^2)y = 0$ with $\nu \notin \mathbb{Z}$.
- $x = 0$ is a regular singular point
  - $p_0 = \lim_{x\to 0}\frac{x}{x^2}x = 1$, $q_0 = \lim_{x\to 0}\frac{x^2-\nu^2}{x^2}x^2 = -\nu^2$.
- Indicial equation: $r(r-1) + r - \nu^2 = r^2 - \nu^2 = 0$, $r = \pm\nu$.
- Using Frobenius Series $y = \sum_0^\infty a_n x^{n+r}$, we get
  - $a_0(r^2 - \nu^2) = 0$, $r = \pm\nu$.
  - $a_1((r+1)^2 - \nu^2) = a_1(1 + 2\nu) = 0$.
    - If $\nu = -\frac{1}{2}$, a_1 is arbitrary
    - Otherwise, $a_1 = 0$.
- When $\nu \notin \mathbb{Z}$, and $\nu \neq -\frac{1}{2}$
  - $a_n = -\frac{a_{n-2}}{(n+\nu)^2-\nu^2}$.
    - If $r = \nu$, $a_n = -\frac{a_{n-2}}{n(n+2\nu)}$.
    - $a_{2n} = \frac{(-1)^n a_0}{n!2^{2n}(1+\nu)\ldots(n+\nu)}$.
    - This gives $y_+(x) = a_0 x^\nu \sum_0^\infty \frac{(-1)^n x^{2n}}{n!2^{2n}(1+\nu)\ldots(n+\nu)}$.
    - And $y_-(x) = a_0 x^\nu \sum_0^\infty \frac{(-1)^n x^{2n}}{n!2^{2n}(1-\nu)\ldots(n-\nu)}$.
- When $\nu = 0$.
  - This gives $r = 0$.
  - $a_m = -\frac{a_{m-2}}{m^2}$, $a_{2m} = \frac{(-1)^m a_0}{2^{2m}(m!)^2}$, $y_1(x) = a_0\sum_0^\infty \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2}$.
  - To get the second solution, use $\frac{\partial}{\partial r}a_0 x^r\left(1 - \frac{x^2}{(2+r)^2} + \frac{x^4}{(2+r)^2(4+r)^2}\ldots\right)$ at $r = 0$.
    - This gives $y_0(x) = a_0 \ln x \sum_0^\infty \frac{(-1)^m x^{2m}}{2^{2m}(m!)^2} + a_0\left(\frac{x^2}{4} - \cdots\right)$.
- When $\nu = \frac{1}{2}$.
  - When $r = \frac{1}{2}$, $a_1\left(1 + 2\cdot\frac{1}{2}\right) = 2a_1 = 0$ gives $a_1 = 0$.
    - $a_n = -\frac{a_{n-2}}{\left(n+\frac{1}{2}\right)^2 - \frac{1}{2}^2} = -\frac{a_{n-2}}{n(n+1)}$.
    - $y_1(x) = a_0 x^{\frac{1}{2}}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \cdots\right) = a_0\frac{\sin x}{x^{1/2}}$.
  - When $r = -\frac{1}{2}$, $a_1(1 + 2r) = a_1(1-1) = 0$, a_1 is arbitrary.
    - $a_n = -\frac{a_{n-2}}{\left(n-\frac{1}{2}\right)^2 - \frac{1}{2}^2} = -\frac{a_{n-2}}{n(n-1)}$.
    - $y_1(x) = a_0 x^{-\frac{1}{2}}\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) = a_0\frac{\cos x}{x^{1/2}}$.
    - Third solution spawned by a_1.
      - $y_3(x) = a_1\frac{\sin x}{x^{1/2}}$.
    - The general solution is $y(x) = a_0\frac{\cos x}{x^{1/2}} + a_1\frac{\sin x}{x^{1/2}}$.

PDEs

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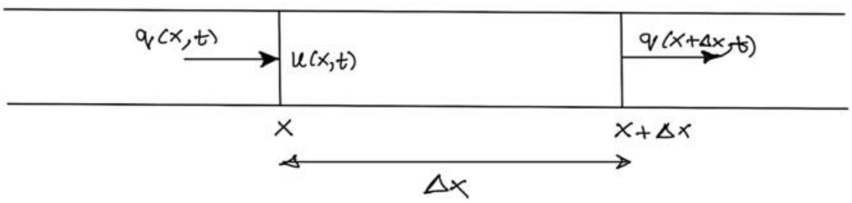
Classifications

- ODEs: $f(x, u(x), u'(x)) = 0$.
- PDEs:
  - First order linear PDEs: $a(x,y)u_x + b(x,y)u_y = c(x,y)u$.
    - Solutions are surfaces
  - Second order linear PDEs: $au_{xx} + 2bu_{xy} + cu_{yy} + du_x + eu_y + fu = g$.
    - a, b are functions of x and y.
    - Solutions are surfaces
    - Recall quadric surfaces: $Ax^2 + 2Bxy + Cy^2 + Dx + Ey = F$.
      - Discriminant $\Delta = B^2 - AC$.
      - If $\Delta < 0$, ellipse
      - If $\Delta > 0$, hyperbola
      - If $\Delta = 0$, parabola
    - Define discriminant $\Delta = b^2 - ac$.
      - If $\Delta < 0$, elliptic, e.g. Laplace's, Poisson equation
      - If $\Delta > 0$, hyperbolic, e.g. wave equation
      - If $\Delta = 0$, parabolic, e.g. heat equation

Conservation law and wave equation

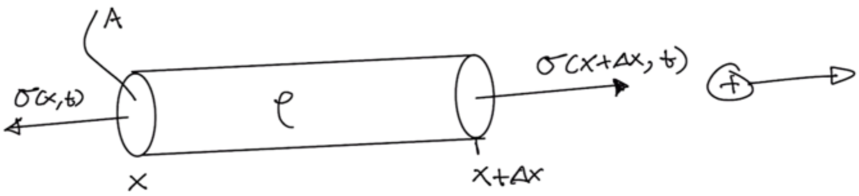


- Let $u(x,t)$ be the density of cars at x at time t, $q(x,t)$ be the flux of cars.
- Conservation principle: cars are neither created nor destroyed
  - Change in # cars in $[x, x+\Delta x]$ over period $[t, t+\Delta t]$=# cars entering - # cars leaving
  - i.e. $u(t+\Delta t)\Delta x - u(x,t)\Delta x = q(x,t)\Delta t - q(x+\Delta x, t)\Delta t$.
- Divide both sides by Δx and Δt, we get $\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$.
- And $q = cu$, so $\frac{\partial u}{\partial t} + c\frac{\partial u}{\partial x} = 0$. (constant coefficient equation)
- Guess $u(x,t) = e^{ikx+\sigma t}$, given that $\sigma = -ikc$.
- In fact $u(x,t) = f(x - ct)$ is a solution

Galilean transform

- $x' = x - ct$, $u(x,t) = f(x-ct) = f(x')$.
- $\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right)u = u_{tt} - c^2u_{xx} = 0$. (wave equations)

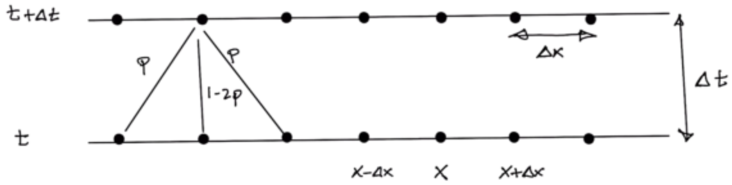
Motion of an elastic bar



- $\sigma(x+\Delta x, t)A - \sigma(x,t)A = \rho A\Delta x\frac{\partial^2 u}{\partial t^2}$.
- Divide both sides by $A\Delta x$, we have $\frac{\partial \sigma}{\partial x} = \rho u_{tt}$.
  - Hooke's law gives $\sigma = Eu_x$.

- Then $u_{tt} = \frac{E}{\rho} u_{xx}$.

Random walks and heat equation



- Let $u(x,t)$ be the density of fruit flies on the tree at x at time t.
  - $u(x, t+\Delta t) = pu(x-\Delta x, t) + (1-2p)u(x,t) + pu(x+\Delta x, t)$.
- $\frac{\partial u}{\partial t} = D\frac{\partial^2 u}{\partial x^2}$. (heat equation)
- In 2D, $\frac{\partial u}{\partial t} = D\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$.
- Obtaining the heat equation from a conservation law $\frac{\partial u}{\partial t} + \frac{\partial q}{\partial x} = 0$.
  - Fourier's law: $q = -\alpha^2 \frac{\partial u}{\partial x}$.
  - Then $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$.
- Solution: $u(x,t) = e^{ikx}e^{-\alpha^2 k^2 t}$.

Finite difference method
- Taylor series gives:
  - First derivative approximation: $\frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \frac{\Delta x}{2!} f''(x) + \cdots$.
    - Or $\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = f'(x) + \frac{\Delta x^2}{3!} f^{(3)}(x) + \cdots$. (More accurate)
  - Second derivative approximation $\frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} = f''(x) + \frac{\Delta x^2}{12} f^{(4)}(x) + \cdots$.
- Forward difference approximation
  - $\frac{f(x+\Delta x) - f(x)}{\Delta x} = f'(x) + \frac{\Delta x}{2!} f''(x)$ (order Δx).
- Central difference approximation
  - $\frac{f(x+\Delta x) - f(x-\Delta x)}{2\Delta x} = f'(x) + \frac{\Delta x^2}{3!} f^{(3)}(x)$ (order Δx^2).
- Approximation to second derivative by central differences
  - $\frac{f(x+\Delta x) - 2f(x) + f(x-\Delta x)}{\Delta x^2} = f''(x) + \frac{\Delta x^2}{12} f^{(4)}(x)$.
- 1D heat equation $\frac{\partial u}{\partial t} = \alpha^2 \frac{\partial^2 u}{\partial x^2}$, $u(0,t) = u(1,t) = 0, u(x,0) = f(x)$.
  - Discretion: $\frac{u(x,t+\Delta t) - u(x,t)}{\Delta t} \approx \alpha^2 \frac{u(x+\Delta x,t) - 2u(x,t) + u(x-\Delta x,t)}{\Delta x^2}$.
  - This gives $u_n^{k+1} = u_n^k + \alpha^2 \left(\frac{\Delta t}{\Delta x^2}\right)\left(u_{n+1}^k - 2u_n^k + u_{n-1}^k\right)$.
    - k for time step, n for position.
- Swift-Honhenberg equation $\frac{\partial u}{\partial t} = \epsilon u - \left(1 + \frac{\partial^2}{\partial x^2}\right)^2 u - u^3$.
- With heat equation in 2D, if $\frac{\partial u}{\partial t} = 0$, we have $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (Laplace's equation)
  - $u(x_n, y_m) \approx u_{n,m}$.
  - Discretion $\frac{u_{n+1,m} - 2u_{n,m} + u_{n-1,m}}{\Delta x^2} + \frac{u_{n,m+1} - 2u_{n,m} + u_{n,m-1}}{\Delta x^2} = 0$.
  - Suppose $\Delta x = \Delta y$, we have $u_{n,m} = \frac{u_{n+1,m} + u_{n-1,m} + u_{n,m+1} + u_{n,m-1}}{4}$.
  - $u_{n,m}^{k+1} = \frac{u_{n+1,m}^k + u_{n-1,m}^k + u_{n,m+1}^k + u_{n,m-1}^k}{4}$.
- Wave equation $u_{tt} = c^2 u_{xx}$, $u(0,t) = u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$
  - Discretion: $\frac{u_n^{k+1} - 2u_n^k + u_n^{k-1}}{\Delta t^2} = c^2 \frac{u_{n+1}^k - 2u_n^k + u_{n-1}^k}{\Delta x^2}$.

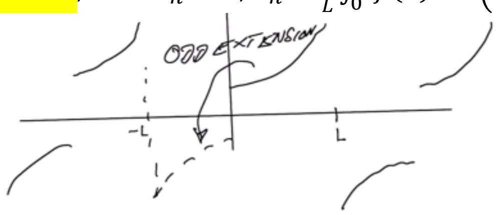
Fourier series

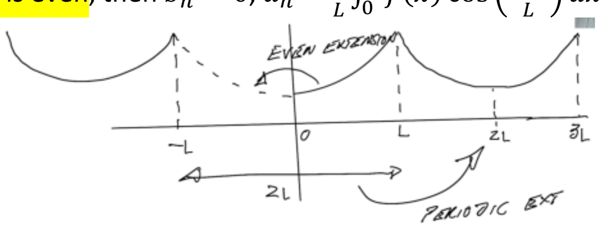
May 27, 2021 1:29 PM

Fourier series example

- $f(x) = x, L = 1.$
  - Then $b_n = \frac{2}{1}\int_0^1 x\sin(n\pi x)\,dx = \frac{2(-1)^{n+1}}{n\pi}$
  - So $f(x) = x \sim \frac{2}{\pi}\sum_1^\infty \frac{(-1)^{n+1}}{n}\sin(n\pi x).$
- Eigen function $f(x) = \sin(3\pi x).$
  - Then $b_n = 2\int \sin(3\pi x)\sin(n\pi x)\,dx = \begin{cases} 0, n \neq 3 \\ 1, n = 3 \end{cases} = \delta_{n3}.$
  - Note: $\delta_{nk} = \begin{cases} 0, n \neq k \\ 1, n = k \end{cases}$ is called the Kroneker delta function.
- Eigenvalue problems in the real world
  - Euler's beam $X'' + \left(\frac{P}{EI}\right)X = 0$, $X(0) = X(L) = 0.$
    - $\frac{P_n}{EI} = -\lambda_n = \left(\frac{n\pi}{L}\right)^2$, then $P_n = EI\left(\frac{n\pi}{L}\right)^2.$
    - $P_1 = \frac{EI\pi^2}{L^2}$ is the critical value (the first mode $\sin\left(\frac{\pi x}{L}\right)$).
  - Quantum mechanics ∞ well $V(x) = \begin{cases} V_0, |x| < L \\ \infty, |x| \geq L \end{cases}$, $\psi'' + \left(\frac{E - V_0}{\hbar^2/2\mathrm{m}}\right)\psi = 0, \psi(0) = \psi(L) = 0$
    - $\frac{E - V_0}{\hbar^2/2\mathrm{m}} = -\lambda = \left(\frac{n\pi}{L}\right)^2$, then $E_n = V_0 + \frac{\hbar^2}{2\mathrm{m}}\left(\frac{n\pi}{L}\right)^2.$

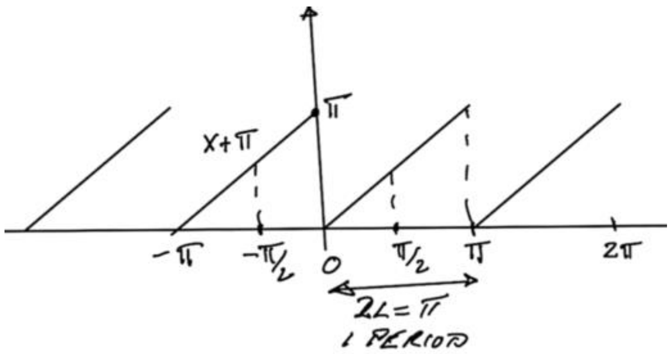
Full Fourier series

- $f(x) = \frac{a_0}{2} + \sum_1^\infty a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right).$
  - $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{C-L}^{C+L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx.$
  - $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{C-L}^{C+L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx.$
- If $f(x)$ is odd, then $a_n = 0$, $b_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx$ and $f(x) = \sum_1^\infty b_n\sin\left(\frac{n\pi x}{L}\right).$
  - 

- If $f(x)$ is even, then $b_n = 0$, $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$ and $f(x) = \frac{a_0}{2} + \sum_1^\infty a_n\cos\left(\frac{n\pi x}{L}\right).$
  - 

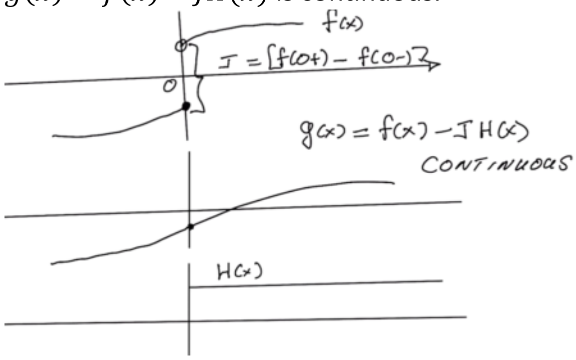
- E.g. $f(x) = \begin{cases} 0, -\pi < x < 0 \\ x, 0 < x < \pi \end{cases}$, find the Fourier series with period 2π, $L = \pi$.
  - $a_0 = \frac{1}{\pi}\int_0^\pi x\,dx = \frac{\pi}{2}.$
  - $a_n = \frac{1}{\pi}\int_0^\pi x\cos(nx)\,dx = -\frac{2}{\pi n^2}$ if n is odd.
  - $b_n = \frac{1}{\pi}\int_0^\pi x\sin(nx)\,dx = \frac{(-1)^{n+1}}{n}.$
  - So $f(x) = \frac{\pi}{4} - \frac{2}{\pi}\sum_0^\infty \frac{\cos(2k+1)x}{(2k+1)^2} + \sum_1^\infty \frac{(-1)^{n+1}}{n}\sin(nx).$
- E.g. find the Fourier sine series for $f(x) = x$ on $[0, \pi]$

- $b_n = \frac{2}{\pi}\int_0^{\pi} x\sin(nx)\,dx = 2\frac{(-1)^n}{n}$.
- $f(x) = 2\sum_1^{\infty}\frac{(-1)^n}{n}\sin(nx)$.

- E.g. Fourier cosine series for $f(x) = x$ on $[0,\pi]$.
  - $a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx = \pi$.
  - $a_n = \frac{2}{\pi}\int_0^{\pi} x\cos(nx)\,dx = -\frac{4}{\pi n^2}$ if n is odd.
  - So $f(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_0^{\infty}\frac{\cos(2k+1)x}{(2k+1)^2}$.
- E.g. determining the Fourier series of period π for the function $f(x) = x$ sampled on $[0,\pi]$.
  - Period: $2L = \pi$.
  - 

  - $a_0 = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)dx = \frac{2}{\pi}\left(\int_{-\frac{\pi}{2}}^{0} x + \pi dx + \int_0^{\frac{\pi}{2}} xdx\right) = \pi$.
  - $a_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = 0$.
  - $b_n = \frac{2}{\pi}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = -\frac{1}{n}$.
  - So $f(x) = \frac{\pi}{2} - \sum_1^{\infty}\frac{\sin(2nx)}{n}$.

Convergence of Fourier series and the Gibbs phenomenon

- Let f and f' be piecewise continuous functions defined on $[-L,L]$ and let f be periodic with period $2L$, then $f(x) = \frac{a_0}{2} + \sum_1^{\infty} a_n\cos\left(\frac{n\pi x}{L}\right) + b_n\sin\left(\frac{n\pi x}{L}\right)$.
  - $a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{C-L}^{C+L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$.
  - $b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \frac{1}{L}\int_{C-L}^{C+L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx$.
- And the Fourier series converges to $f(x)$ at points where f is continuous and to $\frac{1}{2}\left(f(x_{0+}) + f(x_{0-})\right)$ at x_0 where f is discontinuous
- The Gibbs phenomenon: jump is equivalent to a step function
  - Jump: $J = f(x_{o+}) - f(x_{0-})$.
  - $g(x) = f(x) - JH(x)$ is continuous.
  - 

  - All jumps can be expressed by the Heaviside function
  - The Fourier series for the Heaviside function
    - $b_n = \frac{4}{\pi}\frac{1}{2k+1}$, $k = 0,1,\ldots$, $s(x) = \frac{4}{\pi}\sum_0^{\infty}\frac{\sin(2k+1)x}{2k+1}$.
    - $s_N'(x) \to \frac{4(N+1)}{\pi}$ as $x \to 0$.

- If we want $s_N'(x_0) = 0$, then $x_0 = \frac{\pi}{2(N+1)}$,
- Convergence is pointwise but not uniform

Heat equation

May 10, 2021 10:12 AM

Dirichlet boundary value problems

- Given $u_t = \alpha^2 u_{xx}$ $0 < x < L, t > 0$ $u(0,t) = u(L,t) = 0$, $u(x,0) = f(x)$.
- From previous info, $u(x,t) = e^{ikx}e^{-\alpha^2 k^2 t}$
  - k is called the wave number, $e^{ik(x+\Delta)} = e^{ikx}$.
- Need to find the k values so that the soulution matches the boundary conditions. These values are determined by the solution of an eigenvalue problem
- Separation of variables
  - Guess $u(x,t) = X(x)T(t)$.
  - Plug into $u_t = \alpha^2 u_{xx}$, we have $X(x)T'(t) = \alpha^2 X''(x)T(t)$.
  - Divide both sides by $\alpha^2 X(x)T(t)$, $\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)}$.
  - They must equal to a constant λ ($\frac{T'(t)}{\alpha^2 T(t)} = \frac{X''(x)}{X(x)} = t$) because x and t are independent.
  - For $\frac{T'}{\alpha^2 T} = \lambda$, we get $\frac{dT}{T} = \lambda\alpha^2$, $T = Ce^{\alpha^2 \lambda t}$.
  - For $\frac{X''}{X} = \lambda$, $X(0) = X(L) = 0$ (by boundary condition $u(0,t) = u(L,t) = 0$)
    - we get an Eigenvalue problem $X'' = \lambda X$, $X(0) = X(L) = 0$.
    - If $\lambda = \mu^2 > 0$, $X'' - \mu^2 X = 0$, we get $X = A\cosh \mu x + B\sinh \mu x$
      - In this case $0 = X(0) = A$.
      - $0 = X(L) = B\sinh \mu L$, then $B = 0$.
      - Then $X = 0$.
    - If $\lambda = 0$, we get $X = Ax + B$
      - still have the trivial solution $X = 0$.
    - If $\lambda = -\mu^2 < 0$, we get $X = A\cos \mu x + B\sin \mu x$.
      - This gives $A = 0$, $\mu L = n\pi$
      - So $X_n(x) = \sin\left(\frac{n\pi}{L}x\right)$ are eigen functions.
      - Then corresponding eigenvalues are $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$.
  - Final solution $u_n(x,t) = X_n(x)T_n(t) = \sin\left(\frac{n\pi}{L}x\right)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$.
  - Because the PDE is linear, a linear combination of solutions is also a solution.
    - Then we have the general solution $u(x,t) = \sum_1^\infty b_n \sin\left(\frac{n\pi}{L}x\right)e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}$.
    - Using the initial condition $u(x,0) = f(x)$, we get $\sum_1^\infty b_n \sin\left(\frac{n\pi}{L}x\right) = f(x)$.
      - This is the Fourier series expansion of $f(x)$.

Neumann boundary conditions

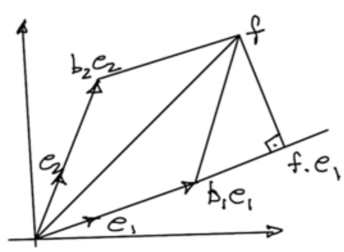
- $u_t = \alpha^2 u_{xx}$, $u_x(0,t) = u_x(L,t) = 0$, $u(x,0) = f(x)$.
- Using separation of variables, we get $T(t) = Ce^{\alpha^2 \lambda t}$.
- And eigenvalue problem gives $X(x) = A\cos \mu x + B\sin \mu x$.
  - When $\lambda = -\mu^2 < 0$, $X'(x) = -A\mu\sin \mu x + B\mu\cos \mu x$, plugging in $X'(0) = X'(L) = 0$.
    - Then $B = 0$, $\mu_n = \frac{n\pi}{L}$, $\lambda_n = -\left(\frac{n\pi}{L}\right)^2$, $X_n(x) = \cos\left(\frac{n\pi x}{L}\right)$.
  - When $\lambda = 0$, eigen function $X_0 = 1$.
  - When $\lambda = \mu^2 > 0$, trivial solution $X = 0$.
  - Overall, $\lambda_n \in \{0\} \cup \left\{-\left(\frac{n\pi}{L}\right)^2\right\}_{n=1}^{\infty}$, and corresponding $X_n \in \{1\} \cup \left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_{n=1}^{\infty}$.
  - Period of eigen functions $\cos\left(\frac{n\pi}{L}\right)$, $P_n = \frac{2L}{n}$.
    - Fundamental period: $P_1 = 2L$.
- General solution: $u(x,t) = A_0 + \sum_1^\infty A_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi x}{L}\right)$.

- Initial condition: $f(x) = u(x,0) = A_0 + \sum_1^\infty A_n \cos\left(\frac{n\pi x}{L}\right)$.
  - $A_0 = \frac{1}{L}\int_0^L f(x)\,dx$.
  - $A_k = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{k\pi x}{L}\right)dx$.
  - Alternative form: let $a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \begin{cases} A_n, n \ge 1 \\ 2A_0, n = 0 \end{cases}$
    - Then, $f(x) = \frac{a_0}{2} + \sum_1^\infty a_n \cos\left(\frac{n\pi x}{L}\right)$.

Periodic boundary conditions and full Fourier series

- $u_t = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta}$.
- If $u(r,\theta) = u(\theta)$ (r constant, $u(r\pi) = u(-r\pi)$), we have $u_t = \frac{1}{r^2}u_{\theta\theta} = u_{ss}$ where $s = r\theta$.
- Let $L = r\pi, s = x$, we get $u_t = \alpha^2 u_{xx}, -L < x < L$
  - Let the BC and IC be $u(-L,t) = u(L,t)$, $u_x(-L,t) = u_x(L,t)$, $u(x,0) = f(x)$.
  - Separation of variable, $T(t) = Ce^{\alpha^2\lambda t}$.
  - And the eigen value problem is $X'' = \lambda X, X(-L) = X(L), X'(-L) = X'(L)$.
    - When $\lambda = \mu^2 > 0, X(x) = A\cosh\mu x + B\sinh\mu x$ and BC gives $X = 0$.
    - When $\lambda = 0, X(x) = Ax + B$, and BC gives $X = 1$.
    - When $\lambda = -\mu^2 < 0, X(x) = A\cos\mu x + B\sin\mu x$ and BC gives $\mu_n = \frac{n\pi}{L}$.
  - Eigenvalues: $\lambda_n \in \{0\} \cup \left\{-\left(\frac{n\pi}{L}\right)^2\right\}$.
  - Eigen functions: $X_n(x) \in \{1\} \cup \left\{A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right\}$.
  - General solution $u(x,t) = A_0 + \sum_1^\infty e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\left\{A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right\}$.
  - The IC gives the full Fourier series $u(x,t) = A_0 + \sum_1^\infty \left\{A_n\cos\left(\frac{n\pi x}{L}\right) + B_n\sin\left(\frac{n\pi x}{L}\right)\right\}$.
    - $A_0 = \frac{1}{2L}\int_{-L}^L f(x)dx = \frac{a_0}{2}$.
    - $A_k = \frac{1}{L}\int_{-L}^L f(x)\cos\left(\frac{k\pi x}{L}\right)dx = a_k$.
    - $B_k = \frac{1}{L}\int_{-L}^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx = b_k$.

Determining the Fourier coefficients

- Expanding a vector f in terms of basis vectors e_1 and e_2.
- 

  $f = b_1e_1 + b_2e_2$

  $f.e_k = b_1 e_1.e_k + b_2 e_2.e_k \quad k=1,2.$

  $$\begin{bmatrix} e_1.e_1 & e_2.e_1 \\ e_1.e_2 & e_2.e_2 \end{bmatrix}\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} f.e_1 \\ f.e_2 \end{bmatrix}$$

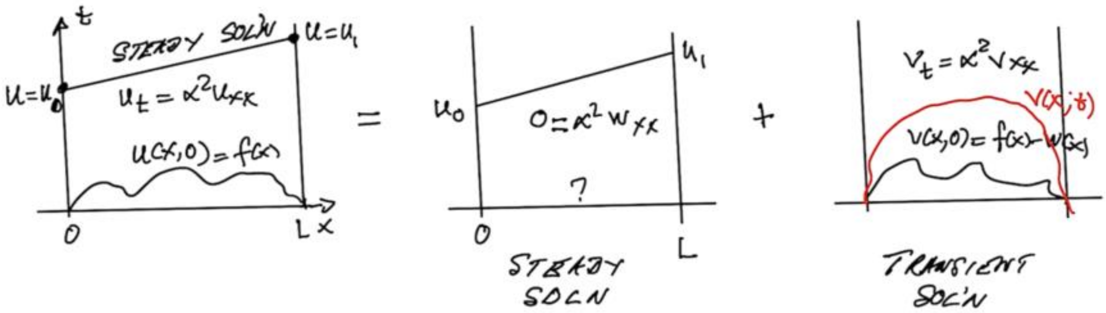
  - If $e_1 \perp e_2$, we have $e_1 \cdot e_2 = 0$, $b_k = \frac{f\cdot e_k}{e_k\cdot e_k}$.
- If $f = [f_1\ f_2 \ldots f_n]$ and $g = [g_1\ g_2 \ldots g_n]$ are vectors of points $f_i = f(x_i)$, then $f\cdot g = \sum_1^n f_i g_i$.
  - Then $f\cdot g \approx \int_0^L f(x)g(x)dx$
- Using above, we have $b_k = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{k\pi x}{L}\right)dx$.

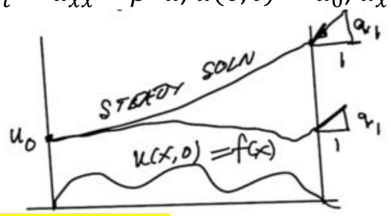
Summary of eigen functions subject to homogeneous conditions (all $n \ge 1$)

- PDE $u_t = \alpha^2 u_{xx}$, IC: $u(x,0) = f(x)$.
- Dirichlet boundary condition $u(0,t) = u(L,t) = 0$:
  - Eigen values: $\mu_n = \frac{n\pi}{L}$.
  - Eigen functions: $X_n = \sin\left(\frac{n\pi x}{L}\right)$.

- Neumann boundary conditions $u_x(0,t) = u_x(L,t) = 0$:
  - Eigen values: $\mu_n \in \{0\} \cup \left\{\frac{n\pi}{L}\right\}_{n=1}^{\infty}$
  - Eigen functions: $X_n \in \{1\} \cup \left\{\cos\left(\frac{n\pi x}{L}\right)\right\}_1^{\infty}$.
- Periodic boundary condition $u(-L,t) = u(L,t), u_x(-L,t) = u_x(L,t)$:
  - Eigen values: $\mu_n \in \{0\} \cup \left\{\frac{n\pi}{L}\right\}_{n=1}^{\infty}$
  - Eigen functions: $X_n \in \{1\} \cup \left\{\cos\left(\frac{n\pi x}{L}\right), \sin\left(\frac{n\pi x}{L}\right)\right\}_1^{\infty}$.
- Mixed boundary condition A $u(0,t) = u_x(L,t) = 0$:
  - Eigen values: $\mu_n = \frac{(2n-1)\pi}{2L}$
  - Eigen functions: $X_n = \sin\left(\frac{(2n-1)\pi x}{2L}\right)$
    - Period: $4L$
- Mixed boundary condition B $u_x(0,t) = u(L,t) = 0$:
  - Eigen values: $\mu_n = \frac{(2n-1)\pi}{2L}$
  - Eigen functions: $X_n = \cos\left(\frac{(2n-1)\pi x}{2L}\right)$

Heat equation subject to inhomogeneities

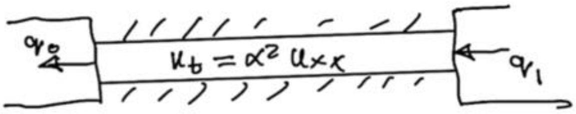
- PDE: $u_t = \alpha^2 u_{xx} + g(x,t), u(0,t) = \phi_0(t), u(L,t) = \phi_1(t), u(x,0) = f(x)$.
- Special case, $g = 0, \phi_0 = u_0, \phi_1 = u_1$ constant, this gives the steady state solution.
- 

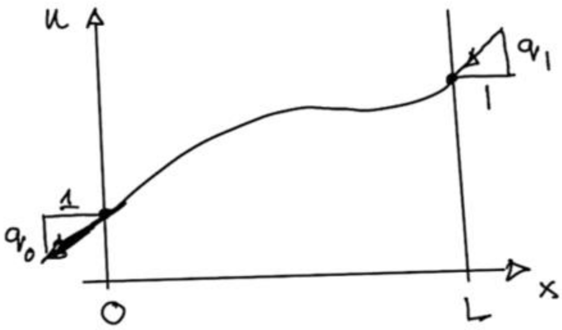
- Initial boundary value problem (IBVP):
  - $u_t = \alpha^2 u_{xx}, u(0,t) = u_0, u(L,t) = u_1, u(x,0) = f(x)$.
  - We want to find a steady state solution $w(x)$, such that $w(x,t) = w(x)$, $w_t = 0$.
    - This gives that $w = Ax + B$,
    - Match BC, $w = \frac{u_1 - u_0}{L}x + u_0$.
  - Transient solution: let $u(x,t) = w(x) + v(x,t)$.
    - $u_t = w_t + v_t = v_t = \alpha^2 u_{xx} = \alpha^2(w_{xx} + v_{xx}) = \alpha^2 v_{xx}$.
    - Boundary condition $u_0 = u(0,t) + w(0) + v(0,t)$, so $v(0,t) = 0$, similarly, $v(L,t) = 0$.
    - Initial condition $v(x,0) = f(x) - w(x) = g(x)$.
    - $v(x,t)$ satisfies a homogenous PDE.
    - $v(x,t) = \sum_1^{\infty} b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$.
      - $b_n = \frac{2}{L}\int_0^L (f(x) - w(x)) \sin\left(\frac{n\pi x}{L}\right) dx$.
  - Finally, $u(x,t) = \frac{u_1 - u_0}{L}x + u_0 + \sum_1^{\infty} b_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi x}{L}\right)$.
    - The transient part goes to 0 as $t \to \infty$.
- Heat equation with a loss term
  - $u_t = u_{xx} - \beta^2 u, u(0,t) = u_0, u_x(L,t) = q_1, u(x,0) = f(x)$.
  - 

  - Steady solution: $w(x)$, with $w_t = 0$

        - $w_t = 0 = w_{xx} - \beta^2 w$.
        - $w(x) = A\cosh\beta x + B\sinh\beta x$, $w_x = A\beta\sinh\beta x + B\beta\cosh\beta x$.
        - This gives $A = u_0$, $B = \frac{\frac{q_1}{\beta} - u_0\sinh\beta L}{\cosh\beta L}$.
        - So $w(x) = u_0\cosh\beta x + \left(\frac{\frac{q_1}{\beta} - u_0\sinh\beta L}{\cosh\beta L}\right)\sinh\beta x = u_0\frac{\cosh\beta(L-x)}{\cosh\beta L} + \frac{q_1}{\beta}\frac{\sinh\beta x}{\cosh\beta L}$.
    - Transient solution
        - Let $u(x,t) = w(x) + v(x,t)$, so $v_t = w_{xx} - \beta^2 w + (v_{xx} - \beta^2 v) = v_{xx} - \beta^2 v$.
        - The conditions are: $v(0,t) = v_x(L,t) = 0$, $v(x,0) = f(x) - w(x)$.
        - Let $v(x,t) = XT$, then $XT' = X''T - \beta^2 XT$, $\frac{T'}{T} + \beta^2 = \frac{X''}{X} = \lambda$
        - Then $T(t) = Ce^{(\lambda-\beta^2)t}$, $X_n = B_n\sin\left(\frac{(2n-1)\pi}{2L}x\right)$, $\lambda_n = -\left(\frac{(2n-1)\pi}{2L}\right)^2$.
        - So $v(x,t) = \sum_{n=1}^{\infty} b_n e^{-\left(\frac{(2n-1)\pi}{2L}\right)^2 t} e^{-\beta^2 t}\sin\left(\frac{(2n-1)\pi}{2L}x\right)$.
            - $b_n = \frac{2}{L}\int_0^L \big(f(x) - w(x)\big)\sin\left(\frac{(2n-1)\pi}{2L}x\right)dx$.

- Inhomogeneous Neumman BC and a particular solution
    - $u_t = \alpha^2 u_{xx}$, $u_x(0,t) = q_0$, $u_x(L,t) = q_1$, $u(x,0) = f(x)$.



    - 

    - Steady solution: $w(x)$, with $w_t = 0$.
        - $w_{xx} = 0$, $w(x) = Ax + B$, but this gives $q_0 = w_x(0) = A = w_x(L) = q_1$.
        - Unless $q_0 = q_1$, there is no steady solution.
    - Particular solution $w(x,t) = Ax^2 + Bx + Ct$.
        - $C = w_t = \alpha^2 w_{xx} = 2\alpha^2 A$, so $C = 2A\alpha^2$.
        - $q_0 = w_x(0,t) = B$, $q_1 = w_x(L,t) = 2AL + B = 2AL + q_0$, then $A = \frac{q_1 - q_0}{2L}$.
        - $C = 2A\alpha^2 = \alpha^2\frac{q_1 - q_0}{L}$.
        - $w(x,t) = \left(\frac{q_1 - q_0}{2L}\right)x^2 + q_0 x + \left(\alpha^2\frac{q_1 - q_0}{L}\right)t$.
    - Let $u(x,t) = w(x,t) + v(x,t)$.
        - $v_t = \alpha^2 v_{xx}$, $v_x(0,t) = v_x(L,t) = 0$, $v(x,0) = f(x) - w(x,0)$.
        - $X_n \in \{1\} \cup \{\cos\left(\frac{n\pi x}{L}\right)\}$.
        - So $v(x,t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n e^{-\alpha^2\left(\frac{n\pi}{L}\right)^2 t}\cos\left(\frac{n\pi}{L}x\right)$.
            - $a_n = \frac{2}{L}\int_0^L \big(f(x) - w(x,0)\big)\cos\left(\frac{n\pi}{L}x\right)dx$.

- Heat equation with space varying source.
    - $u_t = u_{xx} + x$, $u(0,t) = 0$, $u(\pi,t) = u_1$, $u(x,0) = f(x)$.
    - Steady solution $w(x)$, $w_t = 0$.
        - $0 = w_{xx} + x$.
        - This gives $w(x) = -\frac{x^3}{6} + Ax + B$.
        - $B = 0$, $u_1 = -\frac{\pi^3}{6} + A\pi$, $A = \frac{u_1}{\pi} + \frac{\pi^3}{6}$.
        - So $w(x) = -\frac{x^3}{6} + \left(\frac{u_1}{\pi} + \frac{\pi^3}{6}\right)x$.
    - Transient solution:

- $v_t = u_{xx} + x = (w_{xx} + x) + v_{xx}$, $v_t = v_{xx}$, $v(0,t) = v(\pi,t) = 0$, $v(x,0) = f(x) - w(x)$.
- $v(x,t) = \sum_1^\infty b_n e^{-n^2 t} \sin(nx)\, dx$.
  - $b_n = \frac{2}{\pi}\int_0^\pi (f(x) - w(x)) \sin(nx)\, dx$.

- Second method (using an eigenfunction expansion):
  - Look for simplest function $w(x)$ that satisfies the boundary condition.
    - $w(x) = \frac{u_1 x}{\pi}$.
  - Get rid of the inhomogeneous BC by letting $u(x,t) = w(x) + v(x,t)$.
    - $v_t = v_{xx} + x$, $v(0,t) = 0$, $v(\pi,t) = 0$, $v(x,0) = f(x) - w(x)$.
    - Eigen values/functions for the homogeneous BC: $\lambda_n = -n^2$, $X_n = \sin(nx)$.
    - Expand $s(x) = x = \sum_1^\infty s_n \sin(nx)$, $s_n = 2\frac{(-1)^{n+1}}{n}$.
    - Let $v(x,t) = \sum_1^\infty v_n(t) \sin(nx)$.
      - Then $v_t = \sum_1^\infty v_n'(t) \sin(nx)$, $v_{xx} = \sum_1^\infty v_n(t)(-n^2) \sin(nx)$.
    - So $0 = v_t - v_{xx} - x = \sum_1^\infty (v_n' + n^2 v_n - s_n) \sin(nx)$.
    - Then $v_n'(t) + n^2 v_n = s_n$ (since $\sin(nx)$ are lineary independent).
    - So $v_n(t) = \frac{s_n}{n^2} + c_n e^{-n^2 t}$.
    - $v(x,t) = \sum_1^\infty (\frac{s_n}{n^2} + c_n e^{-n^2 t}) \sin(nx)$.
    - Apply the IC, $f(x) - w(x) = \sum_1^\infty \left(\frac{s_n}{n^2} + c_n\right) \sin(nx)$.
      - $c_n = f_n - s_n \frac{u_1}{\pi} - \frac{s_n}{n^2}$.

- Time and space varying source.
  - $u_t = u_{xx} + e^{-t} \sin 2x$, $0 \le x \le \pi$, $u(0,t) = u(\pi,t) = 0$, $u(x,0) = 0$.
  - Eigenvalues: $\lambda_n = -\left(\frac{n\pi}{\pi}\right)^2 = -n^2$, $X_n(x) = \sin(nx)$
  - Expand the source in terms of the eigen functions.
    - $e^{-t} \sin 2x = \sum_{n=1}^\infty s_n(t) \sin(nx)$, so $s_n(t) = \begin{cases} e^{-t}, n = 2 \\ 0, n \ne 2 \end{cases} = e^{-t}\delta_{n2}$.
  - Expand $u(x,t)$ as a series of eigen functions.
    - Let $u(x,t) = \sum_{n=1}^\infty u_n(t) \sin(nx) = \sum_{n=1}^\infty T_n(t) X_n(x)$.
    - $u_t = \sum_{n=1}^\infty u_n'(t) \sin(nx)$, $u_{xx} = \sum_{n=1}^\infty u_n(t)(-n^2) \sin(nx)$.
  - Then $0 = u_t - u_{xx} - e^{-t}\sin(2x) = \sum_{n=1}^\infty (u_n'(t) + n^2 u_n - e^{-t}\delta_{n2}) \sin(nx)$.
  - Then we have $u_n'(t) + n^2 u_n = e^{-t}\delta_{n2}$, since the terms are linearly independent.
  - So $u_n(t) = \frac{1}{n^2 - 1} e^{-t}\delta_{n2} + C_n e^{-n^2 t}$.
  - Then $u(x,t) = \sum_{n=1}^\infty \left(\frac{1}{n^2-1} e^{-t}\delta_{n2} + C_n e^{-n^2 t}\right) \sin(nx)$.
  - Using the IC, $C_n = \begin{cases} -\frac{1}{3}, n = 2 \\ 0, n \ne 2 \end{cases}$, $u(x,t) = \frac{1}{3}(e^{-t} - e^{-4t}) \sin(2x)$.

- Time dependent boundary conditions
  - $u_t = u_{xx}$, $0 < x < L$, $u(x,0) = f(x)$.
  - Dirichelet: $u(0,t) = \phi_0(t)$, $u(L,t) = \phi_1(t)$.
    - Simplest functions that match BC, $w(x,t) = A(t)x + B(t)$.
      - $\phi_0(t) = w(0,t) = B(t)$, $\phi_1(t) = w(L,t) = A(t)L + \phi_0(t)$, so $A(t) = \frac{\phi_1(t) - \phi_0(t)}{L}$.
      - $w(x,t) = \frac{\phi_1(t) - \phi_0(t)}{L} x + \phi_0(t)$.
    - Let $u(x,t) = w(x,t) + v(x,t)$, $v(x,0) = f(x) - w(x,0)$.
      - $u_t = w_t + v_t = \frac{\phi_1'(t) - \phi_0'(t)}{L} x + \phi_0'(t) = u_{xx} = w_{xx} + v_{xx} = v_{xx}$.
      - So $v_t = v_{xx} - \left(\frac{\phi_1'(t) - \phi_0'(t)}{L} x + \phi_0'(t)\right)$, with homogeneous boundary conditions.
      - We can then get the solutions by eigen expansion on $v(x,t)$.
  - Neumann: $u_x(0,t) = q_0(t)$, $u_x(L,t) = q_1(t)$.
    - Functions that match BC, $w(x,t) = A(t)x^2 + B(t)x$.
      - $q_0(t) = w_x(0,t) = B(t)$, $q_1(t) = w_x(L,t) = 2A(t)L + q_0(t)$.

        - $w(x,t) = \frac{q_1(t)-q_0(t)}{2L}x^2 + q_0(t)x$.
    - Let $u(x,t) = w(x,t) + v(x,t)$, $v(x,0) = f(x) - w(x,0)$.
        - $u_t = w_t + v_t = u_{xx} = w_{xx} + v_{xx}$.
        - So $v_t = v_{xx} + \left(\frac{q_1-q_0}{L}\right) - \left(\frac{q_1'-q_0'}{2L}x^2 + q_0'x\right)$, with homogeneous boundary conditions.
- Mixed boundary: $u(0,t) = \phi_0(t)$, $u_x(L,t) = q_1(t)$.
    - Let $w(x,t) = A(t)x + B(t)$.
        - $w(x,t) = q_1(t)x + \phi_0(t)$.
    - Let $u(x,t) = w(x,t) + v(x,t)$, $v(0,t) = 0$, $v_x(L,t) = 0$.
        - $u_t = w_t + v_t = u_{xx} = w_{xx} + v_{xx}$.
        - So $v_t = v_{xx} - \left(q_1'(t)x + \phi_0'(t)\right)$ with homogeneous boundary conditions.
- Mixed boundary: $u_x(0,t) = q_0(t)$, $u(L,t) = \phi_1(t)$.
    - Let $w(x,t) = A(t)x + B(t)$.
        - $w(x,t) = q_0(t)(x-L) + \phi_1(t)$.
    - Let $u(x,t) = w(x,t) + v(x,t)$, $v(0,t) = 0$, $v_x(L,t) = 0$.
        - $u_t = w_t + v_t = u_{xx} = w_{xx} + v_{xx}$.
        - So $v_t = v_{xx} - \left(q_0'(t)(x-L) + \phi_1'(t)\right)$ with homogeneous boundary conditions.

E.g. $u_t = u_{xx}$, $u(0,t) = \frac{t^2}{2}$, $u_x\left(\frac{\pi}{2},t\right) = 1$, $u(x,0) = x$

- Let $w(x,t) = A(t)x + B(t)$, $\frac{t^2}{2} = w(0,t) = B(t)$, $1 = w_x\left(\frac{\pi}{2},t\right) = A(t)$.
    - So $w(x,t) = x + \frac{t^2}{2}$.
- Let $u(x,t) = w(x,t) + v(x,t)$, then $v(0,t) = v_x\left(\frac{\pi}{2},t\right) = 0$, $v(x,0) = u(x,0) - w(x,0) = 0$
    - $u_t = w_t + v_t = u_{xx} = w_{xx} + v_{xx}$.
    - So $v_t = v_{xx} - t$.
    - $\lambda_n = -\left(\frac{(2n-1)\pi}{2(\pi/2)}\right)^2 = -(2n-1)^2$, $X_n = \sin\big((2n-1)x\big)$.
    - Let $s(x,t) = -t = \sum_1^\infty s_n(t)\sin\big((2n-1)x\big)$.
        - This gives $s_n(t) = \frac{4}{\pi}\int_0^{\frac{\pi}{2}} -t\sin\big((2n-1)x\big)\,dx = \frac{-4t}{(2n-1)\pi}$.
    - Let $v(x,t) = \sum_1^\infty v_n(t)\sin\big((2n-1)x\big)$.
        - $v_t = \sum_1^\infty v_n'(t)\sin\big((2n-1)x\big)$, $v_{xx} = \sum_1^\infty v_n(t)\big(-(2n-1)^2\big)\sin\big((2n-1)x\big)$.
        - Then $0 = v_t - v_{xx} + t = \sum_1^\infty \left(v_n' + (2n-1)^2 v_n - \frac{4t}{(2n-1)\pi}\right)\sin\big((2n-1)x\big)$.
        - So we must have $v_n' + (2n-1)^2 v_n = \frac{4t}{(2n-1)\pi}$.
        - $v_n = \frac{4}{\pi(2n-1)}\left(\frac{t}{(2n-1)^2} - \frac{1}{(2n-1)^4} + c_n e^{-(2n-1)^2 t}\right)$.
        - $v(x,0) = \sum_1^\infty \frac{4}{\pi(2n-1)}\left(-\frac{1}{(2n-1)^4} + c_n\right)\sin\big((2n-1)x\big)$.
            - So $c_n = \frac{1}{(2n-1)^4}$.
        - $v(x,t) = \sum_1^\infty \frac{4}{\pi(2n-1)}\left(\frac{t}{(2n-1)^2} + \frac{e^{-(2n-1)^2 t} - 1}{(2n-1)^4}\right)\sin\big((2n-1)x\big)$.
    - $u(x,t) = x + \frac{t^2}{2} + \sum_1^\infty \frac{4}{\pi(2n-1)}\left(\frac{t}{(2n-1)^2} + \frac{e^{-(2n-1)^2 t} - 1}{(2n-1)^4}\right)\sin\big((2n-1)x\big)$.

Wave equation

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1D wave equation

- $u_{tt} = c^2 u_{xx}$.
- The exponential solution: $u = e^{ik(x \pm ct)}$ (oscillation).
  - Lines $x \pm ct = x_0$ are characteristics.
- General solution to the wave equation
  - Let $u(x,t) = F(x - ct)$ (wave moving to the right), $u_{tt} = c^2 F''(x - ct)$, $u_{xx} = F''(x - ct)$
  - Same for $F(x + ct)$ (wave moving to the left).
  - So general solution is $u(x,t) = F(x - ct) + G(x + ct)$.
- D'Alembert's solution to 1D wave equation for an infinite string
  - $u_{tt} = c^2 u_{xx}, u(x,0) = f(x), u_t(x,0) = g(x), x \in \mathbb{R}$.
  - Let $u(x,t) = F(x - ct) + G(x + ct), u_t = -cF'(x - ct) + cG'(x + ct)$
    - $u(x,0) = F(x) + G(x) = f(x)$.
    - $u_t(x,0) = -cF'(x) + cG'(x) = g(x)$.
      - $-cF(x) + cG(x) = \int_0^t g(s)ds + A$.
    - This gives $G(x) = \frac{1}{2}f(x) + \frac{1}{2c}\int_0^t g(s)ds + \frac{A}{2c}$.
    - And $F(x) = \frac{1}{2}f(x) - \frac{1}{2c}\int_0^t g(s)ds - \frac{A}{2c}$.
  - General solution: $u(x,t) = \frac{1}{2}\big(f(x - ct) + f(x + ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)ds$.
- Initial boundary value problem on a finite string.
  - $u_{tt} = c^2 u_{xx}, u(0,t) = u(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x), 0 < x < L$.
  - By separation of variable, $XT'' = c^2 X''T$, this gives $\frac{T''}{c^2 T} = \frac{X''}{X}$.
  - Solving eigenvalue problems
    - $X'' = \lambda X$, only nontrivial solution $\lambda = -\mu^2 < 0$, $\mu_n = \frac{n\pi}{L}$, $X_n = \sin\left(\frac{n\pi x}{L}\right)$.
    - $T_n'' + c^2\mu_n^2 T_n = 0$, $T_n = A_n \cos\mu_n ct + B_n \sin\mu_n ct$.
  - General solution, $u(x,t) = \sum_1^\infty \big(A_n \cos\mu_n ct + B_n \sin\mu_n ct\big)\sin\mu_n x$.
  - Using the initial condition, $f(x) = \sum_1^\infty A_n \sin\mu_n x = \sum_1^\infty f_n \sin\mu_n x$.
    - $A_n = f_n = \frac{2}{L}\int_0^L f(x)\sin\mu_n x\, dx$.
  - $g(x) = u_t(x,0) = \sum_1^\infty B_n\mu_n c\sin\mu_n x = \sum_1^\infty g_n \sin\mu_n x$.
    - $B_n\mu_n c = g_n = \frac{2}{L}\int_0^L g(x)\sin\mu_n x\, dx$.
  - So $u(x,t) = \sum_1^\infty \left(f_n \cos\mu_n ct + \frac{g_n}{\mu_n c}\sin\mu_n ct\right)\sin\mu_n x$.
    - $f_n = \frac{2}{L}\int_0^L f(x)\sin\mu_n x\, dx$.
    - $g_n = \frac{2}{L}\int_0^L g(x)\sin\mu_n x\, dx$.
  - Interpretation in terms of D'Alembert's solution
    - Using trig identities, $u(x,t) = \sum_1^\infty f_n \frac{1}{2}\big(\sin\mu_n(x - ct) + \sin\mu_n(x + ct)\big) + \sum_1^\infty \frac{g_n}{\mu_n c}\frac{1}{2}\big(\cos\mu_n(x - ct) - \cos\mu_n(x + ct)\big)$.
    - However, $\sum_1^\infty f_n \sin\mu_n z = f_{2L}^O(z)$, and $\sum_1^\infty g_n \sin\mu_n z = g_{2L}^o(z)$,
      - Where f_{2L}^O means the odd extenstion of f with period $2L$.
      - So first term becomes $\frac{1}{2}\big(f_{2L}^0(x - ct) + f_{2L}^0(x + ct)\big)$.
      - $\int g_{2L}^o(z)dz = -\sum_1^\infty \frac{g_n}{\mu_n}\cos\mu_n z + C$ gives second term $\frac{1}{2c}\int_{x-ct}^{x+ct} g_{2L}^o(z)dz$.
    - So $u(x,t) = \frac{1}{2}\big(f_{2L}^0(x - ct) + f_{2L}^0(x + ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g_{2L}^o(z)dz$.
      - Essentially, we just replace the function with odd extension of period $2L$.
- If giving Neumann boundary condition:
  - $u_{tt} = c^2 u_{xx}, u_x(0,t) = u_x(L,t) = 0, u(x,0) = f(x), u_t(x,0) = g(x)$.
  - We have an even extension

The eigen values and eigen functions are the same as the heat equation

Wave equations with time dependent BC and sources using eigen expansions.

- $u_{tt} = c^2 u_{xx} + s(x,t), u(x,0) = f(x), u_t(x,0) = g(x)$.
- Dirichlet:$u(0,t) = \phi_0(t), u(L,t) = \phi_1(t)$.
  - $w(x,t) = \frac{\phi_1(t)-\phi_0(t)}{L} x + \phi_0(t)$.
  - Let $u(x,t) = v(x,t) + w(x,t), v(0,t) = v(L,t) = 0$
    - $v(x,0) = f(x) - w(x,0), v_t(x,0) = g(x) - w_t(x,0)$.
    - $v_{tt} = c^2 v_{xx} + s(x,t) - \left(\frac{\phi_1''(t)-\phi_0''(t)}{L} x + \phi_0''(t)\right)$.
- Mixed boundary: $u(0,t) = \phi_0(t), u_x(L,t) = q_1(t)$.
  - $w(x,t) = q_1(t)x + \phi_0(t)$.
  - E.g. $u_{tt} = u_{xx} + e^{-t}\sin 5x, u(0,t) = 0, u_x\left(\frac{\pi}{2}, t\right) = t, u(x,0) = 0, u_t(x,0) = \sin 3x + x$.
    - $w(x,t) = tx$.
    - Let $u(x,t) = v(x,t) + w(x,t)$.
      - $v_{tt} = v_{xx} + e^{-t}\sin 5x, v(0,t) = v_x\left(\frac{\pi}{2}, t\right) = t, v(x,0) = 0, v_t(x,0) = \sin 3x$.
      - Eigen: $\lambda_n = -(2n-1)^2, X_n(x) = \sin\big((2n-1)x\big)$.
      - $e^{-t}\sin(5x) = s(x,t) = \sum_1^\infty s_n(t)\sin((2n-1)x)$.
        - $s_n(t) = e^{-t}\delta_{n3}$.
      - Let $v(x,t) = \sum_1^\infty v_n(t)\sin((2n-1)x)$
      - $0 = v_{tt} - v_{xx} - e^{-t}\sin 5x = \sum_1^\infty \left(v_n''(t) + (2n-1)^2 v_n - e^{-t}\delta_{n3}\right)\sin\big((2n-1)x\big)$.
      - We then need to solve $v_n''(t) + (2n-1)^2 v_n = e^{-t}\delta_{n3}$.
        - Homogeneous solution $v_n(t) = A_n\cos(2n-1)t + B_n\sin(2n-1)t$.
        - Particular solution: guess $v_n(t) = c_n e^{-t}$.
          - $c_n = \frac{\delta_{n3}}{1+(2n-1)^2}$.
        - $v_n(t) = A_n\cos(2n-1)t + B_n\sin(2n-1)t + \frac{\delta_{n3}}{1+(2n-1)^2}$.
      - $v(x,t) = \sum_1^\infty \left(A_n\cos(2n-1)t + B_n\sin(2n-1)t + \frac{\delta_{n3}}{1+(2n-1)^2}\right)\sin\big((2n-1)x\big)$.
      - Substituting the IC, we have $A_n = -\frac{\delta_{n3}}{1+(2n-1)^2}, B_n = \frac{\delta_{n2}}{(2n-1)} + \frac{\delta_{n3}}{(1+(2n-1)^2)(2n-1)}$.
    - $u(x,t) = xt + \frac{1}{3}\sin 3t\sin 3x + \left(-\frac{1}{26}\cos 5t + \frac{1}{5\cdot 26}\sin 5t + \frac{e^{-t}}{26}\right)\sin 5x$.
- Mixed boundary: $u_x(0,t) = q_0(t), u(L,t) = \phi_1(t)$.
  - $w(x,t) = q_1(t)(x-L) + \phi_0(t)$.
- Neumann: $u_x(0,t) = q_0(t), u_x(L,t) = q_1(t)$.

Laplace's equation

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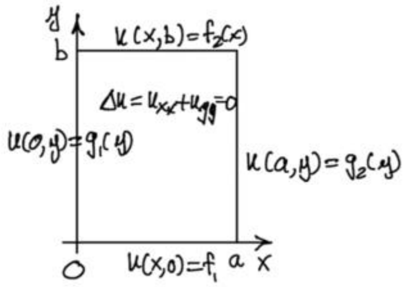
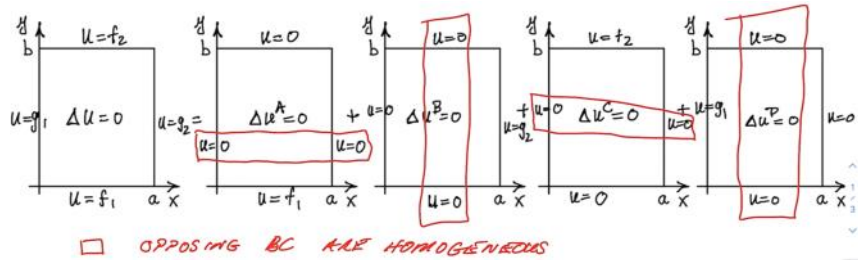
Laplace's equation occurs as a steady solution heat or damped wave equation

- $u_t = \alpha^2(u_{xx} + u_{yy})$, when $u_t = 0$, we have $\Delta u = u_{xx} + u_{yy} = 0$.
- $u_{tt} + \gamma u_t = c^2(u_{xx} + u_{yy})$, when $u_t = u_{tt} = 0$, $\Delta u = u_{xx} + u_{yy} = 0$.
- In polar coordinates $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$.
- Need net flux =0.

Dirichlet problem on a rectangular domain

- 

- $\Delta u = u_{xx} + u_{yy} = 0, u(x,0) = f_1(x), u(x,b) = f_2(x), u(0,y) = g_1(y), u(a,y) = g_2(y)$.
- Split the problem into 4 small problems
    - 

    - Since $\Delta u = \Delta(u^A + u^B + u^C + u^D) = 0$, $u = u^A + u^B + u^C + u^D$.
    - $\Delta u^A = 0, u(x,0) = f_1(x), u(x,b) = u(0,y) = u(a,y) = 0$.
        - $u^A(x,y) = \sum_1^\infty b_n^{f_1} \frac{\sinh\left(\frac{n\pi}{a}(b-y)\right)}{\sinh\left(\frac{n\pi b}{a}\right)} \sin\left(\frac{n\pi x}{a}\right)$.
    - $\Delta u^B = 0, u(a,y) = g_2(x), u(x,b) = u(0,y) = u(x,0) = 0$.
        - $u^B(x,y) = \sum_1^\infty b_n^{g_2} \frac{\sinh\left(\frac{n\pi x}{b}\right)}{\sinh\left(\frac{n\pi a}{b}\right)} \sin\left(\frac{n\pi y}{b}\right)$.
    - $\Delta u^C = 0, u(x,b) = f_2(x), u(x,0) = u(0,y) = u(a,y) = 0$.
        - Let $u^C(x,y) = XY$, $\Delta u^C = X''Y + XY'' = 0$.
        - $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda = -\mu^2$.
            - $X'' + \mu^2 X = 0, X(0) = X(a) = 0, \mu_n = \frac{n\pi}{a}, X_n(x) = \sin\left(\frac{n\pi}{a}x\right)$.
            - $Y'' - \mu^2 Y = 0, Y(0) = 0, Y_n(y) = B_n \sinh\left(\frac{n\pi}{a}y\right)$.
        - $u^C(x,y) = \sum_1^\infty B_n \sinh\left(\frac{n\pi}{a}y\right) \sin\left(\frac{n\pi}{a}x\right)$.
        - With IC, $f_2(x) = u(x,b) = \sum_1^\infty B_n \sinh\left(\frac{n\pi}{a}b\right) \sin\left(\frac{n\pi}{a}x\right) = \sum_1^\infty b_n^{f_2} \sin\left(\frac{n\pi}{a}x\right)$.
            - $b_n^{f_2} = \frac{2}{a}\int_0^a f_2(x) \sin\left(\frac{n\pi x}{a}\right) dx = B_n \sinh\left(\frac{n\pi}{a}b\right)$.
            - $B_n = \frac{b_n^{f_2}}{\sinh\left(\frac{n\pi}{a}b\right)}$.
    - $\Delta u^D = 0, u(0,y) = g_1(x), u(x,b) = u(x,0) = u(a,y) = 0$.
        - Separation of variables.
            - $\mu_n = \frac{n\pi}{b}, Y_n(y) = \sin\left(\frac{n\pi}{b}y\right)$.

- $X_n(x) = \frac{A_n}{\sinh\left(\frac{n\pi a}{b}\right)} \sinh\left(\frac{n\pi}{a}(a-x)\right)$.
  - $u^D(x,y) = \sum_1^\infty \frac{A_n}{\sinh\left(\frac{n\pi a}{b}\right)} \sinh\left(\frac{n\pi}{a}(a-x)\right) \sin\left(\frac{n\pi}{b}y\right)$.
  - With IC, $g_1(y) = u(0,y) = \sum_1^\infty \frac{A_n}{\sinh\left(\frac{n\pi a}{b}\right)} \sinh\left(\frac{n\pi}{a}a\right) \sin\left(\frac{n\pi}{b}y\right)$.
    - $b_n^{g_1} = \frac{2}{b}\int_0^b g_1(y) \sin\left(\frac{n\pi y}{a}\right) dy = \frac{A_n}{\sinh\left(\frac{n\pi a}{b}\right)} \sinh\left(\frac{n\pi}{a}a\right)$.
    - $\frac{A_n}{\sinh\left(\frac{n\pi a}{b}\right)} = \frac{b_n^{g_1}}{\sinh\left(\frac{n\pi a}{b}\right)}$.

Neumann problem

- $\Delta u = 0, u_y(x,t) = u_x(0,y) = u_x(a,y) = 0, u_y(x,0) = f(x)$.
- Need no net heat loss or gain $\int_0^a f(x)dx = 0$.
- Let $u(x,y) = XY, X''Y + XY'' = 0$.
- Then $\lambda_n \in \{0\} \cup \left\{-\left(\frac{n\pi}{a}\right)^2\right\}, X_n \in \{1\} \cup \left\{\cos\left(\frac{n\pi x}{a}\right)\right\}$.
  - When $\lambda = 0, Y = B$.
  - When $\lambda = -\left(\frac{n\pi}{a}\right)^2, Y_n = A_n \frac{\cosh\left(\frac{n\pi}{a}(y-b)\right)}{\cosh\left(\frac{n\pi}{a}b\right)} = D_n \cosh\left(\frac{n\pi}{a}(y-b)\right)$.
- So $u(x,y) = D_0 + \sum_1^\infty D_n \cosh\left(\frac{n\pi}{a}(y-b)\right) \cos\left(\frac{n\pi x}{a}\right)$.
- $f(x) = u_y(x,0) = \sum_1^\infty D_n \left(\frac{n\pi}{a}\right) \sinh\left(\frac{n\pi}{a}(0-b)\right) \cos\left(\frac{n\pi}{a}x\right) = \frac{a_0^f}{2} + \sum_1^\infty a_n^f \cos\left(\frac{n\pi}{a}x\right)$.
  - So we must have $a_0 = 0$. (solvability condition)
  - $D_n = -\frac{a_n^f}{\left(\frac{n\pi}{a}\right)\sinh\left(\frac{n\pi}{a}b\right)}$ where $a_n^f = \frac{2}{a}\int_0^a f(x) \cos\left(\frac{n\pi x}{a}\right) dx$.
- To find D_0 = average value of u at $t = 0$ over $[0,a] \times [0,b]$.
  - $\int_0^b \int_0^a u_t dx\, dy = \int_0^b \int_0^a u_{xx} dx\, dy + \int_0^b \int_0^a u_{yy} dx\, dy = -\int_0^a f(x)dx = 0$.
  - Then $\frac{\partial}{\partial t}\int_0^b \int_0^a u(x,y,t)dx\, dy = 0$, since the integral is constant 0.
  - Then the steady state integral $\int_0^b \int_0^a u(x,y,\infty)dx\, dy = \int_0^b \int_0^a u(x,y,0)dx\, dy = 0$.
    - Also $\int_0^b \int_0^a u(x,y,\infty)dx\, dy = D_0 ab$, since $u(x,y,\infty) = u(x,y)$.

Laplace's equation on a semi infinite strip

- $\Delta u = 0, 0 < x < L, y > 0, u(0,y) = 0, u_x(L,y) = 0, u(x,0) = f(x), u(x,y) \to 0$ as $y \to \infty$.
- Let $u = XY$, then $\frac{X''}{X} = -\frac{Y''}{Y} = \lambda = -\mu^2$.
- $X'' + \mu^2 X = 0, X(0) = X'(L) = 0$.
  - $\mu_n = \frac{(2n-1)\pi}{2L}, X_n = \sin(\mu_n x)$.
- $Y'' - \mu^2 Y = 0$.
  - $Y = Ae^{-\mu y} + Be^{\mu y}$.
  - Since $Y(y \to \infty) = 0$, we have $B = 0$.
- $u(x,y) = \sum_1^\infty A_n e^{-\mu_n y} \sin(\mu_n x)$.
- $f(x) = u(x,0) = \sum_1^\infty A_n \sin \mu_n x = \sum_1^\infty b_n^f \sin\left(\frac{(2n-1)\pi x}{2L}\right)$.
  - So $A_n = b_n^f = \frac{2}{L}\int_0^L f(x) \sin\left(\frac{(2n-1)\pi x}{2L}\right) dx$.

Laplace's equation on circular domains

- $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$.
- Separation of variables $u(r,\theta) = R(r)\Theta(\theta)$, then $R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$.
  - Multiply both sides by $\frac{r^2}{R\Theta}$, $\frac{r^2R''+rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda = \begin{cases} \mu^2, if\ homogeneous\ BC\ in\ \Theta \\ -\mu^2, if\ homogeneous\ BC\ in\ R \end{cases}$.
- For Θ.
  - If $\mu = 0, \Theta'' = 0, \Theta = A\theta + B$.

- If $\mu > 0$, $\Theta = A\cos\mu\theta + B\sin\mu\theta$.
- For R.
  - If $\mu = 0$, $r^2R'' + rR' = 0$, $R(r) = C + D\ln r$.
  - If $\mu > 0$, $r^2R'' + rR - \mu^2R = 0$, Cauchy-Euler equation, $R(r) = Cr^{\mu} + Dr^{-\mu}$.
- Dirichlet: $\Theta(0) = \Theta(\alpha) = 0$ $(u(r,0) = u(r,\theta) = 0)$, $\Theta'' + \mu^2\Theta = 0$.
  - $\mu_n = \frac{n\pi}{\alpha}$, $\Theta_n(\theta) = \sin\left(\frac{n\pi\theta}{\alpha}\right)$.
- Neumann: $\Theta'(0) = \Theta'(\alpha)$ $(u_\theta(r,0) = u_\theta(r,\theta) = 0)$.
  - $\mu_n \in \{0\} \cup \left\{\frac{n\pi}{\alpha}\right\}$, $\Theta_n \in \{1\} \cup \left\{\cos\left(\frac{n\pi\theta}{\alpha}\right)\right\}$.
- Periodic: $\Theta(\pi) = \Theta(-\pi)$, $\Theta'(\pi) = \Theta'(-\pi)$, $(u(r,\pi) = u(r,-\pi), u_\theta(r,\pi) = u_\theta(r,-\pi))$.
  - $\mu_n \in \{1\} \cup \{n\}$, $\Theta_n \in \{1\} \cup \{\cos n\theta, \sin n\theta\}$.
- Mixed: $\Theta(0) = \Theta'(\alpha) = 0$, $u(r,0) = u_\theta(r,\alpha) = 0$.
  - $\mu_n = \frac{(2n-1)\pi}{2\alpha}$, $\Theta_n = \sin\left(\frac{(2n-1)\pi}{2\alpha}\theta\right)$.
- Mixed: $\Theta'(0) = \Theta(\alpha) = 0$, $u_\theta(r,0) = u(r,\alpha) = 0$.
  - $\mu_n = \frac{(2n-1)\pi}{2\alpha}$, $\Theta_n = \cos\left(\frac{(2n-1)\pi}{2\alpha}\theta\right)$.
- General solution: $u(r,\theta) = A_0 + \alpha_0\ln r + \sum_1^\infty(A_n r^{\mu_n} + \alpha_n r^{-\mu_n})\cos\mu_n\theta + \sum_1^\infty(B_n r^{\mu_n} + \beta_n r^{-\mu_n})\sin\mu_n\theta$.

Dirichlet problem e.g.

- Model problem for a crack: $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, $u(r,0) = 0$, $u_\theta(r,\pi) = 0$, $u(a,\theta) = f(\theta)$.
  - Mixed boundary, so $\mu_n = \frac{(2n-1)\pi}{2\pi} = \frac{2n-1}{2}$, $\Theta_n(\theta) = \sin\left(\frac{2n-1}{2}\theta\right)$.
  - We also need $u(r,\theta) < \infty$ as $r \to 0$, so $\beta_n = 0$.
  - $u(r,\theta) = \sum_1^\infty B_n r^{\mu_n}\sin(\mu_n\theta)$.
  - Plug in the initial condition, $f(\theta) = u(a,\theta) = \sum_1^\infty B_n a^{\mu_n}\sin(\mu_n\theta) = \sum_1^\infty b_n^f\sin(\mu_n\theta)$.
    - Then $B_n a^{\mu_n} = b_n^f = \frac{2}{\pi}\int_0^\pi f(\theta)\sin(\mu_n\theta)\,d\theta$.
  - $u(r,\theta) = \sum_1^\infty b_n^f\left(\frac{r}{a}\right)^{\mu_n}\sin(\mu_n\theta)$.
  - Note: $u_r \sim \frac{b_1^f}{2a^{1/2}}r^{-\frac{1}{2}}\sin\frac{\theta}{2}$, the $\frac{b_1^f}{2a^{1/2}}$ is the stress intensity factor.
- Dirichlet problem for a circle: $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, $u(a,\theta) = f(\theta)$, periodic boundary condition.
  - $\mu_n \in \{0\} \cup \{n\}$, $\Theta_n \in \{1\} \cup \{\cos n\theta, \sin n\theta\}$.
  - However, we require $u(r,\theta) < \infty$ as $r \to 0$, so $\alpha_0 = \alpha_n = \beta_n = 0$.
  - So $u(r,\theta) = A_0 + \sum_1^\infty A_n r^n\cos n\theta + B_n r^n\sin n\theta$.
  - $f(\theta) = u(a,\theta) = A_0 + \sum_1^\infty A_n a^n\cos n\theta + B_n a^n\sin n\theta = \frac{a_0^f}{2} + \sum_1^\infty a_n^f\cos n\theta + b_n^f\sin n\theta$.
    - $A_0 = \frac{a_0^f}{2}$, where $a_0^f = \frac{1}{\pi}\int_\pi^{-\pi} f(\theta)d\theta$.
    - $A_n a^n = a_n^f = \frac{1}{\pi}\int_\pi^{-\pi} f(\theta)\cos n\theta\,d\theta$.
    - $B_n a^n = b_n^f = \frac{1}{\pi}\int_\pi^{-\pi} f(\theta)\sin n\theta\,d\theta$.
  - Finally, $u(r,\theta) = \frac{a_0^f}{2} + \sum_1^\infty a_n^f\left(\frac{r}{a}\right)^n\cos n\theta + b_n^f\left(\frac{r}{a}\right)^n\sin n\theta$.
    - It can be written as $\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\phi)\frac{a^2-r^2}{a^2-2ar\cos(\theta-\phi)+r^2}d\phi$ (Poisson formula).

Neumann problem on a circle

- $\Delta u = u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$, $u_r(a,\theta) = f(\theta)$.
- Need $u < \infty$ as $r \to 0$.
  - So we must have $u(r,\theta) = A_0 + \sum_1^\infty A_n r^{\mu_n}\cos\mu_n\theta + B_n r^{\mu_n}\sin\mu_n\theta$.
- Solvability condition: $\int_{-\pi}^{\pi} f(\theta)d\theta = 0$.
- Periodic boundary condition gives $\mu_n \in \{0\} \cup \{n\}$, $\Theta_n \in \{1\} \cup \{\cos n\theta, \sin n\theta\}$.
  - Then $u(r,\theta) = A_0 + \sum_1^\infty A_n r^n\cos n\theta + B_n r^n\sin n\theta$.
- $u_r(r,\theta) = \sum_1^\infty A_n n r^{n-1}\cos n\theta + B_n n r^{n-1}\sin n\theta = f(\theta)$.
  - $a_0^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta = 0$.
  - $a_n^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\,d\theta = A_n n a^{n-1}$.

- $b_n^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = B_n n a^{n-1}$.

- So $u(r,\theta) = A_0 + a\sum_1^\infty \frac{a_n^f}{n}\left(\frac{r}{a}\right)^n \cos n\theta + \frac{b_n^f}{n}\left(\frac{r}{a}\right)^n \sin n\theta$.

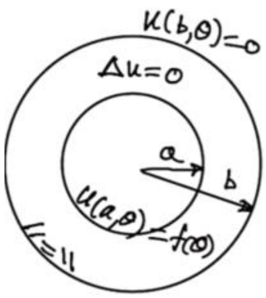
Application to electrical impedance tomography

- $f(\theta) = u_r(a,\theta) = I\left(\delta\left(\theta - \frac{\pi}{2}\right) - \delta\left(\theta + \frac{\pi}{2}\right)\right)$, where I is the current.
- Since $f(\theta) = f(-\theta)$, f is odd, $a_n = 0$.
- $b_n^f = \frac{I}{\pi}\int_{-\pi}^{\pi}\left(\delta\left(\theta - \frac{\pi}{2}\right) - \delta\left(\theta + \frac{\pi}{2}\right)\right)\sin(n\theta)\, d\theta = \frac{I}{\pi}\left(\sin\left(\frac{n\pi}{2}\right) - \sin\left(-\frac{n\pi}{2}\right)\right) = \frac{2I}{\pi}\sin\left(\frac{n\pi}{2}\right)$.
- $u(r,\theta) = A_0 + \frac{2Ia}{\pi}\sum_1^\infty r^n \frac{\sin\left(\frac{n\pi}{2}\right)}{n}\sin n\theta = A_0 + \frac{aI}{2\pi}\ln\frac{a^2+2ar\sin\theta+r^2}{a^2-2ar\sin\theta+r^2}$.

Tunnel or hole

- $u(a,\theta) = f(\theta), r > a$.
- We need $|u| < \infty$ as $r \to \infty$, so $\alpha_0 = A_n = B_n = 0$.
- Periodic boundary condition, so $\mu_n \in \{0\} \cup \{n\}$, $\Theta_n \in \{1\} \cup \{\cos n\theta, \sin n\theta\}$.
- $u(r,\theta) = A_0 + \sum_1^\infty \alpha_n r^{-n}\cos n\theta + \beta_n r^{-n}\sin n\theta$.
- Then $f(\theta) = u(a,\theta) = A_0 + \sum_1^\infty \alpha_n a^{-n}\cos n\theta + \beta_n a^{-n}\sin n\theta = \frac{a_0^f}{2} + \sum_1^\infty a_n^f\cos n\theta + b_n^f\sin n\theta$.
  - $a_0^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta = 2A_0$.
  - $a_n^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta = \alpha_n a^{-n}$.
  - $b_n^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = \beta_n a^{-n}$.
- $u(r,\theta) = \frac{a_0^f}{2} + \sum_1^\infty a_n^f\left(\frac{r}{a}\right)^{-n}\cos n\theta + b_n^f\left(\frac{r}{a}\right)^{-n}\sin n\theta$.

Annulus/washer

- 

- $u(a,\theta) = f(\theta)$, $u(b,\theta) = 0$.
- $u(r,\theta) = A_0 + \alpha_0\ln r + \sum_1^\infty\left(A_n r^{\mu_n} + \alpha_n r^{-\mu_n}\right)\cos\mu_n\theta + \sum_1^\infty\left(B_n r^{\mu_n} + \beta_n r^{-\mu_n}\right)\sin\mu_n\theta$.
- Still have periodic boundary condition, $\mu_n \in \{0\} \cup \{n\}$, $\Theta_n \in \{1\} \cup \{\cos n\theta, \sin n\theta\}$.
- $0 = u(b,\theta) = A_0 + \alpha_0\ln b + \sum_1^\infty\left(A_n b^n + \alpha_n b^{-n}\right)\cos n\theta + \sum_1^\infty\left(B_n b^n + \beta_n b^{-n}\right)\sin n\theta$.
  - $\alpha_0 = -\frac{A_0}{\ln b}$.
  - $\alpha_n = -A_n b^{2n}$.
  - $\beta_n = -B_n b^{2n}$.
  - $u(r,\theta) = A_0\left(1 - \frac{\ln r}{\ln b}\right) + \sum_1^\infty A_n b^n\left(\left(\frac{r}{b}\right)^n - \left(\frac{r}{b}\right)^{-n}\right)\cos n\theta + \sum_1^\infty B_n b^n\left(\left(\frac{r}{b}\right)^n - \left(\frac{r}{b}\right)^{-n}\right)\sin n\theta$.
- $f(\theta) = u(a,\theta) = A_0\left(1 - \frac{\ln a}{\ln b}\right) + \sum_1^\infty A_n b^n\left(\left(\frac{a}{b}\right)^n - \left(\frac{a}{b}\right)^{-n}\right)\cos n\theta + \sum_1^\infty B_n b^n\left(\left(\frac{a}{b}\right)^n - \left(\frac{a}{b}\right)^{-n}\right)\sin n\theta$.
  - $a_0^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)d\theta = 2A_0\left(1 - \frac{\ln a}{\ln b}\right)$.
  - $a_n^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos n\theta\, d\theta = A_n b^n\left(\left(\frac{a}{b}\right)^n - \left(\frac{a}{b}\right)^{-n}\right)$.
  - $b_n^f = \frac{1}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin n\theta\, d\theta = B_n b^n\left(\left(\frac{a}{b}\right)^n - \left(\frac{a}{b}\right)^{-n}\right)$.
- $u(r,\theta) = \frac{a_0^f\left(1-\frac{\ln r}{\ln b}\right)}{2\left(1-\frac{\ln a}{\ln b}\right)} + \sum_1^\infty a_n^f\frac{\left(\frac{r}{b}\right)^n - \left(\frac{r}{b}\right)^{-n}}{\left(\frac{a}{b}\right)^n - \left(\frac{a}{b}\right)^{-n}}\cos n\theta + \sum_1^\infty b_n^f\frac{\left(\frac{r}{b}\right)^n - \left(\frac{r}{b}\right)^{-n}}{\left(\frac{a}{b}\right)^n - \left(\frac{a}{b}\right)^{-n}}\sin n\theta$.

BVP, Sturm-Liouville

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Sturm-Liouville problem

- $Ly = -(p(x)y')' + q(x)y = \lambda r(x)y, \alpha_1 y(0) + \alpha_2 y'(0) = 0, \beta_1 y(l) + \beta_2 y'(l) = 0.$
  - $r(x)$ is the weight function.
- Regular SL problem: $p(x) > 0, r(x) > 0, l < \infty$.
- Singular SL problem: $p(x) \geq 0, r(x) \geq 0$, or $l = \infty$.

Sign conventions for eigenvalues SL problems

- Generally, we need $X'' + \mu^2 X = 0$.
- For SL, we have $-X'' = \lambda X$, then $\lambda = \mu^2$, we have $X'' + \lambda X = 0$.
  - And this gives the sine and cosine functions

Properties of SL problem

- Eigenvalues
  - Eigenvalues λ_j are all real
  - There are an infinite number of eigenvalues with $\lambda_1 < \lambda_2 < \cdots < \lambda_j < \infty$.
  - $\lambda_j > 0$ given that $\frac{\alpha_1}{\alpha_2} < 0, \frac{\beta_1}{\beta_2} > 0, q(x) > 0$.
- Eigenfunctions
  - For each λ_j, there is an eigenfunction $\phi_j(x)$.
  - $\phi_j(x)$ are real and can be normalized so that $\int_0^l r(x)\phi_j^2(x)dx = 1$.
  - Orthogonality $\int_0^l r(x)\phi_j(x)\phi_k(x)dx = 0$ for all $j \neq k$.

Lagrange's Identity: $\int_0^l vLudx - \int_0^l uLvdx = -pvu'\big|_0^v + puv'\big|_0^l = 0$.

Convert an arbitrary second order linear ODE to SL form

- $Ly = -P(x)y'' - Q(x)y' + R(x)y = \lambda y$.
- Multiply both sides by F, $FLy = -FPy'' - FQy' + FRy = \lambda Fy$.
- Consider $-(FPy')' = -FPy'' - (F'P + FP')y'$, we need $FQ = F'P + FP'$.
  - This gives $\frac{dF}{dx} + \left(\frac{P'}{P} - \frac{Q}{P}\right)F = 0$.
  - $F = Ae^{-\int\left(\frac{P'}{P} - \frac{Q}{P}\right)dx} = Ae^{-\ln P} e^{\int \frac{Q}{P}dx} = \frac{A}{P} e^{\int \frac{Q}{P}dx}$. (Abel's formula) This turns a general ODE to SL form.
- E.g. $Ly = x^2y'' + xy' + \lambda y = 0, y'(1) = 0, y(2) = 0$.
  - $F(x) = \frac{1}{x^2} e^{\int \frac{x}{x^2}dx} = \frac{1}{x^2} e^{\ln x} = \frac{x}{x^2} = \frac{1}{x}$.
  - $\frac{1}{x} Ly = xy'' + y' + \frac{\lambda}{x} y = (xy')' + \frac{\lambda}{x} y$.
  - So $-(xy')' = \frac{\lambda}{x} y$.
  - Solve the Cauchy-Euler equation $y = x^r, y' = rx^{r-1}, y'' = r(r-1)x^{r-2}$.
    - Then we need $r^2 + \lambda = 0$.
    - $\lambda = \mu^2 > 0, r = \pm i\mu, y(x) = A\cos\mu\ln x + B\sin\mu\ln x$.
      - The IC gives $B = 0, \mu_n = \frac{(2n-1)\pi}{2\ln 2}, y(x) = \sum_0^\infty A\cos\mu_n \ln x$.
    - $\lambda = 0, r = 0, y = A + B\ln x$.
      - $A = 0, B = 0$, the solution is trivial.
    - $\lambda = -\mu^2 < 0, r = \pm\mu, y(x) = A\cosh\mu\ln x + B\sinh\mu\ln x$.
      - Still the trivial solution

Robin boundary conditions

- Heat loss from both boundaries
- $u_t = \alpha^2 u_{xx}, u_x(0,t) = h_1 u, u_x(l,t) = -h_2 u, h_1, h_2 \geq 0$.

- If $h_2 = 0$. $u_x(l,t) = 0$, flux BC.
- If $h_1 = \infty$, $u(0,t) = 0$, temp BC.

- Let $u(x,t) = X(x)T(t)$, $\frac{T'}{\alpha^2 T} = \frac{X''}{x} = \lambda = -\mu^2$.
  - $T = Ce^{-\mu^2\alpha^2 t}$.
  - $X'' + \mu^2 X = 0$, $X'(0) = h_1 X(0)$, $X'(l) = h_2 X(l)$.
    - $X'(0) = B\mu = h_1 A$, $B = \frac{h_1 A}{\mu}$.
    - $X = A\left(\cos\mu x + \frac{h_1}{\mu}\sin\mu x\right)$.
    - $\left(\frac{h_1 h_2 - \mu^2}{\mu}\right)\sin\mu l + (h_1 + h_2)\cos\mu l = 0$, $\tan\mu l = \frac{(h_1+h_2)\mu}{\mu^2 - h_1 h_2}$ (transcendental equation).
    - $h_1, h_2 \neq 0$.
      - An infinite number of eigenvalues, as $n \to \infty$, $\mu_n l \to n\pi$, or $\mu_n \to \frac{n\pi}{l}$.
      - $X_n = A\left(\cos\mu_n x + \frac{h_1}{\mu_n}\sin\mu_n x\right)$.
    - $h_2 = 0$, $h_1 \neq 0$.
      - $\tan\mu l = \frac{h_2}{\mu}$, $\mu_n \to \frac{n\pi}{l}$.
      - $X_n = \cos\mu_n(l-x)$.
    - $h_1 \to \infty$, $h_2 \neq 0$.
      - $\tan\mu l = \frac{\left(1+\frac{h_2}{h_1}\right)\mu}{\frac{\mu^2}{h_1} - h_2} = -\frac{\mu}{h_2}$, $\mu_n \to \frac{(2n-1)\pi}{2l}$.
      - $X_n = \sin\mu_n x$.

Application of robin BC

- $u_t = \alpha^2 u_{xx}$, $u_x(l,t) = q_0$, $u_x(0,t) = h\big(u(0,t) - u_0\big)$, u_0 is temperature in the room, $u(x,0) = f(x)$.
- Find $w(x)$ that matches the BC: $w(x) = Ax + B$, $q_0 = w_x = A$, $q_0 = h\big(w(0) - u_0\big)$.
  - $A = q_0$, $B = \frac{q_0}{h} + u_0$
  - So $w(x) = q_0\left(x + \frac{1}{h}\right) + u_0$.
- Let $u(x,t) = w(x) + v(x,t)$, then $v_t = \alpha^2 v_{xx}$, $v_x(0,t) = hv(0,t)$, $v_x(l,t) = 0$, $v(x,0) = f(x) - w(x) = g(x)$.
  - Need $\lambda = -\mu^2 < 0$.
  - $T_n = Ce^{-\mu_n^2\alpha^2 t}$.
  - $X_n = A\cos\mu_n x + B\sin\mu_n x$, $X'(0) = hX(0)$, $X'(l) = 0$.
    - $B = \frac{hA}{\mu}$, $X = A\left(\cos\mu x + \frac{h}{\mu}\sin\mu x\right)$.
    - $X'(l) = 0$ gives $\tan\mu_n l = \frac{h}{\mu_n}$, $X_n = \cos\mu_n(l-x)$, $\mu_n \to \frac{n\pi}{l}$.
  - $v(x,t) = \sum_{n=0}^{\infty} A_n e^{-\alpha^2\mu_n^2 t}\cos\mu_n(l-x)$.
  - Match IC, $g(x) = v(x,0) = \sum_0^\infty A_n\cos\mu_n(l-x)$.
    - $\int_0^l g(x)\cos\mu_m(l-x)\,dx = \sum_0^\infty A_n \int_0^l \cos\mu_m(l-x)\cos\mu_n(l-x)\,dx$.
      - $= A_m\int_0^l \cos^2\mu_m(l-x)dx = \frac{A_m}{2}\left(l + \frac{2\sin\mu_m l\cos\mu_m l}{2\mu_m}\right)$.
      - By the transcendental solution $\tan\mu_n l = \frac{h}{\mu_n}$, we have $\frac{1}{\mu_n} = \frac{\sin\mu_n l}{h\cos\mu_n l}$.
      - Then $= \frac{A_m}{2h}(lh + \sin^2\mu_m l)$.
    - So $A_n = \frac{2h}{(lh + \sin^2\mu_n l)}\int_0^l g(x)\cos\mu_n(l-x)\,dx$.
- $u(x,t) = q_0\left(x + \frac{1}{h}\right) + u_0 + \sum_0^\infty A_n e^{-\alpha^2\mu_n^2 t}\cos\mu_n(l-x)$.

SL example of a variable coefficient heat equation with inhomogeneous BC

- $u_t = x^2 u_{xx} + 4xu_x$, $u(1,t) = u(2,t) = 1$, $u(x,0) = 1 - 5x^{-\frac{3}{2}}$.
- Let $w(x) = 1$, then it satisfies the BC.
- Let $u(x,t) = w(x) + v(x,t)$, $v_t = x^2 v_{xx} + 4xv_x$, $v(1,t) = v(2,t) = 0$, $v(x,0) = 5x^{-\frac{3}{2}}$.
- Let $v(x,t) = XT$, then $\frac{T'}{T} = \frac{x^2 X''}{X} + \frac{4xX'}{X} = -\lambda$.

- $T = Ce^{-\lambda t}$.
- $-(x^2X'' + 4xX') = \lambda X, X(1) = X(2) = 0$.
  - Apply SL, let $F = \frac{e^{\int \frac{4x}{x^2}dx}}{x^2} = \frac{e^{4\ln x}}{x^2} = x^2$.
  - We have $x^2LX = -(x^4X'' + 4x^3X') = -(x^4y')' = \lambda x^2 X$.
    - Weight function $r(x) = x^2$.
  - Back to original equation $x^2X'' + 4xX' + \lambda X = 0, X(1) = X(2) = 0$.
    - Let $X = x^r, r(r-1) + 4r + \lambda = 0, r = \frac{-3\pm\sqrt{9-4\lambda}}{2}$.
    - When $\lambda = \frac{9}{4}, r = 0, X = x^{-\frac{3}{2}}(A + B\ln x), A = B = 0$.
    - When $\lambda < \frac{9}{4}, X = x^{-\frac{3}{2}}\left(A\cosh\frac{\sqrt{9-4\lambda}}{2}\ln x + B\sinh\frac{\sqrt{9-4\lambda}}{2}\ln x\right), A = B = 0$.
    - When $\lambda > \frac{9}{4}, X = x^{-\frac{3}{2}}\left(A\cos\frac{\sqrt{4\lambda-9}}{2}\ln x + B\sin\frac{\sqrt{4\lambda-9}}{2}\ln x\right)$.
      - $X(1) = A = 0$.
      - $X(2) = 2^{-\frac{3}{2}}B\sin\frac{\sqrt{4\lambda-9}}{2}\ln 2 = 0$, then $\frac{\sqrt{4\lambda-9}}{2}\ln 2 = n\pi$.
    - So $X_n = x^{-\frac{3}{2}}\sin\left(\frac{n\pi\ln x}{\ln 2}\right)$.
- $v(x,t) = \sum_{n=0}^{\infty} B_n e^{-\lambda_n t}x^{-\frac{3}{2}}\sin(\mu_n \ln x), \mu_n = \frac{n\pi}{\ln 2}$.
- Matching IC, $-5x^{-\frac{3}{2}} = v(x,0) = \sum_{n=0}^{\infty} B_n x^{-\frac{3}{2}}\sin\mu_n \ln x$.
  - Construct an orthogonality property for SL problem with $r(x) = x^2, \phi_j(x) = x^{-\frac{3}{2}}\sin\mu_j \ln x$
  - $\int_1^2 x^2\left(-5x^{-\frac{3}{2}}\right)x^{-\frac{3}{2}}\sin(\mu_m \ln x)\,dx = \sum_{n=0}^{\infty} B_n \int_1^2 x^2\left(x^{-\frac{3}{2}}\sin\mu_m \ln x\right)\left(x^{-\frac{3}{2}}\sin\mu_n \ln x\right)dx$.
  - $= \sum_{n=1}^{\infty} B_n \frac{\ln 2}{2}\delta_{mn} = \frac{B_m \ln 2}{2}$.
  - So $B_m = \frac{10}{\pi}\frac{(-1)^m - 1}{m}$.

- $u(x,t) = 1 + \frac{10}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n-1}{n}e^{-\left(\frac{9}{4}+\left(\frac{n\pi}{\ln 2}\right)^2\right)t}x^{-\frac{3}{2}}\sin\left(\frac{n\pi}{\ln 2}\ln x\right)$.

Nonhomogeneous Sturm-Liouville problems

- $Ly = -(p(x)y')' + q(x)y = \mu r(x)y + f(x)$.
  - $\alpha_1 y(0) + \alpha_2 y'(0) = 0$.
  - $\beta_1 y(1) + \beta_2 y'(1) = 0$.
- If $\mu = \lambda$ (eigenvalues of the homogenuous S-L problem), then the equation doesn't need to have a solution for every $f(x)$. Even if it happens to have a solution, the solution is not unique
- If $\mu \neq \lambda$, then the equation has a unique solution for every $f(x)$.
- Decompose $f(x)$ and $y(x)$ in terms of the eigenfunctions of the homogeneous problem and then solve for the coefficients of the series for $y(x)$.
- E.g. $y'' + 4y = x, y(0) = 0, y'\left(\frac{\pi}{2}\right) = 0$.
  - $\mu = -4$.
  - Homogeneous S-L, $\lambda_n = (2n-1)^2, y_n = \sin((2n-1)x)$.
  - Let $f(x) = x = \sum_1^{\infty} b_n \sin((2n-1)x), y(x) = \sum_1^{\infty} d_n \sin((2n-1)x)$.
  - Plug in the equation and solve for b_n, d_n.