Introduction

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Dynamic system: physical (biological or financial) systems whose state $u(t)$ changes in time in a deteriministic way

- Differential equation initial value problem
  - Equation (defines the system): $\frac{du}{dt} = u' = f(u)$.
  - Initial condition: $u(0) = u_0$.
  - If $f(u)$ has no t dependence, it is called autonomous
    - Solutions are called trajectories.
- Systems with a parameter a, $\frac{du}{dt} = f(u; a)$.
  - How trajectories change with a?
  - How the long term behavior ($\lim_{t\to\infty} u(t)$) changes with a?
    - Changes in limiting behavior with a are called bifurcations.

Discrete dynamical system

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Discrete time

- Time is integer $n = 0,1,2, \dots$.

Dynamic system

- State at time n is X^n.
- Discrete dynamical system: $X^{n+1} = f(X^n; r)$, X^0 is given initial condition.
  - Also called iterative map
  - Confined to a closed bounded interval $[a, b]$. This requires $f(x; r) \in [a, b]$ for all $x \in [a, b]$

Fixed point

- If $f(X_*) = X_*$, then X_* is a fixed point or equilibrium of the map
- If X^0 is close to X_*, it stays close to X_*, then X_* is a stable fixed point
- If X^0 is close to X_*, and $\lim_{n\to\infty} X^n = X_*$, then X_* is an asymptotically stable fixed point (attractor)
- If $\lim_{n\to\infty} X^n = X_*$ for any X^0, then X_* is a global attractor
- We can use numerical root finding to determine X_* more accurately.
  - It is a root of $g(x) = f(x) - x$ with fixed r.
- Can also use Cobweb diagram to determine X_*.

Basin of attraction

- Suppose we have a fixed point X_* such that $X_* = f(X_*)$.
- Consider X^0 near X_* such that $X^0 - X_*$ is small
- Let z^n be the signed distance of X^n to X_*, $z^n = X^n - X_*$
- If $z^n \to 0$ as $n \to \infty$ then $X^n \to X_*$ and X_* is an asymptotically stable fixed point
- The set of X^0 that has this property is called the basin of attraction of X_*

Tangent line approximation

- $f(X^n) = f(X_* + z^n) \approx f(X_*) + f'(X_*)z^n$.
  - Then, $z^{n+1} = f'(X_*)z^n$.
- If $|f'(X_*)| < 1$, $z^n \to 0$ as $n \to \infty$, X_* is an asymptotically stable fixed point
- If $|f'(X_*)| > 1$, X_* is an unstable fixed point(repeller)
- If $|f'(X_*)| = 1$, X_*'s stability cannot be told
- If $|f'(X_*)| \neq 1$, X_* is a hyperbolic fixed point
- If $|f'(X_*)| = 0$, $z^n \to 0$ very quickly, X_* is super stable

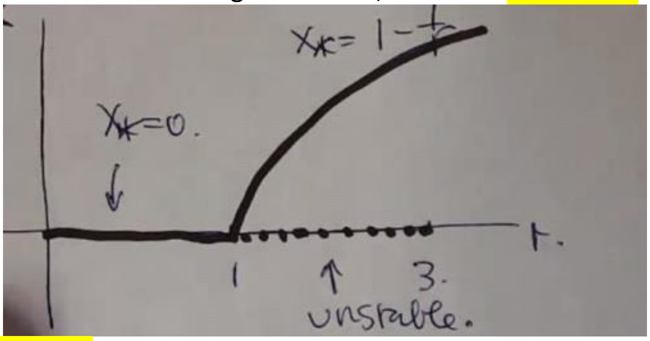
Consider the limiting behavior for all $r \in [-1,1]$ for $X^{n+1} = f(X^n; r) = r \cos X^n$.

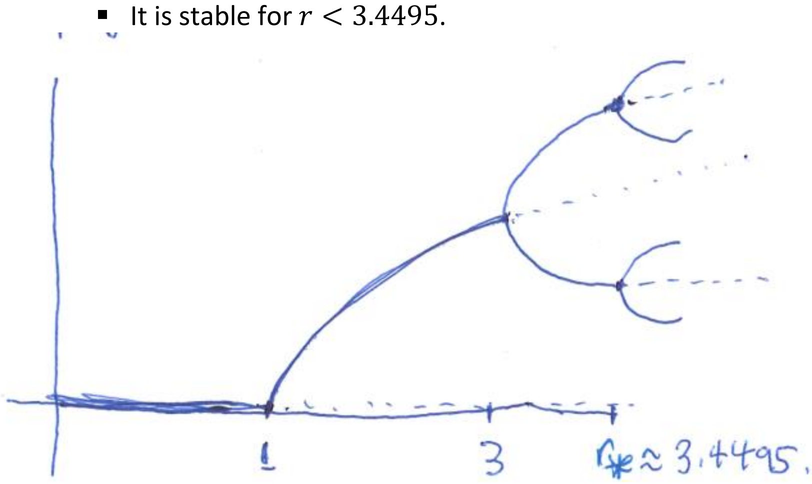
- For each r, there is a single stable fixed point (global attractor) $X_*(r)$.
- No bifurcations
- A bifurcation map shows the behavior of the system as $n \to \infty$ for all r and all X^0.

Newton's method

- Iterative method for root finding
- Find x such that $g(x) = f(x) - x = 0$.
- Requirement: $g'(x_*) \neq 0$ and initial guess x^0 is close enough to the root
- x^1 is computed as the root of the tangent line at $(x^0, g(x^0))$.
  - $y = g(x^0) + g'(x^0)(x - x^0) = 0$.
  - So $x^1 = x^0 - \frac{g(x^0)}{g'(x^0)}$.
- Repeat $x^{n+1} = f(x^n)$ with $f(x) = x - \frac{g(x)}{g'(x)}$.
  - x_* is a (super) stable fixed point of the iterative map.

Logistic map

- $f(x;r) = rx(1-x)$.
  - Map on $[0,1]$ as long as $r \in [0,4]$.
- Fixed points
  - $x = 0$: stable if $r < 1$, unstable if $r > 1$.
  - $x = 1 - \frac{1}{r}$ only in the interval if $r > 1$: stable if $1 < r < 3$.
- The global attractor changes at $r = 1$, there is a bifurcation
  - 

- Periodic orbit for $r > 3$(period 2)
  - $f(p) = q$ and $f(q) = p$.
  - p and q are fixed points of the doubly iterative map $f(f(x)) = x$
    - $f\left(f(q)\right) = q, f\left(f(p)\right) = p$.
    - $p, q = \frac{r+1 \pm \sqrt{(r+1)(r-3)}}{2r}$.
  - Stability of double map $f(f(x))$:
    - Stable if $\left|\frac{d}{dx} f(f(x))\right| = |f'(f(x))f'(x)| < 1$.
      - $|f'(q)f'(p)| < 1$.
    - Unstable if $\left|\frac{d}{dx} f(f(x))\right| = |f'(f(x))f'(x)| > 1$.
    - It is stable for $r < 3.4495$.
- 

Attractor

- Def: an attractor A of a discrete dynamic system is a closed and bounded set with the following properties
  - A is invariant. If $x^0 \in A$, then $f(x^n) \in A$.
  - A attracts an open set U containing A. For all $x^0 \in U$, $dist(x^n, A) \to 0$ as $n \to \infty$.
  - A is minimal (no closed proper subsets of A satisfy the first 2 properties).
- Any hyperbolic stable period m orbit is an attractor
  - For period 2, $f(p) = q, f(q) = p, A = \{p, q\}$.
- For the logistic map, we can also have a chaotic or strange attractor
  - Uncountably infinite number of points
  - Fractal dimension
  - Sensitive dependence on initial conditions

- On average, trajectories separate exponentially ($|x^n - \widehat{x^n}| \approx e^{\lambda n}|x^0 - \widehat{x^0}|$ with $\lambda > 0$)
- For stable orbits, $\lambda < 0$.

Counting

- A countably infinite set has an infinite number of entries that can be put in an ordered list
  - e.g. integers, rational numbers.
- Uncountably infinite set: cannot be put in an ordered set
  - e.g. real number

Dimension

- Interval (a, b) with $b > a$ is one dimensional
- $\{a\}$ and finite collection of points are zero dimensional.
- Box dimension
  - $A \subset [0,1]$. For every ϵ, cover A with the minimum number $N(\epsilon)$ of intervals of length ϵ.
  - If A is a finite number M of points, $N(\epsilon) \leq M$.
  - If A is a subinterval of length L, $N(\epsilon) = \frac{L}{\epsilon}$ (round up).
  - If $N(\epsilon) \sim \frac{c}{\epsilon^d}$, say A has dimension d.
    - With $N \to \infty$, $\epsilon \to 0$, $d = \lim_{\epsilon \to 0} -\frac{\ln N(\epsilon)}{\ln \epsilon}$.

Uncountably infinite set with fractal dimension (Cantor set)

- Length (measure) zero
- Uncountable
- Dimension with $\epsilon = \left(\frac{1}{3}\right)^n$: $d = \frac{\ln 2}{\ln 3}$ is a fractal dimension.
- Like chaotic attractors, the cantor set is self similar
  - $S_\infty \cap \left[0, \frac{1}{3}\right]$ is the same as S_∞ scaled down by $\frac{1}{3}$.

1D Taylor polynomial approximation for a smooth function $f(x)$.

- Tangent line: $T_1(x) = f(a) + f'(a)(x - a)$ valid for x near a.
- Add an error term: $f(x) = T_1(x) + \frac{1}{2}f''(\theta)(x - a)$ for some $\theta \in (a, x)$.
- Quadratic approximation: $T_2(x) = T_1(x) + \frac{1}{2}f''(a)(x - a)^2$.
  - $f(x) = T_2(x) + \frac{1}{6}f'''(\theta)(x - a)^3$.

2D approximation for $f(x, y)$

- First order (tangent plane): $T_1(x, y) = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$.
  - Error size $(x - a)^2 + (y - b)^2$.
- Quadratic: $T_2(x, y) = T_1(x, y) + \frac{1}{2}f_{xx}(x - a)^2 + f_{xy}(x - a)(y - b) + \frac{1}{2}f_{yy}(y - b)^2$.
  - Error size $(x - a)^3 + (y - b)^3$.
- $T_3(x, y) = T_2(x, y) + \frac{1}{6}f_{xxx}(x - a)^3 + \frac{1}{2}f_{xxy}(x - a)^2(y - b) + \frac{1}{2}f_{xyy}(x - a)(y - b)^2 + \frac{1}{6}f_{yyy}(y - b)^3$.
  - Error size $(x - a)^4 + (y - b)^4$.

For any hyperbolic fixed point x_* at r_*

- Fixed: $f(x_*, r_*) = x_*$.
- Hyperbolic: $|f_x(x_*, r_*)| = 1$.
- Use tangent approximation: $f(x, r) = f(x_*, r_*) + f_x(x - x_*) + f_r(r - r_*)$.
- $x \approx x_* + \frac{f_r}{1 - f_x}(r - r_*)$.
  - There is a fixed point for r near r_*, varying approximately linear with r, $\frac{dx}{dr} = \frac{f_r}{1 - f_x}$.
- Stability

- $\frac{d}{dr}\left(f_x(x(r),r)\right) = f_{xx}\frac{f_r}{1-f_x} + f_{xr}$.
- So $\left|\frac{d}{dr}\left(f_x(x(r),r)\right)\right| \neq 1$ for some neighborhood of r near r_*.

- If x_*, r_* is a hyperbolic fixed point, there is a neighborhood of r_*, where there is a unique hyperbolic fixed point $x(r)$ with the same stability as x_*, r_*
- Bifurcation of fixed points can only happen when $f_x = \pm 1$.
  - For bifurcations of fixed points, we require $\begin{cases} f(x,r) - x = 0 \\ f_x(x,r) \pm 1 = 0 \end{cases}$.

Newton's method for root finding ($\begin{cases} f(x,y) = 0 \\ g(x,y) = 0 \end{cases}$)

- Initial guess (x^0, y^0).
- Use tangent approximations to f and g based on (x^0, y^0).
  - $f(x,y) = f(x^0,y^0) + f_x(x - x_0) + f_y(y - y^0)$.
  - $g(x,y) = g(x^0,y^0) + g_x(x - x_0) + g_y(y - y^0)$.
- Take the next approximation to be the root of the linear system
  - $f(x^0,y^0) + f_x(x^1 - x^0) + f_y(y^1 - y^0) = 0$.
  - $g(x^0,y^0) + g_x(x^1 - x^0) + g_y(y^1 - y^0) = 0$.
- This is a linear system $J\begin{pmatrix} x^1 - x^0 \\ y^1 - y^0 \end{pmatrix} = -R^0$
  - $J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ is a 2×2 Jacobian matrix evaluated at (x^0, y^0).
  - $R^0 = \begin{pmatrix} f(x^0,y^0) \\ g(x^0,y^0) \end{pmatrix}$ is the residue.
  - Assume J is invertible, it will be invertible for x^0, y^0 near x_*, y_*.
- Solve $Jz = -R^0$
- Then $\begin{pmatrix} x^1 \\ y^1 \end{pmatrix} = \begin{pmatrix} x^0 \\ y^0 \end{pmatrix} + z$
- Repeat
- This is a vector discrete dynamic system with a super stable fixed point at the root

Types of bifurcation

- Appearance of equilibrium points (saddle-node bifurcations)
  - $f(x_*, r_*) = x_*$.
  - $f_x(x_*, r_*) = 1$.
  - $f_r(x_*, r_*) = a > 0$.
  - $f_{xx}(x_*, r_*) = -b < 0$.
- Pitchfork bifurcation
  - Isolated critical point $x_c(r)$ that changes stability
    - $f_x(x_c(r), r)\begin{cases} < 1, r < r_* \\ = 1, r = r_* \\ > 1, r > r_* \end{cases}$.
  - For this to occur
    - $f_r(x_*, r_*) = 0$.
    - $f_{xx}(x_*, r_*) = 0$.
    - $f_{xr}(x_*, r_*) = a > 0$.
    - $f_{xxx}(x_*, r_*) = -b < 0$.
  - For $r > r_*$, there are two stable fixed points of g, $x_\pm(r) = x_* \pm \sqrt{\frac{6a}{b}(r - r_*)}$.
    - $x_\pm$ cannot be fixed points of f
    - To be a fixed point of g, they must be a periodic orbit of f.
- Flip bifurcation
  - Lone critical point $x_c(r)$ with $\frac{\partial f}{\partial x}(r, x_c(r)) < 0$

    - Change stability with $\frac{\partial f}{\partial x}(r, x_c(r)) = -1$.
    - At the double map, $\frac{\partial g}{\partial x} = 1$.
    - It is supercritical if signs: $g_{xr} > 0$ and $g_{xxx} < 0$.

Model a system as a discrete dynamic system

- x^n: number of members of a UBC club in year n ($n = 0$ at year 1990)
- Parameters
    - a: number of potential new members to the club every year
    - The number of people signing up to year $n+1$ depends on how happy people were in year n
        - $H(0) = 0$.
        - $H(b) = 0$ ($b > 0$).
        - Simplest function: $H(x) = \frac{4}{b^2}x(b-x)$.
    - The fraction F ($0 \le F \le 1$) of people signing up depends on how happy the club was the year before
        - $F(H) = e^{r(H-1)}$.
- $x^{n+1} = f(x^n; a, b, r)$.
    - With parameters above, $f = (a+x)e^{r\left(\frac{4}{b^2}x(b-x)-1\right)}$.
- We can scale (nondimensionalize) to reduce the number of parameters to 2
    - Scale x by b: $x = yb$, then $\frac{4}{b^2}x^n(b-x^n) = 4y^n(1-y^n)$.
    - Then $y^{n+1} = \left(\frac{a}{b} + y^n\right)e^{r(4y^n(1-y^n)-1)}$ and we can replace $\alpha = \frac{a}{b}$.

Scalar dynamics

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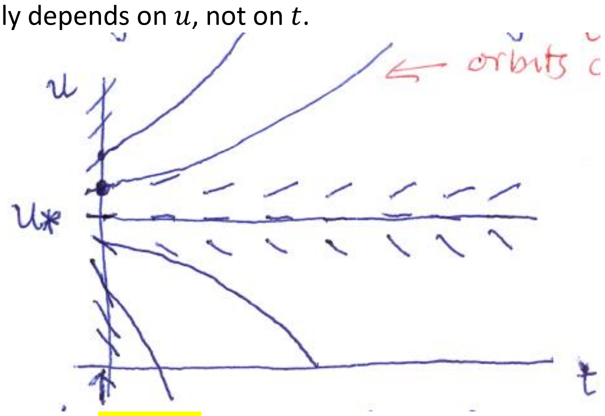
Continuous time dynamic system (scalar quantities) $u(t)$:

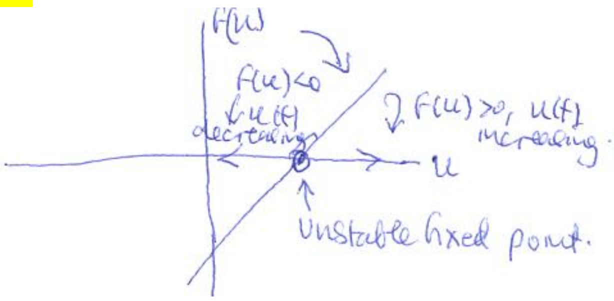
- $\dot{u} = u' = \frac{du}{dt} = f(u; a)$.
  - $f(u; a)$ has no time dependence (autonomous).
- Initial condition: $u(0) = u_0$.
- $u(t)$: solutions(orbits, trajectories).

For $u' = f(u)$, $u(t_0) = u_0$.

- Theorem: if f has a continuous derivative in an open interval containing u_0, then it has a unique solution for t in an open interval containing t_0.

Direction field

- Only depends on u, not on t.
- 

  - u_* is unstable.
  - There are only 3 orbits to consider. Although they look different, they are the same function, shifted in time
  - Phase plot
    - 

- If $f(u) > 0$, then $\frac{du}{dt} > 0$, so $u(t)$ is increasing
- If $f(u) < 0$, then $\frac{du}{dt} < 0$, so $u(t)$ is decreasing
- If $f(u_*) = 0$, for a certain value of u_*, then $u(t) = u_*$ for all t, u_* is an equilibrium solution (fixed point).
- Due to uniqueness, trajectories cannot cross
- Direction fields lead to the simplest numerical method (forward Euler's method) to approximate differential equations

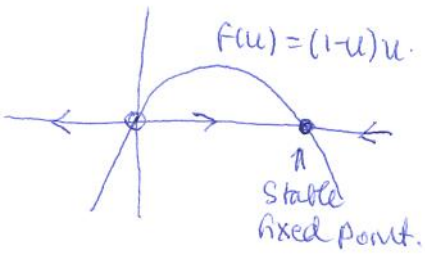
Stability of fixed points

- If u_* is an equilibrium, $f(u_*) > 0$ for $u > u_*$ and $f(u_*) < 0$ for $u < u_*$, i.e. $f'(u_*) > 0$ u_* is unstable.
- Similarly, if $f'(u_*) < 0$, then u_* is stable.
- If $f'(u_*) = 0$, can't tell (possible bifurcation).

Simple analytic solution to $\dot{u} = \lambda u$.

- $u(t) = u_0 e^{\lambda t}$.

Logistic equation

- $\dot{u} = u(1-u)$.
- Fixed points: $f(u) = u(1-u) = 0$, $u = 0$ or $u = 1$.
- $u = 1$ is stable and locally attractive (attracts all positive u).
    - $f'(1) = -1$.
- $u = 0$ is unstable.
    - $f'(0) = 1$.
- $u < 0$ are attracted to $-\infty$.
- Analytic solution: $u(t) = \frac{u_0 e^t}{(1-u_0)+u_0 e^t}$.
- Phase plot
    - 

Limiting behaviors

- If u_* is a fixed point, then $u(t) = u_*$.
- Otherwise, $\lim_{t\to\pm\infty} u(t)$ is one of ∞, $-\infty$, a fixed point of f.

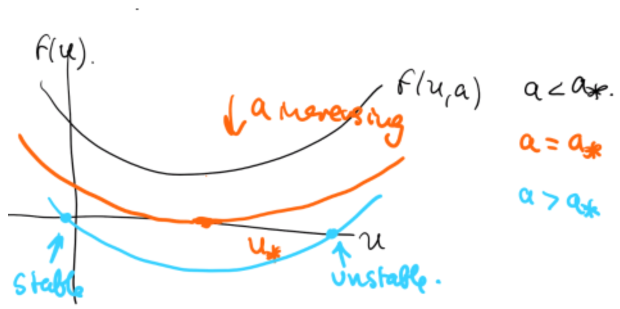
In 1D continuous time, there are no periodic solutions.

Fixed point approximation

- $f'(u_*) \neq 0$, $f(u_*) = 0$, then $\frac{du}{dt} = f(u) \approx f'(u_*)(u - u_*)$.
- Let $v = u - u_*$ be the signed distance to the fixed point.
- $\frac{du}{dt} = \frac{dv}{dt} = f'(u_*)v$, so $v = v_0 e^{f'(u_*)t}$ grows or decays exponentially.
- The time scale of exponential growth/decay is $\frac{1}{|f'(u_*)|}$.
- If $|f'(u_*)|$ is large, trajectories move quickly away/towards the equilibrium

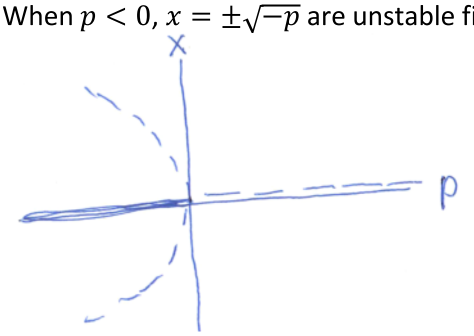
Bifurcation

- Theorem: if u_* is a hyperbolic fixed point for a certain parameter value a_*, then in a neighborhood of a_*, there is a single hyperbolic fixed point $u(a) \leftarrow u(a_*) = u_*$ and u depending continuously on a with the same stability.
- Bifurcations in fixed points can only happen when $\frac{\partial f}{\partial u} = 0$.
- Possible bifurcation points (u, a) satisfy
    - Fixed point: $f(u, a) = 0$.
    - Not hyperbolic: $f_u(u, a) = 0$.
- Types
    - Saddle node (one stable & one unstable)
        - Equilibrium appear where there were none before

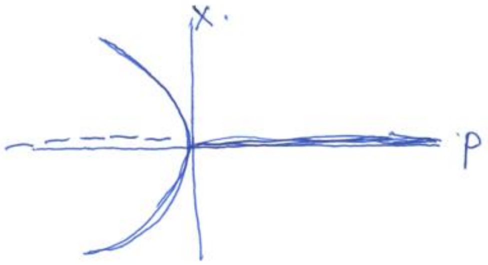
- 

    - $f_a(u_*, a_*) < 0$.
    - $f_{uu}(u_*, a_*) > 0$.
- As long as $f_a(u_*, a_*) \neq 0$ and $f_{uu}(u_*, a_*) \neq 0$, we get a saddle node bifurcation
- Taylor approximation: $f(u, a) = f_a(a - a_*) + \frac{1}{2} f_{uu}(u - u_*)^2$.
    - If $\frac{f_{uu}}{2f_a} > 0$, no fixed point for $\alpha > 0$, $u = u_* \pm \sqrt{-\frac{2f_a}{f_{uu}}(a - a_*)}$ for $\alpha < 0$.
    - Stability of u depends on the sign of f_{uu}.
- Normal form
    - From Taylor approximation, letting $v = x\frac{1}{\sqrt{|f_{uu}|/2}}$, $\alpha = p\frac{-sign(f_{uu})}{f_a}$.
    - $f(x, p) = sign(f_{uu})(x^2 - p)$.

- Trans critical bifurcation
    - Two critical points meet and exchange stability
    - $f_a = 0, f_{uu} = B \neq 0, f_{aa} = C, f_{ua} = D$.
    - Let $v = u - u_*, \alpha = a - a_*$.
    - $f(v, \alpha) = \frac{1}{2}Bv^2 + \frac{1}{2}C\alpha^2 + D\alpha v$.
    - Let $v = x\frac{1}{\sqrt{|B|/2}}$, $\alpha = p\frac{\sqrt{\frac{|B|}{2}}}{D} sign(B)$.
    - $f(x, p) = sign(B)(x^2 + px + \gamma p^2)$, where $\gamma = \frac{BC}{4D^2}$.
        - $x = \frac{1}{2}\left(-p \pm \sqrt{p^2(1 - 4\gamma)}\right)$.
        - If $4\gamma > 1$, no real roots unless $p = 0$.
        - If $4\gamma < 1$, trans critical bifurcation.
            - Two lines with slopes $m_1 = -\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\gamma}$, $m_1 = -\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\gamma}$.
            - This implies $f_{uu} \cdot f_{aa} < f_{ua}^2$.

- Pitchfork bifurcation
    - $f_a = 0, f_{au} = B \neq 0, f_{uu} = 0, f_{uuu} = A \neq 0$.
    - $f(v, \alpha) = Bv\alpha + \frac{1}{6}Av^3$.
    - Normal form: $f(x, p) = x^3 + px = x(x^2 + p)$.
    - $A > 0$. (sub critical)
        - $x = 0$ is a fixed point, stable for $p < 0$, unstable for $p > 0$.
        - When $p < 0$, $x = \pm\sqrt{-p}$ are unstable fixed points.
        - 

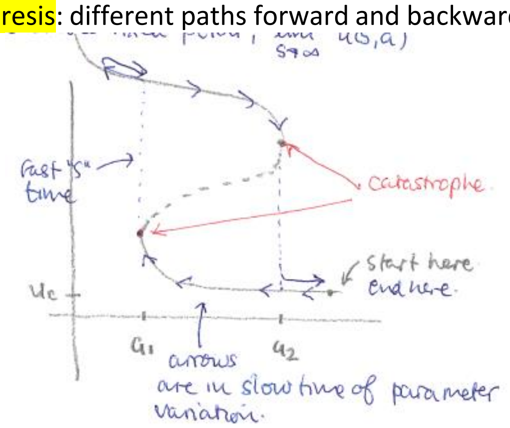
    - $A < 0$.(super critical)
        - $x = 0$, stable for $p > 0$, unstable for $p < 0$.
        - $x = \pm\sqrt{-p}$, stable

- 

Logistic equation

- $\dot{u} = au - \frac{a}{k}u^2$.
- Since $\dot{u} = au\left(1 - \frac{u}{k}\right)$, k is the carrying capacity.
- Scaling by $t = \frac{1}{a}s$, $u = kv$, $\frac{du}{ds} = v - v^2$.

Temperature in a chemical reaction

- Temperature dependent reaction rate: $Ae^{-\frac{E_a}{RT}}$.
  - A: rate constant $(1/s)$.
  - E_a: activation energy (J/mol)
  - R: idea gas constant $R = 8.314\frac{J}{mol\cdot K}$.
  - T: temperature in K.
- With M kg of reactant, net reaction rate $MAe^{-\frac{E_a}{RT}}$.
- Net heat generation: $HMAe^{-\frac{E_a}{RT}}$.
- c: thermal capacity of reactor & reactants(J/K).
- If insulated, $c\frac{dT}{dt} = HMAe^{-\frac{E_a}{RT}}$
  - $T \to \infty$ as $t \to \infty$.
- With coolant: $D(T - T_c)$, $c\frac{dT}{dt} = HMAe^{-\frac{E_a}{RT}} - D(T - T_c)$.
- Scaling by $T = \frac{E_a}{R}u$, $t = sJ$, $J = \frac{CE_a}{RHMA}$, $a = \frac{JD}{C} = \frac{DE_a}{RHMA}$, $\frac{du}{ds} = e^{-\frac{1}{u}} - a(u - u_c)$.
- Hysteresis: different paths forward and backward on bifurcation diagram.
  - 

Numerical approximations of differential equations

- Given $\dot{u} = f(u)$, $u(0) = u_0$.
- Discretize in time computed values.
  - $u^n = u(nk)$.
  - $u^0 = u(0)$ exact.
  - $k = \Delta t = \frac{T}{N}$ (time steps).
- Need to see convergence $\lim_{k\to 0, N\to\infty} \max_{0\le j\le N}\left|u^j - u(jk)\right| = 0$.
- Schemes
  - Approximation of the map $u^n \to u^{n+1}$.
  - $u^{n+1} = u^n + k\frac{du}{dt}(nk) + \frac{k^2}{2}\frac{d^2u}{dt^2}(\theta)$.
  - Numerical approximation: $u^{n+1} = u^n + kf(u^n)$

    - Euler's method/forward Euler/explicit Euler
- Theorem (convergence): $\max_{0\le j\le N}\left|u^j - u(jk)\right| < const \cdot k$.

Vector dynamics

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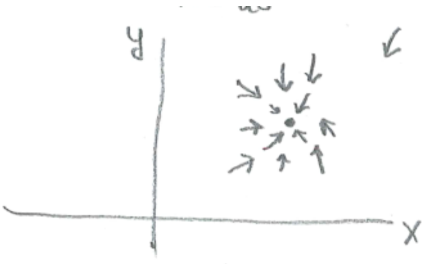
Vector solution

- $X(t) = \big(x(t), y(t)\big)$.
- Autonomous component: $\dot{X} = f(X)$.
  - $\dot{x} = f\big(x, y\big)$.
  - $\dot{y} = g\big(x, y\big)$.

Direction field:

- 

- At every x, y, draw a scaled vector

Linear systems

- $\dot{X} = AX$, A is a 2×2 matrix with real distinct non-zero eigenvalues λ_1, λ_2.
  - Non zero eigenvalues ensure $X = 0$ is the only fixed point.
  - $\lambda_1 \neq \lambda_2$ ensures the eigenvectors $\{v_1, v_2\}$ form a basis.
- General solution: $X = ae^{\lambda_1 t}v_1 + be^{\lambda_2 t}v_2$.
  - a and b can be chosen to match any initial conditions $X(0) = X_0$.
  - $X(0) = av_1 + bv_2 = \big(v_1 | v_2\big)\begin{pmatrix} a \\ b \end{pmatrix}$ a linear system, solvable since v_1 and v_2 are linearly independent.

Fixed points $\big(x_*, y_*\big)$:

- $f\big(x_*, y_*\big) = g\big(x_*, y_*\big) = 0$.
- With λ_1, λ_2 eigenvalues of A.
  - If $\lambda_1 < 0$ and $\lambda_2 < 0$, $X = 0$ is stable (stable node)
  - If $\lambda_1 > 0$ and $\lambda_2 > 0$, $X = 0$ is unstable (unstable node)
  - If $\lambda_1 < 0$ and $\lambda_2 < 0$, $X = 0$ is unstable (saddle point)
  - If $\lambda_1 \lambda_2 = 0$, potential bifurcation.

Linear second order systems

- Introduce $y = \dot{x}$, can convert into $\begin{pmatrix} x \\ y \end{pmatrix}' = A\begin{pmatrix} x \\ y \end{pmatrix}$.
- e.g.
  - $\ddot{x} + 4\dot{x} + 3x = 0$, with $y = \dot{x}$, $\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} 0 & 1 \\ -3 & -4 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$.
  - $\dddot{x} + \sin \ddot{x} + \dot{x}^2 + 3x^2 = 0$, with $y = \dot{x}$, $z = \ddot{x}$, $\begin{pmatrix} x' = y \\ y' = z \\ z' = -\sin z - y^2 - 3x^2 \end{pmatrix}$.
- System of second order equations
  - $\ddot{x} + y\dot{x} + y^2 = 0$, $\ddot{y} + x(\dot{x})^2 + \sin y = 0$.
  - Let $u = \dot{x}$, $v = \dot{y}$, $\dot{u} = \ddot{x}$, $\dot{v} = \ddot{y}$.
  - A system of four equations
    - $\dot{x} = u$.
    - $\dot{y} = v$.
    - $\dot{u} = -yu - y^2$.

- $\dot{v} = -xu^2 - \sin y$.

Nonlinear systems

- Fixed points
  - Find linear approximation around the fixed point
    - $f(x,y) = f(x_*,y_*) + f_x(x - x_*) + f_y(y - y_*)$.
    - $g(x,y) = g(x_*,y_*) + g_x(x - x_*) + g_y(y - y_*)$.
  - Let $u = x - x_*, v = y - y_*$.
    - $\begin{pmatrix} u \\ v \end{pmatrix}' = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix}$.
    - $J = \begin{pmatrix} f_x & f_y \\ g_x & g_y \end{pmatrix}$ is the Jacobian matrix evaluated at x_*, y_*.
  - If the eigenvalues of J are real and nonzero, we can determine the stability of (x_*, y_*) as before.
    - If $\lambda_1 < 0$ and $\lambda_2 < 0$, $X = 0$ is stable (stable node)
    - If $\lambda_1 > 0$ and $\lambda_2 > 0$, $X = 0$ is unstable (unstable node)
    - If $\lambda_1 < 0$ and $\lambda_2 > 0$, $X = 0$ is unstable (saddle point)

Gradient flows

- Given potential $V(x,y)$ and $f(x,y) = -V_x$, $g(x,y) = -V_y$, $\dot{X} = -\nabla V$ is the gradient flow.
- Note: $\frac{dV}{dt} = -V_x^2 - V_y^2 = -|\nabla V|^2$.
  - V is always decreasing on trajectories, except at fixed points.
  - So, gradient flows cannot have periodic orbits.
- Fixed points.
  - $\nabla V = 0$ (critical points for V).
  - Consider the Jacobian matrix: $J = \begin{pmatrix} -V_{xx} & -V_{xy} \\ -V_{xy} & -V_{yy} \end{pmatrix}$.
    - It is symmetric, and thus always have real eigenvalues with orthogonal eigenvectors
  - $\lambda_1 \lambda_2 = \det J = V_{xx} V_{yy} - V_{xy}^2 = D$.
  - If $D > 0$, then either $\lambda_1 \& \lambda_2 > 0$ or $\lambda_1 \& \lambda_2 < 0$, node (can only occur when V_{xx} and V_{yy} have the same sign)
    - If $V_{xx} < 0$, then $\lambda_1 > 0$, $\lambda_2 > 0$, unstable, local max.
    - If $V_{xx} > 0$, then $\lambda_1 < 0$, $\lambda_2 < 0$, stable, local min.
  - If $D < 0$, then $\lambda_1 \lambda_2 < 0$, saddle point.
  - If $D = 0$, cannot tell
- Contours for V.
  - Contours are approximately ellipses near local min/max
  - Contours are approximately hyperbolas near a saddle point
  - ∇ is perpendicular to contour lines.

Steepest descent method for finding local minima of $V(x)$.

- Local minima have $\nabla V = 0$, could do root finding using $J = H$ ($n \times n$ matrix) and solving a linear system iteratively.
  - If n is too large, too computationally expensive.
- Idea: follow V downhill with a gradient flow $\dot{X} = \nabla$.
- Use forward Euler time stepping, $X^{n+1} = X^n - k\nabla V(X^n)$.
  - k is the time step adjusted to get the fastest convergence to the minima possible.

Complex eigenvalues

- $e^{(a+bi)t} = (\cos bt + i \sin bt) e^{at}$.
- Growth or decay determined by $a = Re(\lambda)$.
  - $Re(\lambda) < 0$ decay.
    - $x = 0$ is a stable equilibrium (spiral sink).
    - This can be directly known from eigenvalues (eigenvectors are not needed)

- $Re(\lambda) > 0$ growth.
- If $Re(\lambda) = 0$, oscillations only, neither growth nor decay.
  - $x = 0$ is stable but not asymptotically stable.
- Oscillatory (angular frequency) determined by $b = Im(\lambda)$.
  - Frequency: $\frac{Im(\lambda)}{2\pi}$.
  - Period: $\frac{2\pi}{Im(\lambda)}$.

General solutions

- Complex form $x = c_1 e^{\lambda_1 t} v_1 + c_2 e^{\lambda_2 t} v_2$.
  - c_1 and c_2 are conjugates.
  - To decide whether the spiral is clockwise or counter-clockwise, consider the system on x axis.
- Real form $x = d_1 Re(e^{\lambda_1 t} v_1) + d_2 Im(e^{\lambda_1 t} v_1)$.

Fixed points

- Attracting if all points start close enough to x_* tend to x_* as $t \to \infty$.
- An attracting fixed point has a basin of attraction
  - All points which we start at and tend to x_* as $t \to \infty$.
- If all points are in the basin of attraction, we say x_* is a global attractor.
- x_* is Liapunov stable if all trajectories that start close enough to x_* remain close enough to it for all time
- Liapunov stable but not attracting (like center) is called neutrally stable
- Liapunov stable and attracting (spiral) is called asymptotically stable
- For non-linear 2D problems, we can have attracting fixed points that are not Liapunov stable

Some techniques to understand nonlinear systems

- Heteroclinic trajectory: trajectories connecting saddle nodes.
- Could plot some solutions (numerical approximations) starting at different initial conditions
- Trajectories passing through (x, y) have tangent vector $t = \big(f(x, y), g(x, y)\big)$. Could plot these on a grid (vector field). Could also plot the unit vectors
- Could plot nullclines: curves $C_1: f(x, y) = 0$, $C_2: g(x, y) = 0$.
  - Intersections of C_1 and C_2 are fixed points.
  - Trajectories crossing C_1 must be vertical, going up if $g > 0$, down if $g < 0$.

Things we want to find

- Fixed points & periodic orbits & stability
- Basins of attractions of attracting features
- Bifurcations (changes in the structure of phase plane portrait with parameter variation)

Rabbit and sheep model

- x is the scaled rabbit population, y is the scaled sheep population
- $\dot{x} = f(x, y) = x(3 - x - 2y)$.
- $\dot{y} = g(x, y) = y(2 - x - y)$.
- Two things to do with the fixed points
  - Eigen analysis of Jacobian matrix at fixed points
  - Plot nullclines $f(x, y) = 0$, $g(x, y) = 0$.
- 4 fixed points
  - $(0,0)$, $J = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$, $\lambda_1 = 3$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = 2$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, nodal source.
  - $(0,2)$, $J = \begin{pmatrix} -1 & 0 \\ -2 & -2 \end{pmatrix}$, $\lambda_1 = -1$, $v_1 = \begin{pmatrix} 1 \\ -2 \end{pmatrix}$, $\lambda_2 = -2$, $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, nodal sink.
  - $(3,0)$, $J = \begin{pmatrix} -3 & -6 \\ 0 & -1 \end{pmatrix}$, $\lambda_1 = -3$, $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, $\lambda_2 = -1$, $v_2 = \begin{pmatrix} 3 \\ -1 \end{pmatrix}$, nodal sink.
  - $(1,1)$, $J = \begin{pmatrix} -1 & -2 \\ -1 & -1 \end{pmatrix}$, $\lambda_1 = -1 + \sqrt{2}$, $v_1 = \begin{pmatrix} -\sqrt{2} \\ 1 \end{pmatrix}$, $\lambda_2 = -1 - \sqrt{2}$, $v_2 = \begin{pmatrix} \sqrt{2} \\ 1 \end{pmatrix}$, saddle.

- Nullclines:.
    - $x = 0, y = -\frac{x}{2} + \frac{3}{2}$.
    - $y = 0, y = -x + 2$.

Conservative systems

- $\ddot{x} = f(x)$ (force only depends on position, undamped spring).
    - $V(x) = -\int_0^x f(s)ds$.
- Let $u = \dot{x}$, $\begin{cases} \dot{x} = u \\ \dot{u} = f(x) = -V'(x) \end{cases}$.
- Consider $E(t) = \frac{1}{2}\left(u(t)\right)^2 + V\left(x(t)\right)$.
    - $\frac{dE}{dt} = 0$, E is conserved in time (constant along trajectories).
- $E(x, u) = \frac{1}{2}u^2 + V(x)$.
    - Fixed points: $u = 0, V'(x_*) = 0$.
    - Assume $V''(x_*) \neq 0$ (local min or max).
    - $J = \begin{pmatrix} 0 & 1 \\ -V'' & 0 \end{pmatrix}$, $\lambda^2 + V'' = 0$.
    - At min of V, $V'' > 0, \lambda = \pm i\sqrt{-V''}$ (linear center).
        - $E \approx E(0, x_*) + \frac{1}{2}u^2 + \frac{1}{2}V''(x_*)(x - x_*)^2$, x_* is stable but not asymptotically stable.
        - Contours are ellipses
    - At max of V, $V'' < 0, \lambda = \pm\sqrt{-V''}$, E has a saddle point.
        - $E \approx V(x_*) + \frac{1}{2}u^2 + \frac{1}{2}V''(x_*)(x - x_*)^2$.
        - Contours are hyperbolas

1D Index theory

- $\dot{x} = f(x, a)$ smooth. Consider an interval $[0, b]$ with $f(0) \neq 0$ and $f(b) \neq 0$.
- Define the index of f on $[0, b]$ as $Ind(f, 0, b) = \frac{f(0)f(b)}{|f(0)||f(b)|} = \begin{cases} 1, f(0)f(b) > 0 \\ -1, f(0)f(b) < 0 \end{cases}$.
- It is continuous and constant in a neighborhood of b that contains no fixed points.
- If $c \in (0, b)$ and $f(c) \neq 0$, then $Ind(f, 0, b) = Ind(f, 0, c) \cdot Ind(f, c, b)$.
- If $[0, b]$ contains only hyperbolic fixed points and there are p of them, then $Ind(f, 0, b) = (-1)^p$.
- Note: the index is continuous in the parameter space as long as fixed points do not cross 0 or b.
- After a bifurcation the number of hyperbolic fixed points must be even if they were even before, and odd if they were odd before
    -

bifurcation	Before	After
SN	0	2
Transcritical	2	2
Pitchfork	3	1

2D index theory

- $\dot{x} = f(x, a)$ smooth. Consider a closed curve C in the phase plane. It doesn't have to be a trajectory, but $f(x) \neq C$ for all $x \in C$.
- Let $\phi(x) \in [0, 2\pi)$ be the angle that corresponds to f on C
    - $\phi = \text{atan2}\left(g(x, y), f(x, y)\right)$.
    - $\phi = \text{Arg}(f + ig)$.
- Index of C is defined as $I(f, C) = \frac{1}{2\pi}[\phi]_C$, where $[\phi]_C$ is the change in ϕ as x go around C counter-clockwise.
    - $I(f, C)$ is an integer

- It is the net number of counter-clockwise revolutions made by the vector field as x go around C counter-clockwise.
- If C is continuously deformed to C' without crossing any fixed points, $I(f,C) = I(f,C')$ because it is continuous and is an integer

- Properties
  - If C does not contain any fixed points, $I(f,C) = 0$.
  - If C is a periodic orbit, $I(f,C) = 1$.
    - Every periodic orbit must contain at least 1 fixed point.
  - If $t \to -t$ (backwards in time), $f \to -f$, $\phi \to \phi + \pi$, index does not change $I(f,C) = I(-f,C)$.
  - We can define the index of an isolated fixed point I_p, nodal source/sink, spiral source/sink all have index 1, saddle has index -1.
    - If the fixed points are hyperbolic, $I_p = \pm 1$.
- If C surrounds n isolated fixed points $p_1, \dots, p_n$, then $I(f,C) = \sum_{i=1}^{n} I_{p_i}$.
- Index of C does not change with dynamic system parameters as long as no fixed point cross the curve
  - When $a > 0$, no fixed point, $I(C) = 0$, then $I(C) = 0$ for $a < 0$.
- Bifurcations with index theory
  - Subcritical pitchfork (total index -1)
$a < 0$ one saddle	$a > 0$, 2 saddles + 1 node
  - Supercritical pitchfork (total index 1)
$a < 0$ one node	$a > 0$, 2 nodes + 1 saddle
  - Trans-critical (total index 0)
Saddle + node	Node + saddle
  - Hopf bifurcation (total index 1)
Spiral sink	Spiral source

Limiting behaviors

Discrete $x^{n+1} = f(x^n)$	Continuous 1D $\dot{x} = f(x)$	Continuous 2D, $\dot{x} = f(x)$
$\|x^n\| \to \infty$	$x(t) \to \pm\infty$	$\|x\| \to \infty$
$x^n \to x_*$ (fixed point)	$x^n \to x_*$ (fixed point)	$x^n \to x_*$ (fixed point)
$x^n \to$periodic orbit ($\{x_1, x_2, \dots, x_n\}$)	No periodic orbits	Has periodic orbit
Chaotic (strange) attractor	No chaos	No chaos

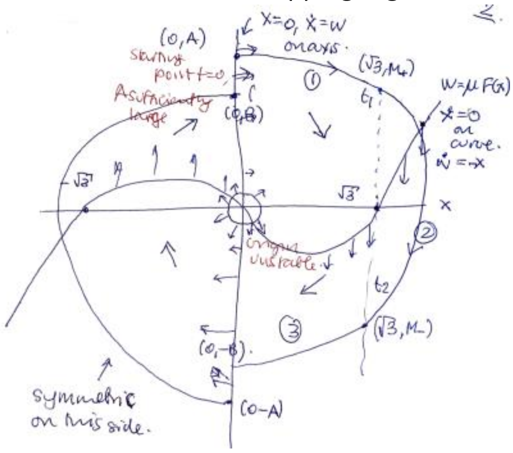
Poincare-Bendixson theorem

- If the following 3 assumptions are satisfied, then R contains a closed orbit(periodic orbit).
  - R is a closed bounded subset of the plane
  - R contains no fixed points (can be an annulus)
  - there is a trajectory T that is confined in R.
- Let ξ_n be the distance of x_n to L, the map $\xi \to$ next crossing of L is called the Poincare map, which can be used to determine the stability of the periodic orbit.
- The P-B provided periodic orbit must be at least one-sided stable
- We can use P-B theorem to prove the existence of periodic orbits using a trapping region R (closed and bounded annulus, in which at the boundaries, all trajectories point inwards)
  - If the trapping region contains no fixed point, then it contains a periodic orbit

Converting from cartesian coordinates to polar coordinates

- $\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}$.
- $\dot{\theta} = \frac{x\dot{y} - y\dot{x}}{r^2}$.

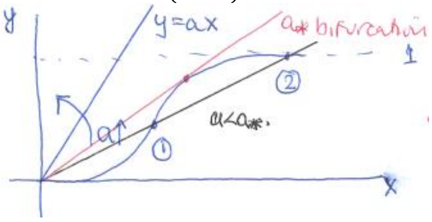
Van der Pol oscillator

- $\ddot{x} + \mu(x^2 - 1)\dot{x} + x = 0$, $\mu > 0$.
- Write in terms of first order systems
  - Let $F(x) = \frac{x^3}{3} - x$, $F'(x) = x^2 - 1$, $w = \dot{x} + \mu F(x)$
  - $\dot{x} = w - \mu F(x)$, $\dot{w} = -x$.
- Only fixed point, $x = w = 0$
  - $J = \begin{pmatrix} \mu & 1 \\ -1 & 0 \end{pmatrix}$, $\lambda = \frac{\mu}{2} \pm \sqrt{\frac{\mu^2}{4} - 1}$.
  - Spiral source for $0 < \mu < 2$.
  - Nodal source for $\mu > 2$.
- Consider $\frac{d}{dt}(x^2 + w^2) = -2\mu\left(\frac{x^4}{3} - x^2\right)$.
  - Circles won't work for a trapping region.
  - 

- To show $B < A$.
  - Consider $I = -2\mu \int_0^{t_3} \frac{x^4}{3} - x^2 dt$.
  - $I_1 = -2\mu \int_0^{t_1} \frac{x^4}{3} - x^2 dt \sim \frac{1}{A}$.
  - $I_2 = -2\mu \int_{t_1}^{t_2} \frac{x^4}{3} - x^2 dt \to -\infty$.
  - $I_3 = -2\mu \int_{t_2}^{t_3} \frac{x^4}{3} - x^2 dt \sim \frac{1}{A}$.
- In polar coordinates
  - $\dot{r} = -\frac{\mu x F(x)}{r}$.
  - $\dot{\theta} = -1 + \frac{\mu w F(x)}{r^2}$.
- When $\mu = 0$.
  - $\begin{pmatrix} \dot{x} \\ \dot{w} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} x \\ w \end{pmatrix}$, $\lambda = \pm i$, origin is a linear center.
  - $\dot{r} = 0$, $\dot{\theta} = -1$.
  - Circles period 2π, clockwise.
- For small $\mu > 0$, we get a periodic orbit at radius 2
  - $r \approx r_* + \mu r_1(t)$.
  - $\theta \approx -t + \mu\theta_1(t)$.
  - Boundary value problem:
    - $\dot{r} = -\frac{\mu x F(x)}{r}$, $\dot{\theta} = -1 + \frac{\mu w F(x)}{r^2}$.
    - $\theta(0) = 0$, $\theta(T) = 2\pi$, $r(0) = r(T)$.
  - Use approximation, $\dot{r} = \mu \dot{r_1} = -\mu \cos t\, F(r_* \cos t)$.
    - $0 = r_1(2\pi) - r_1(0) = \int_0^{2\pi} \left(\frac{r_*^3}{3}\cos^4 t - r_* \cos^2 t\right) dt = \pi r_* \left(\frac{r_*}{2} - 1\right)$.
    - So $r_* = 2$.

2D bifurcation example

- $\dot{x} = -ax + y$.

- $\dot{y} = \frac{x^2}{1+x^2} - y$.
- Plot the nullclines where $\dot{x} = 0$ or $\dot{y} = 0$.
    - $y = ax, y' = a$.
    - $y = \frac{x^2}{1+x^2}, y' = \frac{2x}{(1+x^2)^2}$.
    - 

- Saddle node bifurcation at a_*.
    - One eigendirection at a_*, x_*, y_* has eigenvalue 0.
    - The other eigendirection will have nonzero eigenvalue
- Find a_*, x_*, y_*. (intersection of nullclines)
    - $x_* = 1, a_* = \frac{1}{2}, y_* = \frac{1}{2}$.
- Evaluate the Jacobian at the bifurcation point.
    - $J_* = \begin{pmatrix} -\frac{1}{2} & 1 \\ \frac{1}{2} & -1 \end{pmatrix}$.
    - $\lambda_1 = 0, v_1 = \begin{pmatrix} 1 \\ \frac{1}{2} \end{pmatrix}$.
    - $\lambda_2 = -\frac{3}{2}, v_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.
- 2 is nodal sink, 1 is saddle

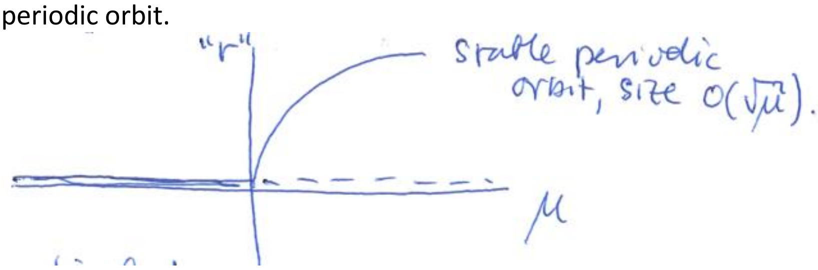
Summary of 2D bifurcation

- If a hyperbolic fixed point with real eigenvalues changes type or appears at a_*, then J_* has eigenvalue $\lambda_1 = 0$ with eigenvector v_1, and nonzero λ_2 with eigenvector v_2.
- Near a_*(both sides), solutions either grow or decay in v_2 direction.
- There is a 1D bifurcation (saddle node, transcritical, pitchfork) in v_1 direction.

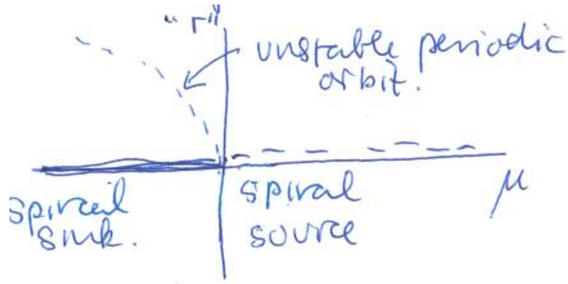
Hopf bifurcation

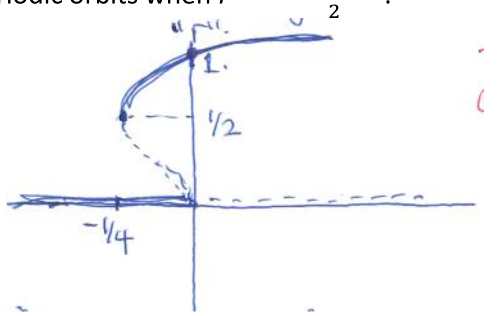
- Spiral source ↔ spiral sink.
- Only possible with complex eigenvalues $\lambda = \alpha + \beta i$.
    - $\alpha(a_*) = 0$ (changes between positive and negative).
    - $\beta(a_*) \neq 0$.

Example in polar coordinates

- Assume $\dot{\theta} = 1$ in all cases.
- Supercritical hopf bifurcation
    - $\dot{r} = \mu r - r^3$.
    - If $\mu < 0, \dot{r} < 0$ for all $r, r = 0$ is spiral sink (global attractor).
    - If $\mu > 0, \dot{r} > 0$ for $0 < r < \sqrt{\mu},\ r = 0$ is unstable spiral source, $r = \sqrt{\mu}$ is a stable periodic orbit.
    - 

- Subcritical Hopf bifurcation
    - $\dot{r} = \mu r + r^3$.
    - If $\mu > 0, \dot{r} > 0$ for all $r > 0, r = 0$ is a spiral source.
    - If $\mu < 0, \dot{r} < 0$ for $0 < r < \sqrt{-\mu}, r = 0$ is spiral sink, $r = \sqrt{-\mu}$ is unstable periodic orbit.

- 

- Subcritical but with hysterisis.
  - $\dot{r} = \mu r + r^3 - r^5$.
  - Periodic orbits when $r^2 = \frac{1 \pm \sqrt{1+4\mu}}{2}$.
  - 

  - Near $r = 0$, $\mu = 0$, still a subcritical Hopf bifurcation.