Curves

January 11, 2021 8:49 AM

Hyper trig:

- $\sinh t = \dfrac{e^t - e^{-t}}{2}$
- $\cosh t = \dfrac{e^t + e^{-t}}{2}$
- $\cosh^2 t - \sinh^2 t = 1$
- $\sinh' t = \cosh t, \cosh' t = \sinh t$

General concepts:

- Def: a parametrized curve is a differentiable map $\alpha: I \to \mathbb{R}^n$ of an open interval $I = (a, b) \subset \mathbb{R}$.
- The image set $\alpha(I) \subset \mathbb{R}^n$ is called the trace of α
- α is regular if $\alpha'(t) \neq 0$ for all $t \in I$.
  - If α is regular, then there is a tangent line to the curve at every point
  - Any point t where $\alpha'(t) = 0$ is called a singular point

Arc length

- The arclength function of α from the point t_0 is $s(t) = \int_{t_0}^{t} |\alpha'(u)| du$ (the length of the part of the curve from $\alpha(t_0)$ to $\alpha(t)$)
- The arclength is invariant under reparameterization
- $s(t)$ is differentiable and $\frac{ds}{dt} = |\alpha'(t)|$
  - We say that $\alpha(t)$ is parametrized by arclength if t is the arclength from some point
  - Every regular curve can be parametrized by arclength, and $\alpha(s)$ has the property $|\alpha'(s)| = 1$(unit speed parametrization)
  - If $\alpha(t)$ is regular, then $s(t)$ has an inverse function $t(s)$ and $\frac{dt(s)}{ds} = \frac{1}{\frac{ds(t)}{dt}}$

Curvature:

- Def: let $\alpha: I \to \mathbb{R}^3$ parametrized by arclength $|\alpha'(s)| = 1$, $\kappa(s) = |\alpha''(s)|$ is the curvature of α at s. It measures how rapidly the curve pulls away from its tangent line at s
- For straight line, curvature is 0 (does not bend)
- For circles, curvature is the same at each point (constant bending)
- Note: when using arclength parametrization, $\alpha''(s)$ is orthogonal to $\alpha'(s)$
- Measures deviation of curve from being a line

Unit tangent vector: $T(s) = \alpha'(s)$

Unit normal vector: $N(s) = \frac{\alpha''(s)}{|\alpha''(s)|} = \frac{\alpha''(s)}{\kappa(s)}$

Osculating plane at s: plane determined by T and N

- Assume $\alpha'' \neq 0$(Frenet curve), then $B(s) = T(s) \times N(s)$ is normal to the osculating plane (binormal vector)
- $|B'(s)|$ is the rate of change of the angle o normal of neighboring osculating planes with the osculating plane at s

Torsion:

- $B'(s) = \frac{d}{dt}\big(T(s) \times N(s)\big) = T' \times N + T \times N' = T \times N'$ (since T' is parallel to N)
  - $B' \perp T$ and $B' \perp B$, $B' \parallel N$
  - So define torsion $\tau(s)$ such that $B'(s) = -\tau(s)N(s)$
- Measures deviation of curve from lying in a plane

Summary (Frenet equations): if $\alpha(s)$ parametrized by arclength with $\alpha''(s) \neq 0$

- $T' = \kappa\, N$
- $B' = -\tau N$
- $N' = B' \times T + B \times T' = \tau N \times T + B \times \kappa N = \tau B - \kappa T$
  - $N = B \times T$
- (T, N)-plane: osculating plane (plane that best fits the curve)
- (N, B)-plane: normal plane (unique plane normal to $\alpha(s)$,$\alpha'(s)$ at s)
- (T, B)-plane: rectifying plane (plane orthogonal to curvature vector)
  - The projection onto this plane straightens or rectifies $\alpha(s)$ in the sense that up to second order, the projected curve is a line
- If α is a Frenet curve, then $\tau = 0$ $\alpha \Leftrightarrow$ is a plane curve

Curves in $\mathbb{R}^2$: the curvature can be given a sign

- If $T = \alpha'$ is the unit tangent vector, then N_s=vector obtained by rotating T counter clockwise by $\frac{\pi}{2}$ (signed normal)
- Then $\alpha'' = T' = \kappa_s N_s$ gives the signed curvature κ_s
- Note: sign of curvature changes if we change the orientation of the curve
- Then Frenet equations in $\mathbb{R}^2$ become:
  - $T' = \kappa_s N_s$
  - $N_s{}' = -\kappa_s T$

Curves in $\mathbb{R}^n$:

- Let $\alpha: I \to \mathbb{R}^n$ regular, n-times continuously differentiable curve parametrized by arclength, α is a Frenet curve if for all s, $\alpha'(s), \alpha''(s), \dots, \alpha^{(n-1)}(s)$ are linearly independent, then there exists a unique Frenet-n-frame if
  - $e_1, \dots, e_n$ orthonormal vectors positively oriented
  - For $k = 1, \dots, n-1, \alpha^{(k)} \in span\{e_1(s), \dots, e_k(s)\}$
  - The inner product $< \alpha^{(k)}(s), e_k >> 0$ for all $k = 1, \dots, n-1$
- We can obtain the n-frame via Gram-Schmidt process
  - $e_1 = \alpha', e_2 = \frac{\alpha''}{|\alpha''|}, \dots, e_{n-1} = \frac{\alpha^{(n-1)} - \sum_{i=1}^{n-2} < \alpha^{(n-1)}, e_i > e_i}{|\alpha^{(n-1)} - \sum_{i=1}^{n-2} < \alpha^{(n-1)}, e_i > e_i|}$
  - e_n determined by (i)
- Frenet equations:
  Let α be a Frenet in $\mathbb{R}^n$ with Frenet-n frame. Then there exists differentiable function $R_i: I \to \mathbb{R}, i = 1, \dots, n-1$ (ith Frenet curvature) along the curve with $k_1, \dots k_{n-2} > 0$, such that $\begin{pmatrix} e_1 \\ e_2 \\ \dots \\ e_n \end{pmatrix}' = \begin{pmatrix} 0 & k_1 & \dots 0 \\ -k_1 & \dots & k_{n-1} \\ 0 & -k_{n-1} & 0 \end{pmatrix} \begin{pmatrix} e_1 \\ e_2 \\ \dots \\ e_n \end{pmatrix}$
  - E.g. in $\mathbb{R}^3$, we have $\begin{pmatrix} T \\ N \\ B \end{pmatrix}' = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}$
  - As torsion in $\mathbb{R}^3$, we can give k_{n-1} a sign

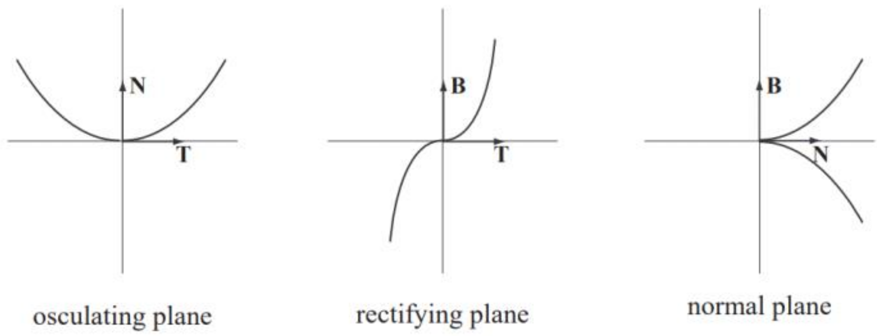
Calculate curvature and torsion by $\alpha(t)$

- $\kappa(t) = \frac{|\alpha' \times \alpha''|}{|\alpha'|^3}$
- $\tau(t) = \frac{\det(\alpha', \alpha'', \alpha''')}{|\alpha' \times \alpha''|^2}$
- Binormal vector $B(t)$ is parallel to $r'(t) \times r''(t)$

Local canonical form

- Using Taylor series and the following expressions,
  we can get $\alpha(s) = \alpha(0) + \left(s - \frac{k^2}{6}s^3\right)T + \left(\frac{ks^2}{2} + \frac{k's^3}{6}\right)N + \frac{k\tau s^3}{6}B + R$
  - $\alpha'(0) = T$
  - $\alpha''(0) = kN$

- $\alpha'''(0) = -k^2 T + k' N + k\tau B$
- $\lim_{s \to 0} \frac{R}{s^3} = 0$

- Rotate and translate the curve such that $\alpha(0) = 0, T = (1,0,0), N = (0,1,0), B = (0,0,1)$, we have
  - $x(s) = s - \frac{k^2}{6} s^3 + R_x$
  - $y(s) = \frac{k}{2} s^2 + \frac{k'}{6} s^3 + R_y$
  - $z(s) = \frac{k\tau}{6} s^3 + R_z$
  - In TN-plane(osculating): a quadratic $(x(s), y(s))$
  - In NB-plane(normal): not regular $(y(s), z(s))$
  - In TB-plane(rectifying): up to second order, the projected curve is a line $(x(s), z(s))$
  - 

- Interpretation of sign of torsion
  - Component of α in the B direction is $\frac{k\tau}{6} s^3$
  - If $\tau > 0$, as s increases, the curve is crossing the osculating plane toward the positive side
  - If $\tau < 0$, the curve is crossing the osculating plane in the negative B direction
  - The curve twists out of the osculating plane, τ measures the twisting or torsion

Osculating circle
- The osculating circle at s is the osculating plane at s, with center on the line in the direction of $N(s)$ and radius $\frac{1}{k(s)}$ that lies on the concave side of α
- It is the circle of closest fit to α at s
- The osculating circle is the unique circle $\beta(s)$ parametrized by arclength with $\beta(s) = \alpha(s), \beta'(s) = \alpha'(s)$, and $\beta''(s) = \alpha''(s)$

Characterize certain curves by properties of their curvature and torsion
- $k = 0$ straight line
- $k \neq 0$ and $\tau = 0$ plane curve
- $\tau = 0, k = const > 0$ circle
- $\tau = const \neq 0, k = const$ helix

Fundamental theorem of curves:
- If $k \neq 0$, the functions k and τ completely describe the curve geometrically
- Definition: $F: \mathbb{R}^n \to \mathbb{R}^n$ is an isometry (or rigid motion) if $|F(v) - F(w)| = |v - w|$ for all $v, w \in \mathbb{R}^n$
  - F is an isometry $\Leftrightarrow F(v) = Av + b$ where $A \in O(n)$ (orthogonal $n \times n$ matrix) and $b \in \mathbb{R}^n$
    - An $n \times n$ matrix is orthogonal if $A^T A = I$, or the columns of A are orthonormal vectors
    - Orthogonal matrices preserve the dot product i.e. $Av \cdot Aw = v \cdot w$
    - The curvature and torsion of a Frenet curve are invariant under orientation preserving isometries
- Let $k(s) > 0$ and $\tau(s)$, $s \in (a, b)$ be differentiable function. Let $s_0 \in (a, b), q_0 \in \mathbb{R}^3$ and let T_0, N_0, B_0 be orthonormal vectors. Then there exists a unique regular curve $\alpha: (a, b) \to$

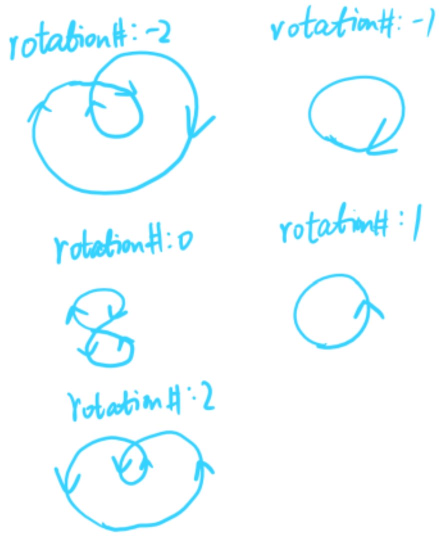
- $\alpha(s_0) = q_0$
- T_0, N_0, B_0 is the Frenet frame of α at s_0
- $k(s)$ is the curvature and $\tau(s)$ is the torsion of α
- Uniqueness: assume $\alpha, \beta: I \to \mathbb{R}^3$ satisfying $k_\alpha(s) = k_\beta(s)$ and $\tau_\alpha(s) = \tau_\beta(s)$ then β is the image of α under a rigid motion of $\mathbb{R}^3$
  - Exists an orthogonal matrix A with $\det A > 0$ and $b \in \mathbb{R}^3$ such that $\beta(s) = A\alpha(s) + b$
- In general, given k and τ, it is difficult to explicitly solve the Frenet equations. For plane curves, can explicitly determine the curve in terms of its curvature
  - Let $\alpha: (a, b) \to \mathbb{R}^2$ be plane curve parametrized by arclength
    $\alpha'(s) = (\cos\theta(s), \sin\theta(s))$, where $\theta(s)$ is the angle $\alpha'(s)$ makes with the positive $x - axis$ measured counter clockwise
    Then $\alpha''(s) = \frac{d\theta}{ds}(-\sin\theta(s), \cos\theta(s))$ where $(-\sin\theta(s), \cos\theta(s))$ is the normal vector
    $k_s(s) = \frac{d\theta}{ds}$ is the rate of change of the angle the tangent makes with the horizontal, $\theta(s) = \theta(s_0) + \int_{s_0}^{s} k_s(u)du$

Fundamental theorem for plane curve

Given $k: (a, b) \to \mathbb{R}$ differentiable, $s_0 \in (a, b), q_0 \in \mathbb{R}^2, T_0 = (\cos\theta_0, \sin\theta_0)$, there exists a unique $\alpha: (a, b) \to \mathbb{R}^2$ parametrized by arclength such that

- $\alpha(s_0) = q_0$
- $\alpha'(s_0) = T_0$
- $k(s)$ is the signed curvature
- $\alpha(s) = q_0 + \left(\int_{s_0}^{s} \cos\theta(u)\, du, \int_{s_0}^{s} \sin\theta(u)\, du\right)$ where $\theta(u)$ defined as above

Global properties of curves

- Def: α: $[a, b] \to \mathbb{R}^2$ is a closed curve if $\alpha(a) = \alpha(b)$, $\alpha'(a) = \alpha'(b)$,… (α and all its derivatives agree at a and b)
  - Def: α is simple if it has no further self-intersections
    $(t_1, t_2 \in [a, b), t_1 \neq t_2 \Rightarrow \alpha(t_1) \neq \alpha(t_2))$
- Let $\alpha: [0, l] \to \mathbb{R}^2$ be a closed curve parametrized by arc length. Define $\theta: [0, l] \to \mathbb{R}$ by $\theta(s) = \int_0^l k_s(s)ds$, θ is differentiable and $\theta'(s) = k_s(s)$. $\int_0^l k_s(s)ds = 2\pi I, I \in \mathbb{Z}$ since the curve is closed. The integer I is the rotation index (number)
  - $I = \frac{1}{2\pi}\int_0^l k_s(s)ds$
  - It measures total rotation of tangent vector as you go around the curve
  - 

  - Theorem of turning tangents: the rotation index of a simple closed curve is ± 1

where the sign depends on the orientation

- Isoperimetric inequality
  - Let C be a simple closed plane curve of length l and let A be the area bounded by C. Then $A \leq \frac{1}{4\pi} l^2$
  - Equality if and only if C is a circle

Surfaces: Local theory

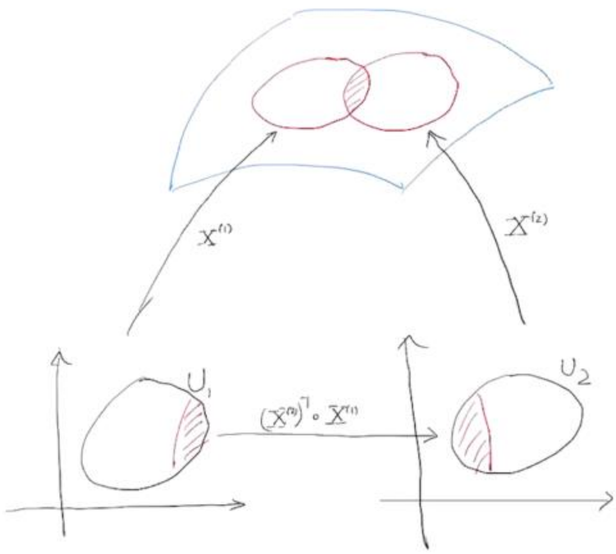
January 12, 2021 1:56 PM

Def: a parametrized surface element is a differentiable map $X\colon U \to \mathbb{R}^3, U \subset \mathbb{R}^2$ an open set with $X(u,v) = \big(x_1(u,v), x_2(u,v), x_3(u,v)\big)$

- X is regular if $\frac{\partial X}{\partial u}$ and $\frac{\partial X}{\partial v}$ are linearly independent (i.e. $\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} \neq 0$) for all $(u,v) \in U$
- Paraboloid: $X(u,v) = (u, v, u^2 + v^2)$
- Helicoid: $X(u,v) = (v\cos u, v\sin u, au)$ where $a \neq 0$ constant, $u \in (0,2\pi), v \in \mathbb{R}$
    - Through each point of a helix, draw a line parallel to xy −plane and through z −axis
    - Minimal, ruled surface
- Sphere: $X(\phi,\theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi), \phi \in (0,\pi), \theta \in (0,2\pi)$.
    - $\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v} = \sin\phi\, X$, $\left|\frac{\partial X}{\partial u} \times \frac{\partial X}{\partial v}\right| = |\sin\phi|$ is regular when $\phi \neq k\pi$
- Surface of revolution: let $\alpha\colon I \to \mathbb{R}^3$ $\alpha(u) = (0, f(u), g(u))$ be a curve, $f > 0$, rotate α about z −axis, $X(u,v) = \big(f(u)\cos v, f(u)\sin v, g(u)\big), u \in I, v \in (0,2\pi)$
- Note: all the regular surfaces miss some point

Def: a subset $S \subset \mathbb{R}^3$ is a regular surface if for each $p \in S$, there is an open set $V \subset \mathbb{R}^3$ containing p, an open set $U \subset \mathbb{R}^2$ and a differentiable map $X\colon U \to \mathbb{R}^3$ such that

- $X(U) = V \cap S$
- X is regular
- X is one-to-one and X^{-1} is continuous (i.e. the map X is a homeomorphism)
    - $\left(X^{(2)}\right)^{-1} \circ X^{(1)}$ is a diffeomorphism (differentiable with differentiable inverse)



Examples of regular surfaces

- Sphere $S^2 = \{(x,y,z)\colon x^2 + y^2 + z^2 = 1\}$
    - Let $U = \{(x,y) \in \mathbb{R}^2\colon x^2 + y^2 < 1\}$ $f(x,y) = \left(x, y, \sqrt{1 - x^2 - y^2}\right)$ is a regular parametrization of upper hemisphere

Tangent plane T_pS to S at $p = f(u_0, v_0)$ is the subspace of $\mathbb{R}^3$ spanned by $\frac{\partial f}{\partial u}(u_0, v_0)$ and $\frac{\partial f}{\partial v}(u_0, v_0)$

A unit normal to S at p is a unit vector normal to the tangent plan T_pS. Given a parametrization $f\colon U \subset \mathbb{R}^2 \to S \subset \mathbb{R}^3$, we obtain a unit normal vector at each $q \in f(U)$ by $n = \frac{\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}}{\left|\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}\right|}$

First fundamental form

- The inner (dot) product of $\mathbb{R}^3$ induces an inner product on $T_pS \subset \mathbb{R}^3$ by restriction, called the first fundamental form I_p

- $I_p: T_pS \times T_pS \to \mathbb{R}$ $I_p(v,w) = v \cdot w$, where v, w are two tangent vectors

- properties:
  - Symmetric ($I_p(v,w) = I_p(w,v)$)
  - Bilinear ($I_p(av_1 + bv_2, w) = aI_p(v_1,w) + bI_p(v_2,w)$)
  - Positive definite ($I_p(v,v) \geq 0, I_p(v,v) = 0$ if and only if $v = 0$)
- Suppose $\{v_1, v_2\}$ is a basis of T_pM, the matrix representation of I_p with respect to the basis is $\begin{pmatrix} I_p(v_1,v_1) & I_p(v_1,v_2) \\ I_p(v_2,v_1) & I_p(v_2,v_2) \end{pmatrix}$.
  - If $u = a_1v_1 + a_2v_2, w = b_1v_1 + b_2v_2$,
    then $I(u,w) = (a_1\ a_2)\begin{pmatrix} I_p(v_1,v_1) & I_p(v_1,v_2) \\ I_p(v_2,v_1) & I_p(v_2,v_2) \end{pmatrix}\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$
- Given a parametrization $f: U \subset \mathbb{R}^2 \to \mathbb{R}^3$, the matrix representation of I with respect to basis $\left\{\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right\}$ of T_pS is denoted $\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} I_p\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial u}\right) & I_p\left(\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}\right) \\ I_p\left(\frac{\partial f}{\partial v}, \frac{\partial f}{\partial u}\right) & I_p\left(\frac{\partial f}{\partial v}, \frac{\partial f}{\partial v}\right) \end{pmatrix} = \begin{pmatrix} f_uf_u & f_uf_v \\ f_vf_u & f_vf_v \end{pmatrix}$
- Def: two surfaces S_1 and S_2 are locally isometric if for each $p \in S_1$, there are parametrizations $f_1: U \to S_1, f_2: U \to S_2$ ($p \in f_1(U)$) such that $f_2 \circ f_1^{-1}: f_1(U) \to f_2(U)$ preserves the first fundamental forms
  - Plane and cylinder are locally isometric
- Importance: By knowing I, we can calculate geometric quantities (length, angle, area) without further reference to the ambient $\mathbb{R}^3$
  - Arclength of a parametrized curve in a surface: $\alpha: (a,b) \to S$ be a curve in S, the arclength of $\alpha: s(t) = \int_0^t |\alpha'(t)| dt = \int_0^t \sqrt{I(\alpha', \alpha')} dt = \int_0^t \sqrt{E\left(\frac{du}{dt}\right)^2 + 2F\frac{du}{dt}\frac{dv}{dt} + G\left(\frac{dv}{dt}\right)^2}\, dt$
    - Element of arclength is $ds^2 = Edu^2 + 2Fdudv + Gdv^2$
  - Angle: $\cos\theta = \frac{f_u \cdot f_v}{|f_u||f_v|} = \frac{F}{\sqrt{EG}}$
    - A parametrization $f: U \to S$ is a conformal (orthogonal) parametrization if $F(u,v) = 0$ and $E = G$, ($\theta = \frac{\pi}{2}$) for all $(u,v) \in U$. i.e. the coordinate curves are orthogonal
  - Area: $\iint_U |f_u \times f_v| dudv = \iint_U \sqrt{EG - F^2} dudv$
    - Note: $\left|\frac{\partial f}{\partial u} \times \frac{\partial f}{\partial v}\right|^2 + \left|\frac{\partial f}{\partial u} \cdot \frac{\partial f}{\partial v}\right|^2 = \left|\frac{\partial f}{\partial u}\right|^2 \left|\frac{\partial f}{\partial v}\right|^2$ does not depend on the parametrization

Def: A surface $S \subset \mathbb{R}^3$ is orientable if there exists a differentiable field of unit normal vectors $N: S \to \mathbb{R}^3$

- Differentiable means $N \circ f: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ differentiable
- The choice of such a field is called an orientation of S
  - There are always two choices: outward/inward based on the direction of normal vectors
- Cube is not regular, thus is not orientable (piecewise orientable on each surface)
- Any parametrized surface element $f: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ is oriented by $n = \frac{f_u \times f_v}{|f_u \times f_v|}$
- Unit sphere $S^2 = \{p \in \mathbb{R}^3: |p| = 1\}$ is orientable
  - $N(p) = p$ is the outward unit normal
- The smooth surface Mobius strip is not orientable

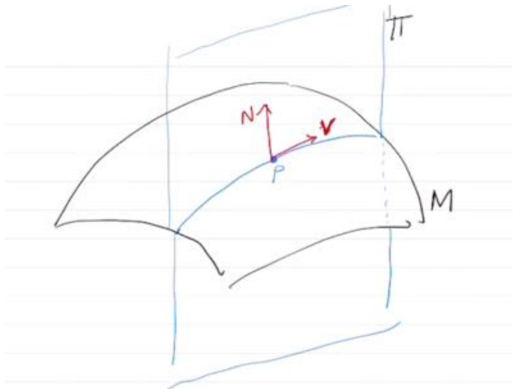
Def: Let $S \subset \mathbb{R}^3$ be a surface with orientation N. The map $N: S \to S^2 \subset \mathbb{R}^3$ ($p \to N(p)$)is called the Gauss map (S^2 is the unit sphere/set of all unit vectors in $\mathbb{R}^3$)

The shape operator at p is the map $S_p: T_pS \to \mathbb{R}^3$ $S_p(V) = -(D_VN)(p)$

- V is the tangent vector to the surface
- Measures the rate of change of N in the direction V at p
- $S_p(V) = -(N \circ \alpha)'(p)$, where α is any curve in S with $\alpha(0) = p, \alpha'(0) = V$
  - V can actually be any tangent vector, not necessarily unit
  - Since N is unit vector, $N(\alpha(t)) \cdot N(\alpha(t)) = 1$
    So $(N \circ \alpha)'(0) \cdot (N \circ \alpha)(0) = 0$,

i.e. $D_V N(p) \cdot N(p) = 0$, $D_V N(p) \in T_p S$
  - Prop: $S_p: T_p S \to T_p S$ is a self-ajoint (symmetric) linear map
    - $S_p(V) \cdot W = V \cdot S_p(W)$

Def: the second fundamental form of S at p is $II: T_p S \times T_p S \to \mathbb{R}$, $II(v, w) = I(S_p(V), W) = S_p(V) \cdot W$
- $II(V, V)$ gives the curvature
- Note: II is symmetric bilinear form
  - Shape operator is linear and dot product is bilinear
  - $II(V, W) = II(W, V)$ (S_p is self adjoint)
- Matrix representation with respect to the basis $\{f_u, f_v\}$ of $T_p S$: $II = \begin{pmatrix} l & m \\ m & n \end{pmatrix}$
  - $l = II(f_u, f_u) = -N_u \cdot f_u = N \cdot f_{uu}$, $m = II(f_u, f_v) = N \cdot f_{uv}$, $n = II(f_v, f_v) = N \cdot f_{vv}$
- Planes have zero second fundamental form, reverse is true
- Sphere oriented inward : $II = I$, $II(v, v) = |v| = 1$

Curves on surfaces and curvature
Let $\alpha: I \to M$ be a regular curve on the surface M parametrized by arclength with $\alpha'' \neq 0$
- $k_n = II(\alpha', \alpha') = \alpha'' \cdot N = \pm k$ is the normal curvature (curvature in a direction $v \in T_p M$) of α
  - Curvature of the curve that comes from the curvature of the surface
- Meusnier theorem: all curves on a given surface having at a given point p the same tangent line, have same normal curvature at p
  - k_n is determined by the surface and does not depend on the curve
  - $k_n = II(\alpha', \alpha') = \alpha'' \cdot N = k \cdot N = k \cos\theta$ (θ is the angle between the curve normal and surface normal)
- Consider a curve which a normal section of the surface intersects at p (a slice of M with a plane π through p parallel to the normal N to M at p)
  - The curve of intersection is the normal section
  - 

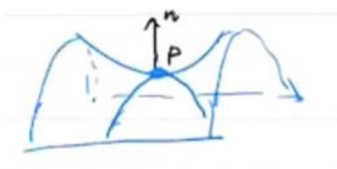
  - Then such a curve is a plane curve through p with $n(p) = \pm N(p)$, $k(p) = |k_n(p)| = |II(v, v)|$
  - For a plane, all normal sections are straight lines, thus, normal curvatures are 0 ($II = 0$)
  - For spheres with inward normal, normal sections are great circles through p(plane curves of radius 1), $k_n = \mathrm{II}(\mathrm{w}, \mathrm{w}) = \mathrm{I}(\mathrm{w}, \mathrm{w}) = 1$
  - For cylinder, normal sections vary from a circle ($k_n = 1$) to a straight line ($k_n = 0$)
- The minimum normal curvature at p, $k_1(p)$ and maximum $k_2(p)$ are called the principal curvatures of M at p, the corresponding directions e_1, e_2 are called the principal directions
  - e_1, e_2 are critical points of the function $w \to II(w, w)$ over all $w \in T_p M$ with $|w| = 1$
- Let $w \in T_p M$ with $|w| = 1$. Then w is a principal direction if and only if w is an eigen vector of the shape operator $S_p: T_p M \to T_p M$. The associated eigen values are the principal curvatures

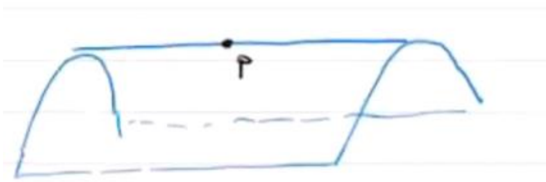
Some side notes of linear algebra
- V: n-dimensional vector space with inner product $<\cdot,\cdot>$
- $A: V \to V$ linear transformation, self-adjoint, $< Av, w > = < v, Aw > \forall v, w \in V$
- $B: V \times V \to \mathbb{R}$ associated symmetric bilinear form $B(v, w) = < Av, w >$
- Spectral theorem: there exists orthonormal basis $e_1, e_2, \ldots, e_n$ of V such that $Ae_1 = \lambda_i e_i$, and $e_1, \ldots, e_n$ are critical points of $v \to B(v, v)$ overall unit vectors v, $\lambda_i = B(e_i, e_i)$ is the eigen value

Principal curvatures and directions

- Principal curvatures: k_1, k_2 are min and max curvatures, i.e. $v \to II(v,v), |v| = 1$
- Principal directions: e_1, e_2 are corresponding directions
  - Critical points of $v \to II_p(v,v)$ over all unit vectors $v \in T_pM$
  - The principal directions are eigen vectors of $S_p: T_pM \to T_pM$
- Remark: either $k_1 = k_2$ and every direction is a principal direction or there exist exactly two (up to sign) principal directions orthogonal to each other
  - $k(p) = 0$ gives $k_n(p) = 0$
  - $k = |\alpha''| \geq \alpha'' \cdot n = k_n$(normal curvature)
  - $\alpha'' = (\alpha'')^T + (\alpha'')^\perp$
    - $(\alpha'')^T$ is the geodesic curvature
    - $(\alpha'')^\perp$ is the normal curvature
- Remark: given $w \in T_pM, |w| = 1, w = \cos\theta\, e_1 + \sin\theta\, e_2$,
  we have $II_p(w,w) = k_1 \cos^2\theta + k_2 \sin^2\theta$ (Euler's formula)

Gauss and mean curvature

- Gauss curvature: $K(p) = k_1(p)k_2(p) = \det S_p = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}$
- Mean curvature: $H(p) = \frac{1}{2}\left(k_1(p) + k_2(p)\right)$
- They are independent of the basis
- the signs of k_1, k_2 changes if we change the orientation of M
  - K does not change, H changes sign
- A point p on a surface is called
  - Elliptic if $K(p) > 0$ (k_1, k_2 have the same sign)
    - 
  - Hyperbolic if $K(p) < 0$ (k_1, k_2 have the opposite sign)
    - 

  - Parabolic if $K(p) = 0$ ($k_1 k_2 = 0$, one is zero, the other is non-zero)
    - 

  - Planar if $k_1 = k_2 = 0$
  - Umbilic if $k_1 = k_2 \neq 0$
- Expressions: given matrix of $S_p: T_pM \to T_pM$ relative to basis $\{f_u, f_v\}$,
  - $S_p = I^{-1}II = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} l & m \\ m & n \end{pmatrix}$
  - Then $K(p) = det(S_p)$ and $H(p) = \frac{1}{2} tr S_p = \frac{1}{2}(a + d)$
    - $K(p) = \dfrac{\det II}{\det I} = \dfrac{ln - m^2}{EG - F^2}$
    - Tr means the trace
  - The principal curvatures can be found by $\det(S - kI) = 0$
    - This gives that $k^2 - 2Hk + K = 0$
  - If S is diagonal, f_t, f_θ are principal directions, the diagonal are the principal curvatures

- Note, if $S_p = \begin{pmatrix} a & c \\ b & d \end{pmatrix}$:
  - $n_u = -(af_u + bf_v)$.
  - $n_v = -(cf_u + df_v)$,

A regular curve $\alpha: I \to M$ is called a line of curvature if $\frac{\alpha'(t)}{|\alpha'(t)|}$ is a principal direction for all $t \in I$ i.e. $S(\alpha') = k\alpha'$, α' is an eigen vector of the shape operator

- For a surface of revolution, f is line of curvature parametrization (t-curves, θ-curves are lines of curvature)
- $n' = \lambda\alpha'$ for the line of curvature, where n is the surface normal.

Surface of revolution of constant Gauss curvature

- Assume t is the arclength parameter of the curve $\alpha(t) = \big(r(t), 0, h(t)\big)$
- Constant Gauss curvature gives that $r'' + k_0 r = 0$
- When $k_0 = 0$, $r(t) = at + b$, $h(t) = \pm\sqrt{1 - a^2}t + c$
  - $a = 0$, $\alpha(t)$ is a straight line, the surface is a cylinder
  - $a = 1$, $\alpha(t)$ is a straight line, the surface is a plane
  - $a \in (0,1)$, surface is a cone
- $k_0 > 0$, sphere, elliptic integrals, oblate sphere

Def: $V \in T_pS$ is an asymptotic direction if the normal curvature in the direction V is zero. $II_p(V, V) = 0$.
A regular curve α in M is an asymptotic curve if $\alpha'(t)$ is an asymptotic direction for all t

- At an elliptic point p, $K_p > 0$, k_1 and k_2 have the same sign and are nonzero, there is no asymptotic direction

A ruled surface is a surface that can be parametrized as $f(s,t) = \alpha(t) + sX(t)$ where $X(t)$ is a vector field along $\alpha(t)$

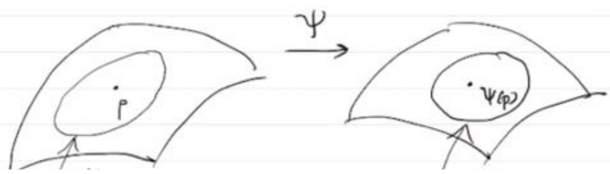
- The line $t = const$ are called generators
- A curve $s = const$ is called a directrix
- Examples: plane, cylinder, helicoid, cone, hyperboloid of one sheet
- Any ruled surface has Gauss curvature $K \leq 0$
- A ruled surface is developable if the Gauss map n is constant along generators
  - Examples: plane, cone, cylinder
  - A ruled surface is developable $\Leftrightarrow \frac{dn}{ds} = 0 \Leftrightarrow K = 0$
  - $S(V) = -n'$.

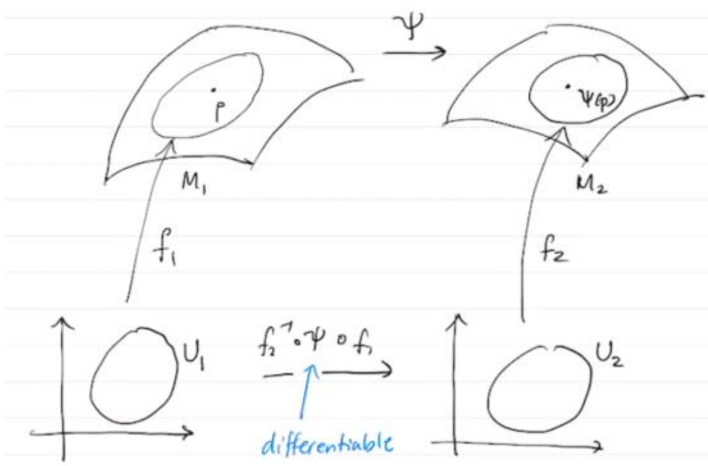
A surface with mean curvature $H = 0$ is called a minimal surface

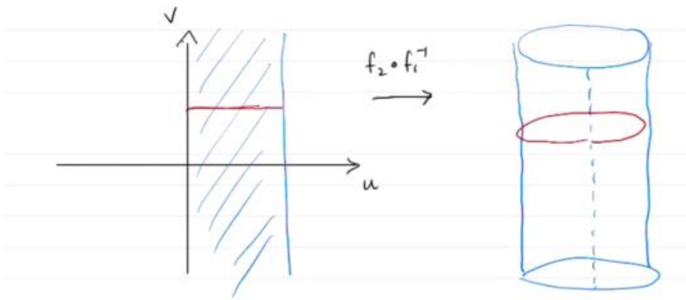
- Helicoid, catenoid, plane
- Note $H = 0 \Leftrightarrow k_1 = -k_2 \Leftrightarrow K = -k^2 \leq 0$ (every point is hyperbolic)
- It minimizes area locally
- Minimal surfaces are critical points of the area function
- Let $f: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ be a surface with boundary curve C,$f_t(u,v) = f(u,v) + th(u,v)n(u,v)$
  - Here $th(u,v)$ is a differentiable function that vanishes on the boundary

Intrinsic geometry of surfaces

- A property of a surface is intrinsic if it depends only on the first fundamental form
- Def: let M_1, M_2 be surfaces. An isometry from M_1 to M_2 is a one-to-one, onto, differentiable map $\psi: M_1 \to M_2$ such that for any curve $\alpha: [a,b] \to M_1$, the length of α equals the length of $\psi \circ \alpha$
  - Remark: $\psi: M_1 \to M_2$ is differentiable if for each $p \in M_1$, there are parametrizations $f_1: U_1 \to M_1$ about p and $f_2: U_2 \to M_2$ about $\psi(p)$ such that $f_2^{-1} \circ \psi \circ f_1: U_1 \to U_2$ is differentiable as a function of 2 variables



- 

- M_1 and M_2 are isometric if there is an isometry $\psi: M_1 \to M_2$
  - $\alpha'(t)$ and $(\psi \circ \alpha)'(t)$ have the same length, arc length are the same
  - We define distance between points $p, q \in M$ by $d_M(p,q) = inf\{L(\alpha): \alpha \ is \ a \ curve \ in \ M \ betwen \ p \ and \ q\}$.
- M_1 and M_2 are locally isometric if for each point $p \in M$, there are open sets V_1 about p and V_2 about $\psi(p)$ and an isometry $\psi: V_1 \to V_2$
  - Suppose $f_1: U \to M_1$, $f_2: U \to M_2$ are parametrizations such that $E_1 = E_2$,$F_1 = F_2$, $G_1 = G_2$, then $\psi = f_2 \circ f_1^{-1}$ is a local isometry
  - 

  - Helicoid and catenoid are locally isometric
- Note: isometry cannot be extended to global isometry (Cylinder is not homeomorphic to plane)
- A one-to-one, onto, differentiable map $\psi: M_1 \to M_2$ is conformal if for any $p \in M_1$ and any curves α and β with $\alpha(0) = \beta(0) = p$, we have $(\psi \circ \alpha)'(0) \cdot (\psi \circ \beta)'(0) = \lambda^2 \alpha'(0) \cdot \beta'(0)$where λ is differentiable and $\lambda \neq 0$
  - Conformal maps preserve angles, but length might be stretched
  - Suppose $f_1: U \to M_1$, $f_2: U \to M_2$ are parametrizations such that $E_1 = \lambda^2 E_2$, $F_1 = \lambda^2 F_2$,and $G_1 = \lambda^2 G_2$, $\lambda \neq 0$ is a differentiable function. Then $\psi = f_2 \circ f_1^{-1}$ is a local conformal map
- Given a regular surface M and $p \in M$, there is a local parametrization of M near p such that $E = G = \lambda^2$ and $F = 0$, λ nowhere zero and everywhere differentiable
  - Any surface is locally conformal to the plane
  - Any two regular surfaces are locally conformal

Codazzi and Gauss equations:

- Known the first and second fundamental form, we want to find the parametrization
- Start with writing derivatives of $\{f_u, f_v, \boldsymbol{n}\}$ in terms of $\{f_u, f_v, \boldsymbol{n}\}$

- Simple for $\boldsymbol{n}$, can just use the shape operator
- $f_{uu} = \Gamma^u_{uu} f_u + \Gamma^v_{uu} f_v + l\boldsymbol{n}$.
- $f_{uv} = \Gamma^u_{uv} f_u + \Gamma^v_{uv} f_v + m\boldsymbol{n}$.
- $f_{vu} = \Gamma^u_{vu} f_u + \Gamma^v_{vu} f_v + m\boldsymbol{n}$. $(f_{uv} = f_{vu})$
- $f_{vv} = \Gamma^u_{vv} f_u + \Gamma^v_{vv} f_v + n\boldsymbol{n}$.
- Γ^*_{**} are called the Christoffel symbols
- In matrix forms:
    - $\begin{pmatrix} f_u \\ f_v \\ n \end{pmatrix}_u = \begin{pmatrix} \Gamma^u_{uu} & \Gamma^v_{uu} & l \\ \Gamma^u_{vu} & \Gamma^v_{vu} & m \\ -a & -b & 0 \end{pmatrix} \begin{pmatrix} f_u \\ f_v \\ n \end{pmatrix}$.
    - $\begin{pmatrix} f_u \\ f_v \\ n \end{pmatrix}_v = \begin{pmatrix} \Gamma^u_{uv} & \Gamma^v_{uv} & m \\ \Gamma^u_{vv} & \Gamma^v_{vv} & n \\ -c & -d & 0 \end{pmatrix} \begin{pmatrix} f_u \\ f_v \\ n \end{pmatrix}$.

- Do dot product with f_u, f_v, then we can find expressions for the Christoffel symbols
    - $\begin{pmatrix} \Gamma^u_{uu} \\ \Gamma^v_{uu} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} E_u \\ F_u - \frac{1}{2} E_v \end{pmatrix}$.
    - $\begin{pmatrix} \Gamma^u_{uv} \\ \Gamma^v_{uv} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} \frac{1}{2} E_v \\ \frac{1}{2} G_u \end{pmatrix}$.
    - $\begin{pmatrix} \Gamma^u_{vv} \\ \Gamma^v_{vv} \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}^{-1} \begin{pmatrix} F_v - \frac{1}{2} G_u \\ \frac{1}{2} G_v \end{pmatrix}$.
- So, given E, F, G, l, m, n, we can solve for f_u, f_v, n and then f. But these are PDEs, and have solutions only if certain compatibility conditions are satisfied
    - Integrability conditions: $f_{uuv} = f_{uvu}, f_{vvu} = f_{vuv}, n_{uv} = n_{vu}$.
- Codazzi-Mainardi Equations
    - $l_v - m_u = l\Gamma^u_{uv} + m(\Gamma^v_{uv} - \Gamma^u_{uu}) - n\Gamma^v_{uu}$.
    - $m_v - n_u = l\Gamma^u_{vv} + m(\Gamma^v_{vv} - \Gamma^u_{uv}) - n\Gamma^v_{uv}$.

Theorema Egregium of Gauss

- The Gauss curvature of a surface is determined by the first fundamental form. That is, K can be computed from just E, F, and G and their first and second partial derivatives.
    - Gauss curvature is intrinsic
- Formula:
    - $EK = (\Gamma^v_{uu})_v - (\Gamma^v_{uv})_u + \Gamma^u_{uu}\Gamma^v_{uv} + \Gamma^v_{uu}\Gamma^v_{vv} - \Gamma^v_{uu}\Gamma^u_{uv} - (\Gamma^v_{uv})^2$. The other ones FK, GK are equivalent
    - If $F = 0$, then $K = -\frac{1}{2\sqrt{EG}}\left(\left(\frac{E_v}{\sqrt{EG}}\right)_v + \left(\frac{G_u}{\sqrt{EG}}\right)_u\right)$

Fundamental theorem of surfaces

- Suppose E, F, G, l, m, n are differentiable functions on an open set $U \subset \mathbb{R}^2$ with $E > 0, G > 0$, $EG - F^2 > 0$ and satisfy the Gauss and Codazzi equations
- Then $\forall q \in U$, there is an open set $U' \subset U, q \in U'$ and parametrized surface $f: U' \to \mathbb{R}^3$ that has E, F, G, l, m, n as its first and second fundamental forms
- Moreover, f is unique up to isometries in $\mathbb{R}^3$
- Uniqueness: let $f: U \subset \mathbb{R}^2 \to \mathbb{R}^3$ parametrized surface
    - $\psi: \mathbb{R}^3 \to \mathbb{R}^3$ isometry $\Rightarrow f$ and $\psi \circ f$ have the same first and second fundamental form
    - $f, \overline{f}$ have same first and second fundamental form $\Rightarrow \overline{f} = \psi \circ f$, ψ isometry of $\mathbb{R}^3$
    - Note the second fundamental form can have different signs

Vector fields

- A function $X: M \to \mathbb{R}^3$ is tangent vector field if
    - $X(p) \in T_pM$ for all $p \in M$

- X is differentiable (for any parametrization $f: U \to \mathbb{R}^3$, $X \circ f$ is differentiable)

- Given a vector field X, we can differentiate X
  - If $v \in T_pM$, $(D_vX)(p) = (X \circ \alpha)'(0)$ where α is a curve in M with $\alpha(0) = p$, $\alpha'(0) = v$.

Covariant derivative

- $\nabla_v X = (D_v X)^T$ is the orthogonal projection of $D_v X$ onto $T_p M$
- If $f: U \to M$ is a parametrized surface
  - $\nabla_{f_u} f_u = (f_{uu})^T = \Gamma^u_{uu} f_u + \Gamma^v_{uu} f_v$.
  - $\nabla_{f_v} f_u = (f_{uv})^T = \Gamma^u_{uv} f_u + \Gamma^v_{uv} f_v$.
  - $\nabla_{f_v} f_v = (f_{vv})^T = \Gamma^u_{vv} f_u + \Gamma^v_{vv} f_v$.
- Covariant differentiation is intrinsic
- $D_v X = 0$ if X is a constant vector field

Parallel vector field

- Given a curve α in M, we say a vector field is parallel along α if $\nabla_{\alpha'} X = 0$
  - $(X \circ \alpha)'(t)$ is a multiple of the normal vector $n(\alpha(t))$
  - Note that if $\alpha(t) = (u(t), v(t))$, then $X(\alpha(t)) = f(u(t), v(t))$
- Let $\alpha: I \to M$ be a curve. Given $t_0 \in I$ and $X_0 \in T_{\alpha(t_0)}M$, there exists a unique parallel vector field X along α with $X(t_0) = X_0$
  - $X(\alpha(t))$ is called the parallel translation of X_0 along α
  - If $X(\alpha(t)) = a(t) f_u + b(t) f_v$, we can solve a, b by the system of ODEs
    - $a' + (u'\Gamma^u_{uu} + v'\Gamma^u_{uv})a + (u'\Gamma^u_{uv} + v'\Gamma^u_{vv})b = 0.$
    - $b' + (u'\Gamma^v_{uu} + v'\Gamma^v_{uv})a + (u'\Gamma^v_{uv} + v'\Gamma^v_{vv})b = 0.$
- Remark: If X and Y are parallel vector fields along α, then $X(\alpha(t)) \cdot Y(\alpha(t))$ is a constant
  - A parallel vector field must have constant length and the angle between parallel vector fields remain constant
- If M and $\overline{M}$ are tangent along a curve α, and X is a vector field along α, then the covariant derivative of X is the same for both surfaces
  - X is parallel along α in $M \Leftrightarrow X$ is parallel along α in $\overline{M}$
- Parallel vector field does not usually exist globally

Geodesics (analog of straight line in a surface)

- The unit tangents are parallel (never changes direction)
- A parametrized curve α in a surface M is a geodesic if $\nabla_{\alpha'} \alpha' = (D_{\alpha'} \alpha')^T = (\alpha'')^T = 0$
  - α' is parallel along α
  - α'' is orthogonal to M
- Remark: α is a geodesic, then α is parametrized proportional to arclength
- An unparametrized curve C is said to be a geodesic if its arclength parametrization is a geodesic
  - Great circles on sphere are geodesic
  - Plane lines are geodesic
- Let $\alpha: I \to M$ be a curve parametrized by arclength,
  - we know that we can decompose $\alpha'' = (\alpha'')^T + (\alpha'')^\perp = (\alpha'' \cdot (n \times T))n \times T + (\alpha'' \cdot n)n$
  - $(\alpha'')^T$ is the geodesic curvature
    - call $\{T, n \times T, n\}$ the Darboux frame
    - $(\alpha'')^T$ is always in the $n \times T$ direction
    - Define the geodesic curvature $k_g = \alpha'' \cdot (n \times T)$
  - $(\alpha'')^\perp = n \cdot \alpha'' = II(\alpha', \alpha')n$ is the normal curvature
  - Note that $k^2 = k_g^2 + k_n^2$, since the two basis are orthogonal
- Existence of geodesics: given a point $p \in M$ and $V \in T_pM$, there exists $\epsilon > 0$ and a unique geodesic $\alpha: (-\epsilon, \epsilon) \to M$ with $\alpha(0) = p$, $\alpha'(0) = V$
  - i.e. there is a unique geodesic through any given point of a surface in any given direction
- System of ODEs to solve for geodesic:
  - Let $\alpha(t) = f(u(t), v(t))$, $\alpha' = u' f_u + v' f_v$

- $u'' + \Gamma_{uu}^{u}(u')^2 + 2\Gamma_{uv}^{u}u'v' + \Gamma_{vv}^{u}(v')^2 = 0.$
- $v'' + \Gamma_{uu}^{v}(u')^2 + 2\Gamma_{uv}^{v}u'v' + \Gamma_{vv}^{v}(v')^2 = 0.$

- Isometry maps geodesic to geodesic
- A short enough piece of geodesic is the curve of length between its endpoints
- For a surface of revolution, meridians are always geodesics
  - Parallels are geodesic if and only if $r'(t_0) = 0.$
  - $\alpha(s)$ is a geodesic if and only if $r(t(s))\cos\phi(s) = const.$ (Clairaut's relation)where $\phi(s)$ is the angle between $\alpha'(s)$ and the parallel
    - Geodesics that is not meridians intersects itself infinitely many times

Further topics

January 12, 2021 1:56 PM

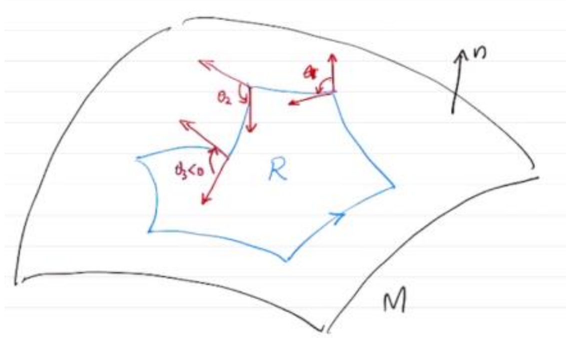
Gauss Bonnet Theorem (simple case)

- A simple closed regular curve bounds a simply-connected region R, then we have:

$$\iint_R K dA + \int_{\partial R} k_g ds = 2\pi$$

  - k_g is the geodesic curvature
  - K is the Gauss curvature
  - $dA = \sqrt{EG - F^2} dudv$ is the area element

Local Gauss Bonnet Theorem

- Suppose R is a simply-connected region with piecewise regular boundary lying in an orthogonal parametrization (F=0)
- If $C = \partial R$ has exterior angles $\theta_j, j = 1, .., n$, then $\iint_R K dA + \int_{\partial R} k_g ds + \sum_{i=1}^n \theta_i = 2\pi$
  - 

  - θ_i is the oriented angle between $\alpha'(t_i^-)$ and $\alpha'(t_i^+)$
- Geodesic $n - gon$: assume $\alpha|_{(s_i, s_{i+1})}$ is geodesic, then $\iint_R K dA = 2\pi - \sum_{i=1}^n \theta_i$
  - $n - gon$ is a polygon of n sides
  - Geodesic $n - gon$ is an $n - gon$ with all sides being geodesics
  - Exterior angle $= \theta_i$, then interior angle $\beta_i = \pi - \theta_i$
  - So for a geodesic triangle, $\iint_R K dA = \beta_1 + \beta_2 + \beta_3 - \pi$
    - $K > 0$ means sum of interior angles is greater than π
    - $K = 0$, sum of interior angles is equal to π
    - $K < 0$, sum of interior angles is less than π
  - If $K \leq 0$ everywhere, then geodesic 2-gons do not exist

Triangulation

- The link between local and global result is provided by triangulations
- Let M be a regular surface, a closed bounded subset $R \subset M$ is regular, if ∂R is the union of simple closed piecewise regular curves that don't intersect
- $T \subset M$ is a triangle if T is homeomorphic to a disk and ∂T has 3 vertices
- A triangulation of a regular region $R \subset M$ is a finite collection of triangles $\{T_1, T_2, \ldots, T_n\}$ such that
  - $\cup_{i=1}^n T_i = R$.
  - If $T_i \cap T_j \neq \emptyset$, then $T_i \cap T_j$ is either a common edge or a common vertex
  - Every regular region R in a regular surface M admits a triangulation

The Euler characteristic of a triangulation R is $\chi = F - E + V$

- F is the number of faces
- E is the number of edges
- V is the number of vertices

- The Euler characteristic does not depend on the triangulation (topological invariant)
- E.g.
  - Disk $\chi = 1$
  - sphere $\chi = 2$
  - Torus $\chi = 0$
  - Two torus $\chi = -2$
  - N-torus $\chi = -2(n-1)$
- Properties
  - Every regular region R of a surface M admits a triangulation
  - The Euler characteristic doesn't depend on the triangulation
- The Euler characteristic allows a topological classification of surfaces in $\mathbb{R}^3$
- $M \subset \mathbb{R}^3$ compact, connected surface without boundary, then $\chi(M) \in \{2,0,-2,\dots,-2(n-1)\}$
- $\chi(M_1) = \chi(M_2)$ if and only if M_1 is homeomorphic to M_2 (there is a bijection from M_1 to M_2)
  - Every compact connected $M \subset \mathbb{R}^3$ without boundary is homeomorphic to a sphere with a certain number g of handles attached $\chi(M) = -2(g-1)$, $g = \frac{2-\chi(M)}{2}$ is called genus

Global Gauss-Bonnet theorem

- Let $R \subset M$ be a regular region of an oriented surface, ∂R consists of closed piecewise regular simple curve (given the positive orientation), $\{\theta_1, \dots, \theta_l\}$ the set of exterior angles of the boundary curve, then $\iint_R KdA + \int_{\partial R} k_g ds + \sum_{i=1}^{n} \theta_i = 2\pi\chi(R)$
- If M is compact orientable surface without boundary, then $\iint_R KdA = 2\pi\chi(R)$
- Consequences:
  - $\chi(M)$ is independent of the choice of triangulation
  - Since $\chi(M)$ is an integer, $\frac{1}{2\pi}\iint_R KdA$ is an integer
  - Gauss-Bonnet theorem asserts the equality of two very differently defined properties
    - Integral of the Gauss curvature (determined by local geometry)
    - Global topological invariant
- A compact surface without boundary of positive curvature is homeomorphic to the sphere
- Define orthonormal vectors, $e_1 = \frac{f_u}{\sqrt{E}}, e_2 = \frac{f_v}{\sqrt{G}}$, then $\phi_{12} = \left(\nabla_{\alpha'} e_1\right) \cdot e_2$
  - $\phi_{12} = \frac{1}{2\sqrt{EG}}\left(-E_v u' + G_u v'\right) + \theta'$. It measures the rate at which e_1 is turning
  - $k_g = \phi_{12} + \theta'$ where θ measures the turning of unit tangent α' relative to e_1.